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Solution and stability of system of quartic functional equations

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Abstract

In this paper, the authors introduced and investigated the general solution of system of quartic functional equations

$$\begin{aligned}
&f(x+y+z)+f(x+y-z)+f(x-y+z)+f(x-y-z)\\
&\quad=2[f(x+y)+f(x-y)+f(x+z)+f(x-z)+f(y+z)+f(y-z)]\\
&\qquad-4[f(x)+f(y)+f(z)],\\
&f(3x+2y+z)+f(3x+2y-z)+f(3x-2y+z)+f(3x-2y-z)\\
&\quad=72[f(x+y)+f(x-y)]+18[f(x+z)+f(x-z)]+8[f(y+z)+f(y-z)]\\
&\qquad+144f(x)-96f(y)-48f(z),\\
&f(x+2y+3z)+f(x+2y-3z)+f(x-2y+3z)+f(x-2y-3z)\\
&\quad=8[f(x+y)+f(x-y)]+18[f(x+z)+f(x-z)]+72[f(y+z)+f(y-z)]\\
&\qquad-48f(x)-96f(y)+144f(z).
\end{aligned}$$

Its generalized Hyers-Ulam stability using Hyers direct method and fixed point method are discussed. Counter examples for non stable cases are also given.

Keywords: Quartic functional equation, Generalized Hyers-Ulam stability, fixed point

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1 Introduction

One of the interesting questions in the theory of functional analysis concerning the stability problem of functional equations had been first raised by S.M. Ulam [28] as follows: When is it true that a mapping satisfying a functional equation approximately must be close to an exact solution of the given functional equation? For very general functional equations, the concept of stability for functional equations arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation?

In 1941, D. H. Hyers [9] gave an affirmative answer to the question of S.M. Ulam for Banach spaces. In 1950, T. Aoki [2] was the second author to treat this problem for additive mappings. In 1978, Th.M. Rassias [20] succeeded in extending Hyers' Theorem by weakening the condition for the Cauchy difference controlled by $||x||^p+||y||^p,\ p\in[0,1)$, to be unbounded.

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In 1982, J.M. Rassias [18] replaced the factor $||x||^p + ||y||^p$ by $||x||^p||y||^q$ for $p, q \in \mathbb{R}$. A generalization of all the above stability results was obtained by P. Gavruta [8] in 1994 by replacing the unbounded Cauchy difference by a general control function $\varphi(x, y)$.

In 2008, a special case of Gavruta's theorem for the unbounded Cauchy difference was obtained by Ravi et. al., [25] by considering the summation of both the sum and the product of two $p-$ norms. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1, 5, 6, 7, 10, 11, 12, 15, 21, 23]) and reference cited there in.

The quartic functional equation

$$F(x+2y) + F(x-2y) + 6F(x) = 4[F(x+y) + F(x-y) + 6F(y)] \tag{1.1}$$

was first introduced by J.M. Rassias [19], who solved its Ulam stability problem. Later P.K. Sahoo and J.K. Chung [26], S.H. Lee et. al., [13] remodified J.M. Rassias' equation as

$$f(2x+y) + f(2x-y) = 4f(x+y) + 4f(x-y) + 24f(x) - 6f(y) \tag{1.2}$$

and obtained its general solution.

Also the generalized Hyers-Ulam-Rassias stability for a 3 dimensional quartic functional equation

$$\begin{aligned} &g(2x+y+z) + g(2x+y-z) + g(2x-y+z) + g(-2x+y+z) + 16g(y) + 16g(z) \\ &= 8[g(x+y) + g(x-y) + g(x+z) + g(x-z)] + 2[g(y+z) + g(y-z)] + 32g(x) \end{aligned} \tag{1.3}$$

in fuzzy normed space was discussed by M. Arunkumar [3]. Several other types of quartic functional equations were introduced and investigated in [4, 16, 22, 24, 27].

In this paper, the authors introduced and investigated the general solution of system of quartic functional equations

$$\begin{aligned} &f(x+y+z) + f(x+y-z) + f(x-y+z) + f(x-y-z) \\ &\quad = 2[f(x+y) + f(x-y) + f(x+z) + f(x-z) + f(y+z) + f(y-z)] \\ &\qquad - 4[f(x) + f(y) + f(z)], \end{aligned} \tag{1.4}$$

$$\begin{aligned} &f(3x+2y+z) + f(3x+2y-z) + f(3x-2y+z) + f(3x-2y-z) \\ &\quad = 72[f(x+y) + f(x-y)] + 18[f(x+z) + f(x-z)] + 8[f(y+z) + f(y-z)] \\ &\qquad + 144f(x) - 96f(y) - 48f(z), \end{aligned} \tag{1.5}$$

$$\begin{aligned} &f(x+2y+3z) + f(x+2y-3z) + f(x-2y+3z) + f(x-2y-3z) \\ &\quad = 8[f(x+y) + f(x-y)] + 18[f(x+z) + f(x-z)] + 72[f(y+z) + f(y-z)] \\ &\qquad - 48f(x) - 96f(y) + 144f(z). \end{aligned} \tag{1.6}$$

Its generalized Ulam - Hyers stability using Hyers direct method and fixed point method are discussed. Counter examples for non stable cases are also given.

In Section 2, we proved the general solutions of (1.4), (1.5) and (1.6) are provided.

In Section 3, the generalized Ulam - Hyers stability of the functional equation (1.5) using Hyers direct method is investigated.

In Section 4, Counter examples of non stable cases are provided.

The generalized Ulam - Hyers stability of the functional equation (1.5) using another substitutions is given in Section 5.

Also, the generalized Ulam - Hyers stability of the functional equation (1.5) using fixed point method is present in Section 6.

2 General Solutions of (1.4), (1.5) and (1.6)

In this section, the general solutions of (1.4), (1.5) and (1.6) are given. Throughout this section, let X and Y be real vector spaces.

Lemma 2.1. *[13] If a mapping $f : X \to Y$ satisfies the functional equation (1.2) for all $x, y \in X$, then $f : X \to Y$ is quartic.*

Proof. Let $f : X \to Y$ satisfies the functional equation (1.2) for all $x, y \in X$. Setting (x, y) by $(0, 0)$ in (1.2), we get $f(0) = 0$. Again setting x by 0 in (1.2), we reach

$$f(-y) = f(y)$$

for all $y \in X$. Therefore f is an even function. Replacing y by 0 and y by $2x$ in (1.2), we obtain

$$f(2x) = 2^4 f(x) \qquad \textit{and} \qquad f(3x) = 3^4 f(x)$$

respectively, for all $x \in X$. In general for any positive integer a, we have

$$f(ax) = a^4 f(x)$$

for all $x \in X$. Hence f is quartic. □

Theorem 2.0. *If the mapping $f : X \to Y$ satisfies the functional equation (1.2) for all $x, y \in X$, then $f : X \to Y$ satisfies the functional equation (1.4) for all $x, y, z \in X$.*

Proof. Let $f : X \to Y$ satisfies the functional equation (1.2) for all $x, y \in X$. Replacing (x, y) by (y, z) in (1.2), we get

$$f(2y+z) + f(2y-z) = 4f(y+z) + 4f(y-z) - 6f(z) + 24f(y) \tag{2.1}$$

for all $y, z \in X$. Replacing z by $x + z$ in (2.1)and using evenness of f , we obtain

$$f(x+2y+z) + f(x-2y+z) = 4[f(x+y+z) + f(x-y+z)] - 6f(x+z) + 24f(y) \tag{2.2}$$

for all $x, y, z \in X$. Replacing z by $-z$ in (2.2), we get

$$f(x+2y-z) + f(x-2y-z) = 4[f(x+y-z) + f(x-y-z)] - 6f(x-z) + 24f(y) \tag{2.3}$$

for all $x, y, z \in X$. Adding (2.2) and (2.3), we reach

$$\begin{aligned} f(x+2y+z) &+ f(x-2y+z) + f(x+2y-z) + f(x-2y-z) \\ &= 4[f(x+y+z) + f(x-y+z) + f(x+y-z) + f(x-y-z)] \\ &\quad - 6[f(x+z) + f(x-z)] + 48f(y) \end{aligned} \tag{2.4}$$

for all $x, y, z \in X$. Interchanging y and z in (2.4), we get

$$\begin{aligned} f(x+y+2z) &+ f(x-y+2z) + f(x+y-2z) + f(x-y-2z) \\ &= 4[f(x+y+z) + f(x-y+z) + f(x+y-z) + f(x-y-z)] \\ &\quad - 6[f(x+y) + f(x-y)] + 48f(z) \end{aligned} \tag{2.5}$$

for all $x, y, z \in X$. Interchanging x and z in (2.5) and using evenness of f, we have

$$\begin{aligned} f(2x+y+z) &+ f(2x-y+z) + f(2x+y-z) + f(2x-y-z) \\ &= 4[f(x+y+z) + f(x-y+z) + f(x+y-z) + f(x-y-z)] \\ &\quad - 6[f(y+z) + f(y-z)] + 48f(x) \end{aligned} \tag{2.6}$$

for all $x, y, z \in X$. Replacing y by $2y$ in (2.6) and using (2.4) and (2.1), we arrive

$$\begin{aligned} f(2x+2y+z) &+ f(2x-2y+z) + f(2x+2y-z) + f(2x-2y-z) \\ &= 4[f(x+2y+z) + f(x-2y+z) + f(x+2y-z) + f(x-2y-z)] \\ &\quad - 6[f(2y+z) + f(2y-z)] + 48f(x) \\ &= 16[f(x+y+z) + f(x-y+z) + f(x+y-z) + f(x-y-z)] \\ &\quad - 24[f(x+z) + f(x-z)] - 24[f(y+z) + f(y-z)] \\ &\quad + 48f(x) + 48f(y) + 36f(z) \end{aligned} \tag{2.7}$$

for all $x, y, z \in X$. Replacing z by $2z$ in (2.7) and using (2.5) and (2.1), we obtain

$$\begin{aligned}
&f(2x+2y+2z)+f(2x-2y+2z)+f(2x+2y-2z)+f(2x-2y-2z)\\
&=16[f(x+y+2z)+f(x-y+2z)+f(x+y-2z)+f(x-y-2z)]\\
&\quad-24[f(x+2z)+f(x-2z)]-24[f(y+2z)+f(y-2z)]\\
&\quad+48f(x)+48f(y)+36f(2z)
\end{aligned}\tag{2.8}$$

for all $x, y, z \in X$. With the help of Lemma 2.1, we desired our result. □

Theorem 2.0. *A mapping $f : X \to Y$ satisfies the functional equation (1.2) for all $x, y \in X$ if, and only if, $f : X \to Y$ satisfies the functional equation (1.5) for all $x, y, z \in X$.*

Proof. Let $f : X \to Y$ satisfies the functional equation (1.2) for all $x, y \in X$. Replacing y by $x + z$ in (1.2) and evenness of f, we obtain

$$f(3x+z)+f(x-z)=4[f(2x+z)+f(z)]+24f(x)-6f(x+z)\tag{2.9}$$

for all $x, z \in X$. Replacing z by $-z$ in (2.9) and using evenness of f, we get

$$f(3x-z)+f(x+z)=4[f(2x-z)+f(z)]+24f(x)-6f(x-z)\tag{2.10}$$

for all $x, z \in X$. Adding (2.9) and (2.10) and using (1.2), we arrive

$$f(3x+z)+f(3x-z)=9[f(x+z)+f(x-z)]+144f(x)-16f(z)\tag{2.11}$$

for all $x, z \in X$. Replacing z by $y + z$ in (2.11), we have

$$f(3x+y+z)+f(3x-y-z)=9[f(x+y+z)+f(x-y-z)]+144f(x)-16f(y+z)\tag{2.12}$$

for all $x, y, z \in X$. Replacing z by $-z$ in (2.12), we get

$$f(3x+y-z)+f(3x-y+z)=9[f(x+y-z)+f(x-y+z)]+144f(x)-16f(y-z)\tag{2.13}$$

for all $x, y, z \in X$. Adding (2.12) and (2.13) and using Theorem 2.0, we have

$$\begin{aligned}
&f(3x+y+z)+f(3x-y-z)+f(3x+y-z)+f(3x-y+z)\\
&=9[f(x+y+z)+f(x-y-z)+f(x+y-z)+f(x-y+z)]\\
&\quad-16[f(y+z)+f(y-z)]+288f(x)\\
&=18[f(x+y)+f(x-y)]+18[f(x+z)+f(x-z)]+2[f(y+z)+f(y-z)]\\
&\quad+252f(x)-36f(y)-36f(z)
\end{aligned}\tag{2.14}$$

for all $x, y, z \in X$. Replacing y by $2y$ in (2.14) and using (2.1), we arrive (1.5) as desired.

Conversely, assume that $f : X \to Y$ satisfies the functional equation (1.5) for all $x, y, z \in X$. Setting $x = y = z = 0$ in (1.5), we obtain $f(0) = 0$. Replacing (x, y, z) by $(0, 0, x)$ in (1.5), we reach $f(-x) = f(x)$ for all $x \in X$. Setting $x = z = 0$ in (1.5), we have $f(2y) = 2^4 f(y)$ for all $y \in X$. Setting $y = z = 0$ in (1.5), we get $f(3x) = 3^4 f(x)$ for all $x \in X$. In general for any positive integer a, we obtain $f(ax) = a^4 f(x)$ for all $x \in X$. Replacing (x, y, z) by $(0, x, y)$ in (1.5) and using evenness of f, we reach (1.2) as desired. □

Theorem 2.0. *If $f : X \to Y$ satisfies the functional equation (1.5), then there exists a unique symmetric multi - additive function $Q : X \times X \times X \times X \to Y$ such that*

$$f(x) = Q(x, x, x, x)$$

for all $x \in X$.

Proof. By Theorem 2.0, if $f : X \to Y$ satisfies the functional equation (1.5), then $f : X \to Y$ satisfies the functional equation (1.2) for all $x, y \in X$. By Theorem 2.1 of [13], we desired our result. □

Corollary 2.0. *If the mapping $f : X \to Y$ satisfies the functional equation (1.5) for all $x, y, z \in X$, then $f : X \to Y$ satisfies the functional equation (1.4) for all $x, y, z \in X$.*

Corollary 2.0. *If the mapping $f : X \to Y$ satisfies the functional equation (1.5) for all $x, y, z \in X$, then $f : X \to Y$ satisfies the functional equation (1.6) for all $x, y, z \in X$.*

Hereafter, through out this paper, let we consider G be a normed space and H be a Banach space. Define a mapping $Df : G \to H$ by

$$\begin{aligned} Df(x,y,z) = f(3x+2y+z) + f(3x+2y-z) + f(3x-2y+z) + f(3x-2y-z) \\ -72[f(x+y)+f(x-y)] - 18[f(x+z)+f(x-z)] \\ -8[f(y+z)+f(y-z)] - 144f(x) + 96f(y) + 48f(z) \end{aligned}$$

for all $x, y, z \in G$.

3 Stability results of (1.2): Direct method

In this section, the generalized Ulam - Hyers stability of the quartic functional equation (1.5) is given.

Theorem 3.0. *Let $j = \pm 1$ and $\psi : G^3 \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\psi\left(6^{nj}x, 6^{nj}y, 6^{nj}z\right)}{6^{4nj}} = 0 \tag{3.1}$$

for all $x, y, z \in G$. Let $f : G \to H$ be a function satisfying the inequality

$$\|Df(x,y,z)\| \le \psi(x,y,z) \tag{3.2}$$

for all $x, y, z \in G$. Then there exists a unique quartic mapping $Q : G \to H$ which satisfies (1.5) and

$$\|f(x) - Q(x)\| \le \frac{1}{6^4} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\xi\left(6^{kj}x\right)}{6^{4kj}} \tag{3.3}$$

where $\xi(x)$ and $Q(x)$ are defined by

$$\xi(x) = \psi(x,x,x) + \frac{1}{2}\psi(x,0,x) + \frac{89}{4}\psi(0,x,0) \tag{3.4}$$

and

$$Q(x) = \lim_{n\to\infty} \frac{f(6^{nj}x)}{6^{4nj}} \tag{3.5}$$

for all $x \in G$, respectively.

Proof. Replacing (x,y,z) by (x,x,x) in (3.2), we get

$$\|f(6x) + f(4x) - 97f(2x)\| \le \psi(x,x,x) \tag{3.6}$$

for all $x \in G$. Again, replacing (x,y,z) by $(x,0,x)$ in (3.2), we obtain

$$\|f(4x) - 8f(2x) - 128f(x)\| \le \frac{1}{2}\psi(x,0,x) \tag{3.7}$$

for all $x \in G$. Finally, replacing (x,y,z) by $(0,x,0)$ in (3.2), we have

$$\|f(2x) - 16f(x)\| \le \frac{1}{4}\psi(0,x,0) \tag{3.8}$$

for all $x \in G$. It follows from (3.6), (3.7), and (3.8) that

$$\begin{aligned} &\|f(6x) - 1296f(x)\| \\ &= \|f(6x) + f(4x) - 97f(2x) - f(4x) + 8f(2x) + 128f(x) + 89f(2x) - 1424f(x)\| \\ &\le \|f(6x) + f(4x) - 97f(2x)\| + \|f(4x) - 8f(2x) - 128f(x)\| + 89\|f(2x) - 16f(x)\| \\ &\le \psi(x,x,x) + \frac{1}{2}\psi(x,0,x) + \frac{89}{4}\psi(0,x,0) \end{aligned} \tag{3.9}$$

for all $x \in G$. Dividing the above inequality by 1296, we obtain

$$\left\| \frac{f(6x)}{6^4} - f(x) \right\| \leq \frac{\xi(x)}{6^4} \tag{3.10}$$

where

$$\xi(x) = \psi(x,x,x) + \frac{1}{2}\psi(x,0,x) + \frac{89}{4}\psi(0,x,0)$$

for all $x \in G$. Now replacing x by $6x$ and dividing by 6^4 in (3.10), we get

$$\left\| \frac{f(6^2x)}{6^8} - \frac{f(6x)}{6^4} \right\| \leq \frac{\xi(6x)}{6^8} \tag{3.11}$$

for all $x \in G$. From (3.10) and (3.11), we obtain

$$\begin{aligned} \left\| \frac{f(6^2x)}{6^8} - f(x) \right\| &\leq \left\| \frac{f(6x)}{6^4} - f(x) \right\| + \left\| \frac{f(6^2x)}{6^8} - \frac{f(6x)}{6^4} \right\| \\ &\leq \frac{1}{6^4}\left[\xi(x) + \frac{\xi(6x)}{6^4} \right] \end{aligned} \tag{3.12}$$

for all $x \in G$. Proceeding further and using induction on a positive integer n, we get

$$\begin{aligned} \left\| \frac{f(6^nx)}{6^{4n}} - f(x) \right\| &\leq \frac{1}{6^4} \sum_{k=0}^{n-1} \frac{\xi(6^kx)}{6^{4k}} \\ &\leq \frac{1}{6^4} \sum_{k=0}^{\infty} \frac{\xi(6^kx)}{6^{4k}} \end{aligned} \tag{3.13}$$

for all $x \in G$. In order to prove the convergence of the sequence $\left\{ \frac{f(6^nx)}{6^{4n}} \right\}$, replace x by 6^mx and dividing by 6^{4m} in (3.13), for any $m, n > 0$, we deduce

$$\begin{aligned} \left\| \frac{f(6^{n+m}x)}{6^{4(n+m)}} - \frac{f(6^mx)}{6^{4m}} \right\| &= \frac{1}{6^{4m}} \left\| \frac{f(6^n \cdot 6^mx)}{6^{4n}} - f(6^mx) \right\| \\ &\leq \frac{1}{6^4} \sum_{k=0}^{n-1} \frac{\xi(6^{k+m}x)}{6^{4(k+m)}} \\ &\leq \frac{1}{6^4} \sum_{k=0}^{\infty} \frac{\xi(6^{k+m}x)}{6^{4(k+m)}} \\ &\to 0 \quad as \quad m \to \infty \end{aligned}$$

for all $x \in G$. Hence the sequence $\left\{ \frac{f(6^nx)}{6^{4n}} \right\}$ is a Cauchy sequence. Since H is complete, there exists a mapping $Q : G \to H$ such that

$$Q(x) = \lim_{n\to\infty} \frac{f(6^nx)}{6^{4n}}, \quad \forall \ x \in G.$$

Letting $n \to \infty$ in (3.13), we see that (3.3) holds for all $x \in G$. To prove that Q satisfies (1.5), replacing (x, y, z) by $(6^nx, 6^ny, 6^nz)$ and dividing by 6^{4n} in (3.2), we obtain

$$\begin{aligned} \frac{1}{6^{4n}} \Big\| & f(6^n(3x+2y+z)) + f(6^n(3x+2y-z)) + f(6^n(3x-2y+z)) + f(6^n(3x-2y-z)) \\ & - 72[f(6^n(x+y)) + f(6^n(x-y))] - 18[f(6^n(x+z)) + f(6^n(x-z))] \\ & - 8[f(6^n(y+z)) + f(6^n(y-z))] - 144f(6^nx) \\ & + 96f(6^ny) + 48f(6^nz) \Big\| \leq \frac{1}{6^{4n}}\psi(6^nx, 6^ny, 6^nz) \end{aligned}$$

for all $x, y, z \in G$. Letting $n \to \infty$ in the above inequality and using the definition of $Q(x)$, we see that

$$\begin{aligned} & Q(3x+2y+z) + Q(3x+2y-z) + Q(3x-2y+z) + Q(3x-2y-z) \\ & = 72[Q(x+y) + Q(x-y)] + 18[Q(x+z) + Q(x-z)] + 8[Q(y+z) + Q(y-z)] \\ & \quad + 144Q(x) - 96Q(y) - 48Q(z). \end{aligned}$$

Hence Q satisfies (1.5) for all $x, y, z \in G$. To prove that Q is unique, let $R(x)$ be another quartic mapping satisfying (1.5) and (3.3), then

$$\begin{aligned}
\|Q(x) - R(x)\| &= \frac{1}{6^{4n}} \|Q(6^n x) - R(6^n x)\| \\
&\leq \frac{1}{6^{4n}} \{\|Q(6^n x) - f(6^n x)\| + \|f(6^n x) - R(6^n x)\|\} \\
&\leq \frac{2}{6^4} \sum_{k=0}^{\infty} \frac{\xi(6^{k+n} x)}{6^{4(k+n)}} \\
&\to 0 \quad as \quad n \to \infty
\end{aligned}$$

for all $x \in G$. Thus Q is unique. Hence for $j = 1$ the theorem holds.

Now, replacing x by $\frac{x}{6}$ in (3.9), we reach

$$\left\| f(x) - 1296 f\left(\frac{x}{6}\right) \right\| \leq \psi\left(\frac{x}{6}, \frac{x}{6}, \frac{x}{6}\right) + \frac{1}{2}\psi\left(\frac{x}{6}, 0, \frac{x}{6}\right) + \frac{89}{4}\psi\left(0, \frac{x}{6}, 0\right) \tag{3.14}$$

for all $x \in G$. The rest of the proof is similar to that of $j = 1$. Hence for $j = -1$ also the theorem holds. This completes the proof of the theorem. $\square$

The following Corollary is an immediate consequence of Theorem 3.0 concerning the Ulam-Hyers [9], Ulam-TRassias [20], Ulam-GRassias [18] and Ulam-JRassias [25] stabilities of (1.5).

Corollary 3.0. *Let ρ and s be nonnegative real numbers. Let $f : G \to H$ be a function satisfying the inequality*

$$\|Df(x,y,z)\| \leq \begin{cases} \rho, & \\ \rho\left\{||x||^s + ||y||^s + ||z||^s\right\}, & s \neq 4; \\ \rho||x||^s||y||^s||z||^s, & 3s \neq 4; \\ \rho\left\{||x||^s||y||^s||z||^s + \left\{||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}\right\}, & 3s \neq 4; \end{cases} \tag{3.15}$$

for all $x, y, z \in G$. Then there exists a unique quartic function $Q : G \to H$ such that

$$\|f(x) - Q(x)\| \leq \begin{cases} \dfrac{\rho}{4|3^3 - 1|}, \\ \dfrac{\rho||x||^s}{4|3^3 - 3^s|}, \\ \dfrac{\rho||x||^{3s}}{4|3^3 - 3^{3s}|} \end{cases} \tag{3.16}$$

for all $x \in G$.

4 Counter examples for non stable cases of (1.5)

Now, we will provide an example to illustrate that the functional equation (1.5) is not stable for $s = 4$ in condition (ii) of Corollary 3.0. Let $\psi : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$\psi(x) = \begin{cases} \mu x^4, & \text{if } |x| < 1 \\ \mu, & \text{otherwise} \end{cases}$$

where $\mu > 0$ is a constant, and define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} \frac{\psi(6^n x)}{6^{4n}} \qquad for\ all \quad x \in \mathbb{R}.$$

Then f satisfies the functional inequality

$$|Df(x,y,z)| \leq \frac{(488 \times 6^8)\mu}{1295} \left(|x|^4 + |y|^4 + |z|^4\right) \tag{4.1}$$

for all $x, y, z \in \mathbb{R}$. Then there do not exist a quartic mapping $Q : \mathbb{R} \to \mathbb{R}$ and a constant $\kappa > 0$ such that

$$|f(x) - Q(x)| \leq \kappa |x|^4 \qquad for\ all \quad x \in \mathbb{R}. \tag{4.2}$$

Proof. Now
$$|f(x)| \leq \sum_{n=0}^{\infty} \frac{|\psi(6^n x)|}{|6^{4n}|} \leq \sum_{n=0}^{\infty} \frac{\mu}{6^{4n}} = \frac{1296\mu}{1295}.$$
Therefore, we see that f is bounded. We are going to prove that f satisfies (4.1).

If $x = y = z = 0$ then (4.1) is trivial. If $|x|^4 + |y|^4 + |z|^4 \geq \frac{1}{6^4}$ then the left hand side of (4.1) is less than $\frac{488 \times 6^8 \mu}{1295}$. Now suppose that $0 < |x|^4 + |y|^4 + |z|^4 < \frac{1}{6^4}$. Then there exists a positive integer k such that
$$\frac{1}{6^{4(k+1)}} \leq |x|^4 + |y|^4 + |z|^4 < \frac{1}{6^{4k}}, \tag{4.3}$$
so that $6^{k-1}x < \frac{1}{6}, 6^{k-1}y < \frac{1}{6}, 6^{k-1}z < \frac{1}{6}$ and consequently
$$\begin{aligned} &6^{k-1}(3x+2y+z), 6^{k-1}(3x-2y+z), 6^{k-1}(3x+2y-z), 6^{k-1}(3x-2y-z),\\ &\quad 6^{k-1}(x+y), 6^{k-1}(x-y), 6^{k-1}(x+z), 6^{k-1}(x-z),\\ &\qquad 6^{k-1}(y+z), 6^{k-1}(y-z), 6^{k-1}(x), 6^{k-1}(y), 6^{k-1}(z) \in (-1,1). \end{aligned}$$
Therefore for each $n = 0, 1, \ldots, k-1$, we have
$$\begin{aligned} &6^n(3x+2y+z), 6^n(3x-2y+z), 6^n(3x+2y-z), 6^n(3x-2y-z),\\ &\quad 6^n(x+y), 6^n(x-y), 6^n(x+z), 6^n(x-z),\\ &\qquad 6^n(y+z), 6^n(y-z), 6^n(x), 6^n(y), 6^n(z) \in (-1,1). \end{aligned}$$
and
$$\begin{aligned} &\psi(6^n(3x+2y+z)) + \psi(6^n(3x+2y-z)) + \psi(6^n(3x-2y+z)) + \psi(6^n(3x-2y-z))\\ &\quad - 72[\psi(6^n(x+y)) + \psi(6^n(x-y))] - 18[\psi(6^n(x+z)) + \psi(6^n(x-z))]\\ &\qquad - 8[\psi(6^n(y+z)) + \psi(6^n(y-z))] - 144\psi(6^n(x)) + 96\psi(6^n(y)) + 48\psi(6^n(z)) = 0 \end{aligned}$$
for $n = 0, 1, \ldots, k-1$. From the definition of f and (4.3), we obtain that
$$\begin{aligned} &\Big| f(3x+2y+z) + f(3x+2y-z) + f(3x-2y+z) + f(3x-2y-z)\\ &\quad - 72[f(x+y) + f(x-y)] - 18[f(x+z) + f(x-z)] - 8[f(y+z) + f(y-z)]\\ &\qquad - 144f(x) + 96f(y) + 48f(z)\Big|\\ &\leq \sum_{n=0}^{\infty} \frac{1}{6^{4n}} \Big| \psi(6^n(3x+2y+z)) + \psi(6^n(3x+2y-z)) + \psi(6^n(3x-2y+z)) + \psi(6^n(3x-2y-z))\\ &\quad - 72[\psi(6^n(x+y)) + \psi(6^n(x-y))] - 18[\psi(6^n(x+z)) + \psi(6^n(x-z))]\\ &\qquad - 8[\psi(6^n(y+z)) + \psi(6^n(y-z))] - 144\psi(6^n(x)) + 96\psi(6^n(y)) + 48\psi(6^n(z))\Big|\\ &= \sum_{n=k}^{\infty} \frac{1}{6^{4n}} \Big| \psi(6^n(3x+2y+z)) + \psi(6^n(3x+2y-z)) + \psi(6^n(3x-2y+z)) + \psi(6^n(3x-2y-z))\\ &\quad - 72[\psi(6^n(x+y)) + \psi(6^n(x-y))] - 18[\psi(6^n(x+z)) + \psi(6^n(x-z))]\\ &\qquad - 8[\psi(6^n(y+z)) + \psi(6^n(y-z))] - 144\psi(6^n(x)) + 96\psi(6^n(y)) + 48\psi(6^n(z))\Big|\\ &\leq \sum_{n=k}^{\infty} \frac{1}{6^{4n}} 488\mu = 488\,\mu \times \frac{1296}{1295 \cdot 6^{4k}} \leq \frac{6^8 \times 488\mu}{1295}(|x|^4 + |y|^4 + |z|^4). \end{aligned}$$
Thus f satisfies (4.1) for all $x \in \mathbb{R}$ with $0 < |x|^4 + |y|^4 + |z|^4 < \frac{1}{6^4}$.

We claim that the quartic functional equation (1.5) is not stable for $s = 4$ in condition (ii) of Corollary 3.0. Suppose on the contrary that there exist a quartic mapping $Q : \mathbb{R} \to \mathbb{R}$ and a constant $\kappa > 0$ satisfying (4.2). Since f is bounded and continuous for all $x \in \mathbb{R}$, Q is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 3.0, Q must have the form $Q(x) = cx^4$ for any x in $\mathbb{R}$. Thus, we obtain that

$$|f(x)| \leq (\kappa + |c|)\,|x|^4. \tag{4.4}$$

But we can choose a positive integer m with $m\mu > \kappa + |c|$.

If $x \in \left(0, \frac{1}{6^{m-1}}\right)$, then $6^n x \in (0,1)$ for all $n = 0, 1, \ldots, m-1$. For this x, we get

$$f(x) = \sum_{n=0}^{\infty} \frac{\psi(6^n x)}{6^{4n}} \geq \sum_{n=0}^{m-1} \frac{\mu(6^n x)^4}{6^{4n}} = m\mu x^4 > (\kappa + |c|)\, x^4$$

which contradicts (4.4). Therefore the quartic functional equation (1.5) is not stable in sense of Ulam, Hyers and Rassias if $s = 4$, assumed in the inequality condition (ii) of (3.16). $\square$

A counter example to illustrate the non stability in Condition (iii) of Corollary 3.0. Let s be such that $0 < s < \frac{4}{3}$. Then there is a function $f : \mathbb{R} \to \mathbb{R}$ and a constant $\lambda > 0$ satisfying

$$|Df(x,y,z)| \leq \lambda |x|^{\frac{4s}{3}}\; |y|^{\frac{4s}{3}}\; |z|^{\frac{4-8s}{3}} \tag{4.5}$$

for all $x, y, z \in \mathbb{R}$ and

$$\sup_{x \neq 0} \frac{|f\,(x) - Q\,(x)|}{|x|^4} = +\infty \tag{4.6}$$

for every quartic mapping $Q : \mathbb{R} \to \mathbb{R}$.

Proof. If we take

$$f\,(x) = \begin{cases} x^4 \ln|x|\,, & if \quad x \neq 0, \\ 0, & if \quad x = 0. \end{cases}$$

Then from the relation (4.6), it follows that

$$\begin{aligned} \sup_{x \neq 0} \frac{|f\,(x) - Q\,(x)|}{|x|^4} &\geq \sup_{\substack{n \in \mathbb{N} \\ n \neq 0}} \frac{|f\,(n) - Q\,(n)|}{|n|^4} \\ &= \sup_{\substack{n \in \mathbb{N} \\ n \neq 0}} \frac{\left|n^4 \ln|n| - n^4\, Q\,(1)\right|}{|n|^4} = \sup_{\substack{n \in \mathbb{N} \\ n \neq 0}} \; |\ln|n| - Q\,(1)| \; = \infty. \end{aligned}$$

We have to prove (4.5) is true.

Case (i): If $x, y, z > 0$ in (4.5) then,

$$\begin{aligned} &|f(3x+2y+z) + f(3x+2y-z) + f(3x-2y+z) + f(3x-2y-z) \\ &\quad - 72[f(x+y) + f(x-y)] - 18[f(x+z) + f(x-z)] - 8[f(y+z) + f(y-z)] \\ &\qquad -144f(x) + 96f(y) + 48f(z)| \\ &= |(3x+2y+z)\ln|3x+2y+z| + (3x+2y-z)\ln|3x+2y-z| \\ &\quad + (3x-2y+z)\ln|3x-2y+z| + (3x-2y-z)\ln|3x-2y-z| \\ &\qquad - 72[(x+y)\ln|x+y| + (x-y)\ln|x-y|] \\ &\qquad\quad - 18[(x+z)\ln|x+z| + (x-z)\ln|x-z|] \\ &\qquad\qquad - 8[(y+z)\ln|y+z| + (y-z)\ln|y-z|] \\ &\qquad\qquad\quad -144(x)\ln|x| + 96(y)\ln|y| + 48(z)\ln|z|| \end{aligned}$$

Set $x = u, y = v, z = w$ it follows that

$$\begin{aligned}
&|f(3x+2y+z)+f(3x+2y-z)+f(3x-2y+z)+f(3x-2y-z)\\
&\quad -72[f(x+y)+f(x-y)]-18[f(x+z)+f(x-z)]-8[f(y+z)+f(y-z)]\\
&\qquad -144f(x)+96f(y)+48f(z)|\\
&=|(3x+2y+z)\ln|3x+2y+z|+(3x+2y-z)\ln|3x+2y-z|\\
&\quad +(3x-2y+z)\ln|3x-2y+z|+(3x-2y-z)\ln|3x-2y-z|\\
&\qquad -72[(x+y)\ln|x+y|+(x-y)\ln|x-y|]\\
&\qquad\quad -18[(x+z)\ln|x+z|+(x-z)\ln|x-z|]\\
&\qquad\qquad -8[(y+z)\ln|y+z|+(y-z)\ln|y-z|]\\
&\qquad\qquad\quad -144(x)\ln|x|+96(y)\ln|y|+48(z)\ln|z||\\
&=|(3u+2v+w)\ln|3u+2v+w|+(3u+2v-w)\ln|3u+2v-w|\\
&\quad +(3u-2v+w)\ln|3u-2v+w|+(3u-2v-w)\ln|3u-2v-w|\\
&\qquad -72[(u+v)\ln|u+v|+(u-v)\ln|u-v|]\\
&\qquad\quad -18[(u+w)\ln|u+w|+(u-w)\ln|u-w|]\\
&\qquad\qquad -8[(v+w)\ln|v+w|+(v-w)\ln|v-w|]\\
&\qquad\qquad\quad -144(u)\ln|u|+96(v)\ln|v|+48(w)\ln|w||\\
&|f(3u+2v+w)+f(3u+2v-w)+f(3u-2v+w)+f(3u-2v-w)\\
&\quad -72[f(u+v)+f(u-v)]-18[f(u+w)+f(u-w)]-8[f(v+w)+f(v-w)]\\
&\qquad -144f(u)+96f(v)+48f(w)|\\
&\leq \lambda|u|^{\frac{4s}{3}}\;|v|^{\frac{4s}{3}}\;|w|^{\frac{4-8s}{3}}=\lambda|x|^{\frac{4s}{3}}\;|y|^{\frac{4s}{3}}\;|z|^{\frac{4-8s}{3}}.
\end{aligned}$$

For the ***Cases:***

$$\begin{aligned}
&(ii): x,y,z<0\\
&(iii): x>0, y,z<0\\
&(iv): x<0, y,z>0\\
&(v): x=y=z=0
\end{aligned}$$

the proof is similar to that of Case (i). □

Now, we will provide an example to illustrate that the functional equation (1.5) is not stable for $s = \frac{4}{3}$ in condition (iv) of Corollary 3.0. The proof of the following example is similar to that of Example 4. Let $\psi : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$\psi(x) = \begin{cases} \mu x^4, & \text{if } |x| < \frac{4}{3} \\ \dfrac{4\mu}{3}, & \text{otherwise} \end{cases}$$

where $\mu > 0$ is a constant, and define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} \frac{\psi(6^n x)}{6^{4n}} \qquad for\ all \quad x \in \mathbb{R}.$$

Then f satisfies the functional inequality

$$|Df(x,y,z)| \leq \frac{488 \times 6^8 \times 4\mu}{3 \cdot 1295} \left(|x|^{\frac{4}{3}}|y|^{\frac{4}{3}}|z|^{\frac{4}{3}} + |x|^4 + |y|^4 + |z|^4\right) \tag{4.7}$$

for all $x, y, z \in \mathbb{R}$. Then there do not exist a quartic mapping $Q : \mathbb{R} \to \mathbb{R}$ and a constant $\kappa > 0$ such that

$$|f(x) - Q(x)| \leq \kappa |x|^4 \qquad for\ all \quad x \in \mathbb{R}. \tag{4.8}$$

5 Stability results of (1.2) using various substitutions

In this section, the generalized Ulam-Hyers stability of (1.5) using various substitutions is investigated. The proofs of the following theorems and corollaries are similar to that Theorem 3.0 and Corollary 3.0. Hence the details of the proofs are omitted.

Theorem 5.0. *Let $j = \pm 1$ and $\psi : G^3 \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\psi\left(4^{nj}x, 4^{nj}y, 4^{nj}z\right)}{4^{4nj}} = 0 \tag{5.1}$$

for all $x, y, z \in G$. Let $f : G \to H$ be a function satisfying the inequality

$$\|Df(x, y, z)\| \le \psi\,(x, y, z) \tag{5.2}$$

for all $x, y, z \in G$. Then there exists a unique quartic mapping $Q : G \to H$ which satisfies (1.5) and

$$\|f(x) - Q(x)\| \le \frac{1}{2 \cdot 4^4} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\zeta\left(4^{kj}x\right)}{4^{4kj}} \tag{5.3}$$

where $\zeta\,(x)$ and $Q(x)$ are defined by

$$\zeta\,(x) = \frac{1}{2}\psi\,(x, 0, x) + 4\psi\,(0, x, 0) \tag{5.4}$$

and

$$Q(x) = \lim_{n\to\infty} \frac{f(4^{nj}x)}{4^{4nj}} \tag{5.5}$$

for all $x \in G$, respectively.

Corollary 5.0. *Let ρ and s be nonnegative real numbers. Let $f : G \to H$ be a function satisfying the inequality*

$$\|Df(x, y, z)\| \le \begin{cases} \rho, & \\ \rho\left\{||x||^s + ||y||^s + ||z||^s\right\}, & s \neq 4; \\ \rho\left\{||x||^s||y||^s||z||^s + \left\{||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}\right\}, & 3s \neq 4; \end{cases} \tag{5.6}$$

for all $x, y, z \in G$. Then there exists a unique quartic function $Q : G \to H$ such that

$$\|f(x) - Q(x)\| \le \begin{cases} \dfrac{5\rho}{2|4^4 - 1|}, \\ \dfrac{5\rho||x||^s}{2|4^4 - 4^s|}, \\ \dfrac{5\rho||x||^{3s}}{2|4^4 - 4^{3s}|} \end{cases} \tag{5.7}$$

for all $x \in G$.

Theorem 5.0. *Let $j = \pm 1$ and $\psi : G^3 \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\psi\left(3^{nj}x, 3^{nj}y, 3^{nj}z\right)}{3^{4nj}} = 0 \tag{5.8}$$

for all $x, y, z \in G$. Let $f : G \to H$ be a function satisfying the inequality

$$\|Df(x, y, z)\| \le \psi\,(x, y, z) \tag{5.9}$$

for all $x, y, z \in G$. Then there exists a unique quartic mapping $Q : G \to H$ which satisfies (1.5) and

$$\|f(x) - Q(x)\| \le \frac{1}{4 \cdot 3^4} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\psi\left(3^{kj}x, 0, 0\right)}{3^{4kj}} \tag{5.10}$$

where $Q(x)$ is defined by

$$Q(x) = \lim_{n\to\infty} \frac{f(3^{nj}x)}{3^{4nj}} \tag{5.11}$$

for all $x \in G$.

Corollary 5.0. *Let ρ and s be nonnegative real numbers. Let $f : G \to H$ be a function satisfying the inequality*

$$\|Df(x,y,z)\| \leq \begin{cases} \rho, & \\ \rho\left\{||x||^s + ||y||^s + ||z||^s\right\}, & s \neq 4; \\ \rho\left\{||x||^s||y||^s||z||^s + \left\{||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}\right\}, & 3s \neq 4; \end{cases} \tag{5.12}$$

for all $x, y, z \in G$. Then there exists a unique quartic function $Q : G \to H$ such that

$$\|f(x) - Q(x)\| \leq \begin{cases} \frac{\rho}{4|3^4 - 1|}, \\ \frac{\rho||x||^s}{4|3^4 - 3^s|}, \\ \frac{\rho||x||^{3s}}{4|3^4 - 3^{3s}|} \end{cases} \tag{5.13}$$

for all $x \in G$.

Theorem 5.0. *Let $j = \pm 1$ and $\psi : G^3 \to [0, \infty)$ be a function such that*

$$\lim_{n \to \infty} \frac{\psi\left(2^{nj}x, 2^{nj}y, 2^{nj}z\right)}{2^{4nj}} = 0 \tag{5.14}$$

for all $x, y, z \in G$. Let $f : G \to H$ be a function satisfying the inequality

$$\|Df(x,y,z)\| \leq \psi(x,y,z) \tag{5.15}$$

for all $x, y, z \in G$. Then there exists a unique quartic mapping $Q : G \to H$ which satisfies (1.5) and

$$\|f(x) - Q(x)\| \leq \frac{1}{4 \cdot 2^4} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\psi\left(0, 2^{kj}x, 0\right)}{2^{4kj}} \tag{5.16}$$

where $Q(x)$ is defined by

$$Q(x) = \lim_{n \to \infty} \frac{f(2^{nj}x)}{2^{4nj}} \tag{5.17}$$

for all $x \in G$.

Corollary 5.0. *Let ρ and s be nonnegative real numbers. Let $f : G \to H$ be a function satisfying the inequality*

$$\|Df(x,y,z)\| \leq \begin{cases} \rho, & \\ \rho\left\{||x||^s + ||y||^s + ||z||^s\right\}, & s \neq 4; \\ \rho\left\{||x||^s||y||^s||z||^s + \left\{||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}\right\}, & 3s \neq 4; \end{cases} \tag{5.18}$$

for all $x, y, z \in G$. Then there exists a unique quartic function $Q : G \to H$ such that

$$\|f(x) - Q(x)\| \leq \begin{cases} \frac{\rho}{4|2^4 - 1|}, \\ \frac{\rho||x||^s}{4|2^4 - 2^s|}, \\ \frac{\rho||x||^{3s}}{4|2^4 - 2^{3s}|} \end{cases} \tag{5.19}$$

for all $x \in G$.

6 Stability results of (1.2): fixed point method

In this section, we apply a fixed point method for achieving stability of the functional equation (1.5) is present. Now, first we will recall the fundamental results in fixed point theory.

Theorem 6.0. *(Banach's contraction principle) Let (X, d) be a complete metric space and consider a mapping $T : X \to X$ which is strictly contractive mapping, that is*

(A1) $d(Tx,Ty) \leq Ld(x,y)$ *for some (Lipschitz constant)* $L < 1$. *Then,*
(i) The mapping T *has one and only fixed point* $x^* = T(x^*)$;
(ii)The fixed point for each given element x^* *is globally attractive, that is*

(A2) $lim_{n\to\infty} T^n x = x^*$, *for any starting point* $x \in X$;
(iii) One has the following estimation inequalities:

(A3) $d(T^n x, x^*) \leq \frac{1}{1-L}\ d(T^n x, T^{n+1}x), \forall\ n \geq 0, \forall\ x \in X;$

(A4) $d(x, x^*) \leq \frac{1}{1-L}\ d(x, x^*), \forall\ x \in X.$

Theorem 6.0. *[14] Suppose that for a complete generalized metric space* (Ω, δ) *and a strictly contractive mapping* $T : \Omega \to \Omega$ *with Lipschitz constant* L. *Then, for each given* $x \in \Omega$ *, either*

$$d(T^n x, T^{n+1}x) = \infty \qquad \forall \qquad n \geq 0,$$

or there exists a natural number n_0 *such that*
(FP1) $d(T^n x, T^{n+1}x) < \infty$ *for all* $n \geq n_0$ *;*
(FP2) The sequence $(T^n x)$ *is convergent to a fixed point* y^* *of* T
(FP3) y^* *is the unique fixed point of* T *in the set* $\Delta = \{y \in \Omega : d(T^{n_0}x, y) < \infty\}$;
(FP4) $d(y^*, y) \leq \frac{1}{1-L} d(y, Ty)$ *for all* $y \in \Delta$.

Hereafter throughout this section, let us consider $\mathcal{G}$ and $\mathcal{H}$ to be a normed space and a Banach space, respectively.

Theorem 6.0. *Let* $f : \mathcal{G} \to \mathcal{H}$ *be a mapping for which there exists a function* $\psi : \mathcal{G}^3 \to [0, \infty)$ *with the condition*

$$\lim_{k\to\infty} \frac{1}{\tau_i^{4k}} \psi(\tau_i^k x, \tau_i^k y, \tau_i^k z) = 0 \tag{6.1}$$

where

$$\tau_i = \begin{cases} 6 & if \quad i = 0; \\ \frac{1}{6} & if \quad i = 1, \end{cases} \tag{6.2}$$

such that the functional inequality

$$\|Df(x,y,z)\| \leq \psi(x,y,z) \tag{6.3}$$

for all $x, y, z \in \mathcal{G}$. *If there exists* $L = L(i)$ *such that the function* $\Phi : \mathcal{G} \to [0, \infty)$ *defined by*

$$\Phi(x) = \xi\left(\frac{x}{6}\right)$$

where

$$\xi(x) = \psi\left(x,x,x\right) + \frac{1}{2}\psi\left(x,0,x\right) + \frac{89}{4}\psi\left(0,x,0\right)$$

has the property

$$\Phi(x) = \frac{L}{\tau_i^4}\,\Phi\left(\tau_i x\right). \tag{6.4}$$

for all $x \in \mathcal{G}$. *Then there exists a unique quartic mapping* $Q : \mathcal{G} \to \mathcal{H}$ *satisfying the functional equation (1.5) and*

$$\|f(x) - Q(x)\| \leq \frac{L^{1-i}}{1-L}\Phi(x) \tag{6.5}$$

for all $x \in \mathcal{G}$.

Proof. Consider the set

$$\Gamma = \{p/p : \mathcal{G} \to \mathcal{H},\ p(0) = 0\}$$

and introduce the generalized metric on Γ,

$$d(p,q) = \inf\{K \in (0,\infty) :\| p(x) - q(x) \|\leq K\Phi(x), x \in \mathcal{G}\}.$$

It is easy to see that (Γ, d) is complete.

Define $\Upsilon : \Gamma \to \Gamma$ by

$$\Upsilon p(x) = \frac{1}{\tau_i^4} p(\tau_i x),$$

for all $x \in \mathcal{G}$. Now $p, q \in \Gamma$,

$$\begin{aligned}
d(p,q) \leq K &\Rightarrow \| \; p(x) - q(x) \; \| \leq K\Phi(x), x \in \mathcal{G}, \\
&\Rightarrow \left\| \frac{1}{\tau_i^4} p(\tau_i x) - \frac{1}{\tau_i^4} q(\tau_i x) \right\| \leq \frac{1}{\tau_i^4} K\Phi(\tau_i x), x \in \mathcal{G}, \\
&\Rightarrow \left\| \frac{1}{\tau_i^4} p(\tau_i x) - \frac{1}{\tau_i^4} q(\tau_i x) \right\| \leq LK\Phi(x), x \in \mathcal{G}, \\
&\Rightarrow \| \; \Upsilon p(x) - \Upsilon q(x) \; \| \leq LK\Phi(x), x \in \mathcal{G}, \\
&\Rightarrow d(\Upsilon p, \Upsilon q) \leq LK.
\end{aligned}$$

This implies $d(\Upsilon p, \Upsilon q) \leq Ld(p,q)$, for all $p, q \in \Gamma$. i.e., T is a strictly contractive mapping on Γ with Lipschitz constant L.

It follows from (3.9), we arrive

$$\|f(6x) - 1296 f(x)\| \leq \xi(x) \tag{6.6}$$

where

$$\xi(x) = \psi\,(x,x,x) + \frac{1}{2}\psi\,(x,0,x) + \frac{89}{4}\psi\,(0,x,0)$$

for all $x \in \mathcal{G}$. It follows from (6.6) that

$$\left\| \frac{f(6x)}{6^4} - f(x) \right\| \leq \frac{\xi(x)}{6^4} \tag{6.7}$$

for all $x \in \mathcal{G}$. Using (6.4) for the case $i = 0$ it reduces to

$$\left\| \frac{f(6x)}{6^4} - f(x) \right\| \leq \; L\Phi(x)$$

for all $x \in \mathcal{G}$,

$$\text{i.e.,} \quad d(\Upsilon f, f) \leq L \Rightarrow d(\Upsilon f, f) \leq L = L^{1-i} < \infty. \tag{6.8}$$

Again replacing $x = \frac{x}{6}$ in (6.6), we get

$$\left\| f(x) - 1296 f\left(\frac{x}{6}\right) \right\| \leq \xi\left(\frac{x}{6}\right) \tag{6.9}$$

for all $x \in \mathcal{G}$. Using (6.4) for the case $i = 1$ it reduces to

$$\left\| f(x) - 1296 f\left(\frac{x}{6}\right) \right\| \leq \; \Phi(x)$$

for all $x \in \mathcal{G}$,

$$\text{i.e.,} \quad d(f, \Upsilon f) \leq 1 \Rightarrow d(f, \Upsilon f) \leq 1 = L^{1-i} < \infty. \tag{6.10}$$

From (6.8) and (6.10), we arrive

$$d(f, \Upsilon f) \leq L^{1-i}.$$

Therefore (FP1) holds.

By (FP2), it follows that there exists a fixed point Q of Υ in Γ such that

$$Q(x) = \lim_{k\to\infty} \frac{f(\tau_i^k x)}{\tau_i^{4k}}, \qquad \forall \; x \in \mathcal{G}. \tag{6.11}$$

We have to prove $Q : \mathcal{G} \to \mathcal{H}$ is quartic. Replacing (x, y, z) by $(\tau_i^k x, \tau_i^k y, \tau_i^k z)$ in (6.3) and dividing by τ_i^{4k}, it follows from (6.1) that

$$\frac{1}{\tau_i^{4k}} \left\| Df(\tau_i^k x, \tau_i^k y, \tau_i^k z) \right\| \leq \frac{1}{\tau_i^{4k}} \psi(\tau_i^k x, \tau_i^k y, \tau_i^k z)$$

for all $x \in \mathcal{G}$. Letting $k \to \infty$ in the above inequality and using the definition of $Q(x)$, we see that

$$DQ(x,y,z) = 0$$

i.e., Q satisfies the functional equation (1.5) for all $x, y, z \in \mathcal{G}$.

By (FP3), Q is the unique fixed point of Υ in the set

$$\Delta = \{Q \in \Gamma : d(f,Q) < \infty\},$$

such that

$$\|f(x) - Q(x)\| \leq K\Phi(x)$$

for all $x \in \mathcal{G}$ and $K > 0$. Finally by (FP4), we obtain

$$d(f,Q) \leq \frac{1}{1-L} d(f, \Upsilon f)$$

this implies

$$d(f,Q) \leq \frac{L^{1-i}}{1-L}$$

which yields

$$\|f(x) - Q(x)\| \leq \frac{L^{1-i}}{1-L}\Phi(x)$$

this completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 6.0 concerning the stability of (1.5).

Corollary 6.0. *Let $f : \mathcal{G} \to \mathcal{H}$ be a mapping and there exist real numbers ρ and s such that*

$$\|Df(x,y,z)\| \leq \begin{cases} \rho, & \\ \rho\left\{||x||^s + ||y||^s + ||z||^s\right\}, & s \neq 4; \\ \rho||x||^s||y||^s||z||^s, & 3s \neq 4; \\ \rho\left\{||x||^s||y||^s||z||^s + \left\{||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}\right\}, & 3s \neq 4; \end{cases} \tag{6.12}$$

for all $x \in \mathcal{G}$. Then there exists a unique quartic function $Q : \mathcal{G} \to \mathcal{H}$ such that

$$\|f(x) - Q(x)\| \leq \begin{cases} \dfrac{95\rho}{4|6^4-1|}, \\ \dfrac{105\rho||x||^s}{4|6^4-6^s|}, \\ \dfrac{\rho||x||^{3s}}{|6^4-6^{3s}|}, \\ \dfrac{109\rho||x||^{3s}}{4|6^4-6^{3s}|} \end{cases} \tag{6.13}$$

for all $x \in \mathcal{G}$.

Proof. Setting

$$\psi(x,y,z) = \begin{cases} \rho, \\ \rho\left\{||x||^s + ||y||^s + ||z||^s\right\}, \\ \rho||x||^s||y||^s||z||^s, \\ \rho\left\{||x||^s||y||^s||z||^s + ||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}. \end{cases}$$

for all $x \in \mathcal{G}$. Now,

$$\frac{1}{\tau_i^{4k}}\psi(\tau_i^k x, \tau_i^k y, \tau_i^k z) = \begin{cases} \frac{\rho}{\tau_i^{4k}}, \\ \frac{\rho}{\tau_i^{4k}}\left\{||\tau_i^k x||^s + ||\tau_i^k y||^s + ||\tau_i^k z||^s\right\}, \\ \frac{\rho}{\tau_i^{4k}}||\tau_i^k x||^s||\tau_i^k y||^s||\tau_i^k z||^s, \\ \frac{\rho}{\tau_i^{4k}}\left\{||\tau_i^k x||^s||\tau_i^k y||^s||\tau_i^k z||^s + ||\tau_i^k x||^{3s} + ||\tau_i^k y||^{3s} + ||\tau_i^k z||^{3s}\right\}. \end{cases}$$

$$= \begin{cases} \to 0 \text{ as } k \to \infty, \\ \to 0 \text{ as } k \to \infty, \\ \to 0 \text{ as } k \to \infty, \\ \to 0 \text{ as } k \to \infty. \end{cases}$$

Thus, (6.1) is holds.

But we have $\Phi(x) = \xi\left(\frac{x}{6}\right)$ where $\xi(x) = \psi(x,x,x) + \frac{1}{2}\psi(x,0,x) + \frac{89}{4}\psi(0,x,0)$ has the property $\Phi(x) = \frac{L}{\tau_i^4}\Phi(\tau_i x)$ for all $x \in \mathcal{G}$. Hence

$$\Phi(x) = \xi\left(\frac{x}{6}\right) = \begin{cases} \frac{95\rho}{4}, \\ \frac{105\rho}{4\cdot 6^s}||x||^s, \\ \frac{\rho}{6^{3s}}||x||^{3s}, \\ \frac{109\rho}{4\cdot 6^{3s}}||x||^{3s}. \end{cases}$$

Now,

$$\frac{1}{\tau_i^4}\Phi(\tau_i x) = \begin{cases} \frac{95\rho}{4\cdot\tau_i^4} \\ \frac{105\rho}{4\cdot 6^s\cdot\tau_i^4}||\tau_i x||^s, \\ \frac{\rho}{6^{3s}\cdot\tau_i^4}||\tau_i x||^{3s}, \\ \frac{109\rho}{4\cdot 6^{3s}\cdot\tau_i^4}||\tau_i x||^{3s}. \end{cases} = \begin{cases} \tau_i^{-1}\Phi(x), \\ \tau_i^{s-4}\Phi(x), \\ \tau_i^{3s-4}\Phi(x), \\ \tau_i^{3s-4}\Phi(x). \end{cases}$$

Hence the inequality (6.4) holds either, $L = 6^{-4}$ if $i = 0$ and $L = 6^4$ if $i = 1$. Now from (6.5), we prove the following cases for condition (i).

Case:1 $L = 6^{-4}$ if $i = 0$

$$\|f(x) - Q(x)\| \le \frac{\left(6^{-4}\right)^{1-0}}{1-6^{-4}}\Phi(x) = \frac{95\rho}{4(6^4-1)}.$$

Case:2 $L = 6^4$ if $i = 1$

$$\|f(x) - Q(x)\| \le \frac{\left(6^{4}\right)^{1-1}}{1-6^{4}}\Phi(x) = \frac{-95\rho}{4(1-6^4)}.$$

Also the inequality (6.4) holds either, $L = 6^{s-4}$ for $s < 4$ if $i = 0$ and $L = 6^{4-s}$ for $s > 4$ if $i = 1$. Now from (6.5), we prove the following cases for condition (ii).

Case:3 $L = 6^{s-4}$ for $s < 4$ if $i = 0$

$$\|f(x) - Q(x)\| \le \frac{\left(6^{(s-4)}\right)^{1-0}}{1-6^{(s-4)}}\Phi(x) = \frac{105\rho||x||^s}{6^4-6^s}.$$

Case:4 $L = 6^{4-s}$ for $s > 4$ if $i = 1$

$$\|f(x) - Q(x)\| \le \frac{\left(6^{4-s}\right)^{1-1}}{1-6^{4-s}}\Phi(x) = \frac{105\rho||x||^s}{6^s-6^4}.$$

Again the inequality (6.4) holds either, $L = 6^{3s-4}$ for $3s < 4$ if $i = 0$ and $L = 6^{4-3s}$ for $3s > 4$ if $i = 1$. Now from (6.5), we prove the following cases for condition (iii).
Case:5 $L = 6^{3s-4}$ for $3s < 4$ if $i = 0$

$$\|f(x) - Q(x)\| \leq \frac{\left(6^{(3s-4)}\right)^{1-0}}{1 - 6^{(3s-4)}}\Phi(x) = \frac{\rho||x||^{3s}}{6^4 - 6^{3s}}.$$

Case:6 $L = 6^{4-3s}$ for $3s > 4$ if $i = 1$

$$\|f(x) - Q(x)\| \leq \frac{\left(6^{4-3s}\right)^{1-1}}{1 - 6^{4-3s}}\Phi(x) = \frac{\rho||x||^{3s}}{6^{3s} - 6^4}.$$

The proof of condition (iv) is similar to that of condition (iii). Hence the proof is complete. □

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