Malaya J. Mat. 3(3)(2015) 277–288

Malaya Journal of Matematik

MJM
an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

Mild solutions for semi-linear fractional order functional stochastic differential equations with impulse effect

Mohd Nadeem[a,*] and Jaydev Dabas[b]

[a,b]*Department of Applied Science and Engineering, IIT Roorkee, Saharanpur Campus, Saharanpur-247001, India.*

Abstract

This paper is concerned with the existence results of mild solution for an impulsive fractional order stochastic differential equation with infinite delay subject to nonlocal conditions. The results are obtained by using the fixed point techniques and solution operator generated by sectorial operator on a Hilbert space.

Keywords: Fractional order differential equation, nonlocal conditions, existence and uniqueness, impulsive conditions, stochastic differential equations.

2010 MSC: 26A33,34B10,34A12,34A37,34K50.

1 Introduction

Recently, fractional differential equations have been proved to be valuable tools in the modeling of many phenomena in various fields of engineering, physics, economics and science. Fractional models have various applications such as nonlinear oscillations of earthquakes, viscoelasticity, electrochemistry, seepage flow in porous media, and electromagnetic, etc. There has been a significant development in fractional differential equations since last few years for more details one can see the papers ([7],[8],[9],[11],[14],[15],[19]) and references cited therein.

The deterministic systems often fluctuate due to environmental noise due to this reason it is important and necessary for researcher to study these systems. These systems are modeled as stochastic differential systems. In many evolution processes impulsive effects exist in which states are changed abruptly at certain moments of time. Therefore the stochastic differential equations with impulsive effects exist in real systems and provide a more accurate mathematical model. For more details one can see the papers ([16],[17],[18]) and references therein.

Further, if we combine the stochastic differential equation with a nonlocal initial condition strengthens the model even further. These fact motivate us to study such model in this paper. The basic tools are used in this paper including fixed-point techniques, the theory of linear semi-groups, results for probability measures, and results for infinite dimensional stochastic differential equations. The results are important from the viewpoint of applications since they cover nonlocal generalizations of integro-differential stochastic differential equation arising in various fields such as electromagnetic theory, population dynamics, and heat conduction in materials with memory, for more detail one can see the papers ([6],[13],[16],[23],[24],[25]) and references therein.

In [4] Bahuguna, considered the following problem

$$\begin{cases} u'(t) + Au(t) = f(t, u(t), u(b_1(t)), u(b_2(t)), \dots, u(b_m(t))), t \in (0, T], \\ h(u) = \phi_0 \ \text{ on } [-\tau, 0], \end{cases}$$

*Corresponding author.

E-mail address: mohdnadeem.jmi@gmail.com (Mohd Nadeem), jay.dabas@gmail.com(Jaydev Dabas).

and found the existence, uniqueness and continuation of a mild solution on the maximal interval of existence. The author also proved some regularity results under various conditions. Chauhan et al. [5] considered the following semi-linear fractional order differential equations with nonlocal condition

$$\frac{d^\alpha}{dt^\alpha}x(t)+Ax(t)=f(t,x(t),x(a_1(t)),\ldots,x(a_m(t))),t\in[0,T],t\neq t_i,$$
$$x(0)+g(x)=x_0,\ \ \Delta x(t_i)=I_i(x(t_i^-)),$$

and discussed the existence and uniqueness results of solutions using the applications of classical fixed point theorems.

Balasubramaniam et al. [2] studied the existence of solutions for the the following semi-linear neutral stochastic functional differential equations

$$\begin{aligned} d[x(t)+F(t,x(t),x(b_1(t)),\ldots,x(b_m(t)))] &= Ax(t)dt+G(t,x(t),x(a_1(t)),\ldots,x(a_n(t)))dw(t),t\in J=[0,b],\\ x(0) &= x_0+g(x), \end{aligned}$$

where A is a infinitesimal generator of an analytic semigroup of bounded linear operators $T(t),t\geq 0$, on a separable Hilbert space. By using fractional power of operators and Sadovskii fixed point theorem, the authors established the existence of mild and strong solutions.

Sakthivel et al. [22] considered the following impulsive fractional stochastic differential equations with infinite delay in the form

$$\begin{cases} D_t^\alpha x(t)=Ax(t)+f(t,x_t,B_1x(t))+\sigma(t,x_t,B_2x(t))\frac{dw(t)}{dt},t\in[0,T],t\neq t_k,\\ \Delta x(t_k)=I_k(x(t_k)),\ k=1,2,\ldots,m\ x(t)=\phi(t),\ \ \phi(t)\in\mathfrak{B}_h, \end{cases}$$

and studied the existence results of mild solutions and established the sufficient conditions for the existence of mild solutions by using fixed point techniques.

Motivated by the works of these author's ([2],[4],[5],[22]), we study the existence of mild solutions of the following semi-linear stochastic fractional functional differential equation of the form:

$$\begin{aligned} {}^cD_t^\alpha x(t) &= Ax(t)+f(t,x_t,x(a_1(t)),\ldots,x(a_m(t)))\\ &\quad +\sigma(t,x_t,x(a_1(t)),\ldots,x(a_m(t)))\frac{dw(t)}{dt},t\in J,t\neq t_k, &(1.1)\\ \Delta x(t_k) &= I_k(x(t_k^-)),k=1,2,\ldots,p, &(1.2)\\ x(t)+g(x) &= \phi(t),\ t\in(-\infty,0], &(1.3) \end{aligned}$$

where $J=[0,T]$ and ${}^cD_t^\alpha$ denotes the Caputo's fractional derivative of order $\alpha\in(0,1)$. $A:D(A)\subset\mathbb{H}\to\mathbb{H}$ is a closed linear sectorial operator defined on a Hilbert space $(\mathbb{H},\|\cdot\|)$. The functions f,σ are given and satisfy some assumptions to be defined later. We assume that $x_t:(-\infty,0]\to\mathbb{H},x_t(s)=x(t+s),s\leq 0$, belong to an abstract phase space $\mathfrak{B}_h$. Here $0\leq t_0<t_1<\cdots<t_p<t_{p+1}\leq T,I_k\in C(\mathbb{H},\mathbb{H}),(k=1,2,\ldots,p)$, are bounded functions, $\Delta x(t_k)=x(t_k^+)-x(t_k^-),x(t_k^+)=\lim_{h\to 0}x(t_k+h)$ and $x(t_k^-)=\lim_{h\to 0}x(t_k-h)$ represent the right and left-hand limits of x(t) at $t=t_k$, respectively, also we take $x(t_i^-)=x(t_i)$.

The nonlocal condition $g:\mathbb{H}\to\mathbb{H}$ is defined as $g(x)=\sum_{k=1}^p c_kx(t_k)$ where $c_k,k=1,\ldots,p$, are given constants and $0<t_1<t_2<\cdots<t_p<T$. Such nonlocal conditions were first introduced by Deng [10]. The initial data $\phi=\{\phi(t),t\in(-\infty,0]\}$ is an $\mathcal{F}_0$-measurable, $\mathfrak{B}_h$-valued random variable independent of $w(t)$ with finite second moments.

To the best of our knowledge, the existence and uniqueness of mild solution for the system $(1.1)-(1.3)$ with non local condition is an untreated topic yet in the literature and this fact is the motivation of the present work.

Our work is divided in four sections, Second section provides the basic definitions and preliminaries results which are used in proving our main results. In the third section, we state and prove the existence results of the considered problem in this the paper. The fourth section includes examples.

2 Preliminaries

Let $\mathbb{H},\mathbb{K}$ be two separable Hilbert spaces and $\mathcal{L}(\mathbb{K},\mathbb{H})$ be the space of bounded linear operators from $\mathbb{K}$ into $\mathbb{H}$. For convenience, we will use the same notation $\|\cdot\|$ to denote the norms in $\mathbb{H},\mathbb{K}$ and $\mathcal{L}(\mathbb{K},\mathbb{H})$, and use $(\cdot,\cdot)$ to denote the inner product of $\mathbb{H}$ and $\mathbb{K}$ without any confusion. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space satisfying that $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$. $\mathbb{W} = (\mathbb{W}_t)_{t\geq 0}$ be a $\mathbb{Q}$-Wiener process defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with the covariance operator $\mathbb{Q}$ such that $Tr\mathbb{Q} < \infty$. We assume that there exists a complete orthonormal system $\{e_k\}_{k\geq 1}$ in $\mathbb{K}$, a bounded sequence of nonnegative real numbers λ_k such that $\mathbb{Q}e_k = \lambda_k e_k,\ k = 1, 2, \ldots$, and a sequence of independent Brownian motions $\{\beta_k\}_{k\geq 1}$ such that

$$(w(t), e)_{\mathbb{K}} = \sum_{k=1}^{\infty} \sqrt{\lambda_k}(e_k, e)_{\mathbb{K}}\beta_k(t), e \in \mathbb{K}, t \geq 0.$$

Let $\mathcal{L}_2^0 = \mathcal{L}_2(\mathbb{Q}^{\frac{1}{2}}\mathbb{K}, \mathbb{H})$ be the space of all Hilbert Schmidt operators from $\mathbb{Q}^{\frac{1}{2}}\mathbb{K}$ to $\mathbb{H}$ with the inner product $< \varphi, \psi >_{\mathcal{L}_2^0} = Tr[\varphi\mathbb{Q}\psi*]$.

Now, we introduce abstract space phase $\mathfrak{B}_h$. Assume that $h : (-\infty, 0] \to (0, \infty)$ with $l = \int_{-\infty}^0 h(t)dt < \infty$, a continuous function. An abstract phase $\mathfrak{B}_h$ defined by

$$\begin{aligned} \mathfrak{B}_h \quad = \quad & \{\phi : (-\infty, 0] \to \mathbb{H},\ \text{for any } a > 0,\ (E|\phi(\theta)|^2)^{1/2} \text{ is bounded and measurable function on} [-a, 0] \text{ with} \\ & \phi(0) = 0 \text{ and } \int_{-\infty}^0 h(s) \sup_{s\leq\theta\leq 0} (E|\phi(\theta)|^2)^{1/2} ds < \infty\}. \end{aligned}$$

If $\mathfrak{B}_h$ is endowed with the norm

$$\|\phi\|_{\mathfrak{B}_h} = \int_{-\infty}^0 h(s) \sup_{s\leq\theta\leq 0} (E|\phi(\theta)|^2)^{1/2} ds,\ \phi \in \mathfrak{B}_h,$$

then $(\mathfrak{B}_h, \|\cdot\|_{\mathfrak{B}_h})$ is a Banach space ([20],[21]).

Now we consider the space

$$\begin{aligned} \mathfrak{B}'_h \quad = \quad & \{x : (-\infty, T] \to \mathbb{H} \text{ such that } x|_{J_k} \in C(J_k, \mathbb{H}) \text{ and there exist} \\ & x(t_k^+) \text{ and } x(t_k^-) \text{ with } x(t_k) = x(t_k^-), x_0 = \phi \in \mathfrak{B}_h, k = 1, 2, \ldots, p\}, \end{aligned}$$

where $x|_{J_k}$ is the restriction of x to $J_k = (t_k, t_{k+1}], k = 0, 1, 2, \ldots, p$. The function $\|\cdot\|_{\mathfrak{B}'_h}$ to be a semi-norm in $\mathfrak{B}'_h$, it is defined by

$$\|x\|_{\mathfrak{B}'_h} = \|\phi\|_{\mathfrak{B}_h} + \sup_{s\in[0,T]} (E\|x(s)\|^2)^{1/2}, x \in \mathfrak{B}'_h.$$

Lemma 2.1. ([2]) Assume that $x \in \mathfrak{B}'_h$, then for $t \in J$, $x_t \in \mathfrak{B}_h$. Moreover,

$$l(E\|x(t)\|^2)^{1/2} \leq l \sup_{s\in[0,t]} (E\|x(s)\|^2)^{1/2} + \|x_0\|_{\mathfrak{B}_h},\ \text{where } l = \int_{-\infty}^0 h(s)ds < \infty.$$

Definition 2.1. The Reimann-Liouville fractional integral operator for order $\alpha > 0$, of a function $f : \mathbb{R}^+ \to \mathbb{R}$ and $f \in L^1(\mathbb{R}^+, X)$ is defined by

$$\mathbb{J}_t^0 f(t) = f(t),\ \mathbb{J}_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \quad \alpha > 0,\ t > 0,$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2. Caputo's derivative of order $\alpha > 0$ for a function $f : [0, \infty) \to \mathbb{R}$ is defined as

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds = \mathbb{J}^{n-\alpha} f^{(n)}(t),$$

for $n - 1 < \alpha < n,\ n \in N$. If $0 < \alpha < 1$, then

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f^{(1)}(s)ds.$$

Obviously, Caputo's derivative of a constant is equal to zero.

Definition 2.3. A two parameter function of the Mittag Lefller type is defined by the series expansion

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} = \frac{1}{2\pi \iota} \int_c \frac{\mu^{\alpha-\beta} e^{\mu}}{\mu^{\alpha} - z} d\mu, \alpha, \beta > 0, z \in \mathbb{C},$$

where c is a contour which starts and ends at $-\infty$ and encircles the disc $|\mu| \leq |z|^{\frac{1}{\alpha}}$ counter clockwise. The most interesting properties of the Mittag Lefller functions are associated with their Laplace integral

$$\int_0^{\infty} e^{-\lambda t} t^{\beta-1} E_{\alpha,\beta}(\omega t^{\alpha}) dt = \frac{\lambda^{\alpha-\beta}}{\lambda^{\alpha} - \omega}, Re\lambda > \omega^{\frac{1}{\alpha}}, \omega > 0.$$

Definition 2.4. [12] A closed and linear operator A is said to be sectorial if there are constants $\omega \in \mathbb{R},\ \theta \in [\frac{\pi}{2}, \pi],\ M > 0$, such that the following two conditions are satisfied:

(1) $\sum_{(\theta,\omega)} = \{\lambda \in C : \lambda \neq \omega, |arg(\lambda - \omega)| < \theta\} \subset \rho(A)$,

(2) $\|R(\lambda, A)\|_{L(X)} \leq \frac{M}{|\lambda-\omega|}, \lambda \in \sum_{(\theta,\omega)}$.

Definition 2.5. [1] Let A be a closed and linear operator with the domain $D(A)$ defined in a Banach space X. Let $\rho(A)$ be the resolvent set of A. We say that A is the generator of an α-resolvent family if there exist $\omega \geq 0$ and a strongly continuous function $T_{\alpha} : \mathbb{R}_+ \to \mathcal{L}(X)$, where $\mathcal{L}(X)$ is a Banach space of all bounded linear operators from X into X and the corresponding norm is denoted by $\|.\|$, such that $\{\lambda^{\alpha} : Re\lambda > \omega\} \subset \rho(A)$ and

$$(\lambda^{\alpha} I - A)^{-1} = \int_0^{\infty} e^{\lambda t} T_{\alpha}(t) x dt, \quad Re\lambda > \omega, x \in X,$$

where $T_{\alpha}(t)$ is called the α-resolvent family generated by A.

Definition 2.6. [11] Let A be a closed and linear operator with the domain $D(A)$ defined in a Banach space X and $\alpha > 0$. We say that A is the generator of a solution operator if there exist $\omega \geq 0$ and a strongly continuous function $S_{\alpha} : \mathbb{R}_+ \to \mathcal{L}(X)$, such that $\{\lambda^{\alpha} : Re\lambda > \omega\} \subset \rho(A)$ and

$$\lambda^{\alpha-1}(\lambda^{\alpha} I - A)^{-1} = \int_0^{\infty} e^{\lambda t} S_{\alpha}(t) x dt, \quad Re\lambda > \omega, x \in X,$$

where $S_{\alpha}(t)$ is called the solution operator generated by A.

Theorem 2.1. [26](Schauder fixed point theorem) If U is a closed , bounded, convex subset of a Banach space X and the mapping $T : U \to U$ is completely continuous, then T has a fixed point in U.

Definition 2.7. A measurable $\mathcal{F}_t-$ adapted stochastic process $x : (-\infty, T] \to \mathbb{H}$ is called a mild solution of the system (1.1)-(1.3) if $x(0) = \phi(0) - g(x) \in \mathfrak{B}_h$ on $(-\infty, 0], \Delta x|_{t=t_k} = I_k(x(t_k^-)), k = 1, 2, \ldots, p$, the restriction of $x(\cdot)$ to the interval $[0, T) \backslash t_1, \ldots, t_p$, is continuous and $x(t)$ satisfies the following fractional integral equation

$$x(t) = \begin{cases} S_{\alpha}(t)(\phi(0) - g(x)) + \int_0^t T_{\alpha}(t-s) f(s, x_s, x(a_1(s)), \ldots, x(a_m(s))) ds & \\ + \int_0^t T_{\alpha}(t-s) \sigma(s, x_s, x(a_1(s)), \ldots, x(a_m(s))) dw(s), & t \in [0, t_1], \\ S_{\alpha}(t-t_1)[x(t_1^-) + I_1(x(t_1^-))] + \int_{t_1}^t T_{\alpha}(t-s) f(s, x_s, x(a_1(s)), \ldots, x(a_m(s))) ds & \\ + \int_{t_1}^t T_{\alpha}(t-s) \sigma(s, x_s, x(a_1(s)), \ldots, x(a_m(s))) dw(s), & t \in (t_1, t_2], \\ \ldots & \\ S_{\alpha}(t-t_p)[x(t_p^-) + I_p(x(t_p^-))] + \int_{t_p}^t T_{\alpha}(t-s) f(s, x_s, x(a_1(s)), \ldots, x(a_m(s))) ds & \\ + \int_{t_p}^t T_{\alpha}(t-s) \sigma(s, x_s, x(a_1(s)), \ldots, x(a_m(s))) dw(s), & t \in (t_p, T], \end{cases} \tag{2.4}$$

where

$$S_{\alpha}(t) = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} \lambda^{\alpha-1} (\lambda^{\alpha} I - A)^{-1} d\lambda,\ T_{\alpha}(t) = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} (\lambda^{\alpha} I - A)^{-1} d\lambda,$$

are called analytic solutions operator and $\alpha-$resolvent family and Γ is a suitable path lying on $\sum_{\theta,\omega}$ for more details one can see [11].

Further we introduce the following assumptions to establish our results:

(H0) If $\alpha \in (0,1)$ and $A \in \mathbb{A}^{\alpha}(\theta_0, \omega_0)$ then for any $x \in \mathbb{H}$ and $t > 0$ we have $\|T_\alpha(t)\| \leq Me^{\omega t}$ and $\|S_\alpha(t)\| \leq Ce^{\omega t}(1+t^{\alpha-1})$, $\omega > \omega_0$. Thus we have

$$\|T_\alpha(t)\| \leq \widetilde{M}_T \text{ and } \|S_\alpha(t)\| \leq t^{\alpha-1}\widetilde{M}_S,$$

where $\widetilde{M}_T = \sup_{0\geq t\geq T} \|T_\alpha(t)\|$ and $\widetilde{M}_S = \sup_{0\geq t\geq T} Ce^{\omega t}(1+t^{1-\alpha})$(for more details, see [12]).

(H1) There exist a constants $L_g > 0$, such that $E\|g(x)-g(y)\|^2_{\mathbb{H}} \leq L_g\|x-y\|^2_{\mathbb{H}}$.

(H2) The nonlinear maps $f : J \times \mathfrak{B}_h \times \mathbb{H}^m \to \mathbb{H}$ and $\sigma : J \times \mathfrak{B}_h \times \mathbb{H}^m \to \mathcal{L}(\mathbb{K}, \mathbb{H})$ are continuous and there exist constants L_f, L_σ, such that

$$\begin{aligned} E\|f(t,\varphi,x_1,x_2,\ldots,x_m) - f(t,\psi,y_1,y_2,\ldots,y_m)\|^2_{\mathbb{H}} &\leq L_f[\|\varphi-\psi\|^2_{\mathfrak{B}_h} + \sum_{i=1}^{m} E\|x_i-y_i\|^2_{\mathbb{H}}], \\ E\|\sigma(t,\varphi,x_1,x_2,\ldots,x_m) - \sigma(t,\psi,y_1,y_2,\ldots,y_m)\|^2_{\mathcal{L}(\mathbb{K},\mathbb{H})} &\leq L_\sigma[\|\varphi-\psi\|^2_{\mathfrak{B}_h} + \sum_{i=1}^{m} E\|x_i-y_i\|^2_{\mathbb{H}}], \end{aligned}$$

for all $(x_1, x_2, \ldots, x_m)$ and $(y_1, y_2, \ldots, y_m) \in \mathbb{H}^m$, $t \in J$ and $\varphi, \psi \in \mathfrak{B}_h$.

(H3) The functions $I_k : \mathbb{H} \to \mathbb{H}$ are continuous and there exists $L_k > 0$, such that

$$E\|I_k(x) - I_k(y)\|^2_{\mathbb{H}} \leq L_k E\|x-y\|^2_{\mathbb{H}},$$

$x, y \in \mathbb{H},\ k = 1, 2, \ldots, p,\ L = \max\{L_k\} > L_g$.

3 Existence and uniqueness of solutions

Theorem 3.2. Let the assumptions (H0)-(H3) are satisfied and

$$\Theta = \left[3\widetilde{M}_S^2(1+L) + 3\widetilde{M}_T^2\frac{T^{2\alpha}}{\alpha^2}L_f(l+m) + 3\widetilde{M}_T^2\frac{T^{2\alpha-1}}{2\alpha-1}L_\sigma(l+m)\right] < 1,$$

then the problem (1.1)-(1.3) has a unique mild solution $x \in \mathbb{H}$ on J.

Proof. First we convert the problem (1.1)-(1.3) into a fixed point problem. Consider the operator $P : \mathfrak{B}'_h \to \mathfrak{B}'_h$ defined by

$$(Px)(t) = \begin{cases} S_\alpha(t)(\phi(0) - g(x)) + \int_0^t T_\alpha(t-s)f(s,x_s,x(a_1(s)),\ldots,x(a_m(s)))ds & \\ + \int_0^t T_\alpha(t-s)\sigma(s,x_s,x(a_1(s)),\ldots,x(a_m(s)))dw(s), & t \in [0,t_1], \\ S_\alpha(t-t_1)[x(t_1^-) + I_1(x(t_1^-))] + \int_{t_1}^t T_\alpha(t-s)f(s,x_s,x(a_1(s)),\ldots,x(a_m(s)))ds & \\ + \int_{t_1}^t T_\alpha(t-s)\sigma(s,x_s,x(a_1(s)),\ldots,x(a_m(s)))dw(s), & t \in (t_1,t_2], \\ \cdots & \\ S_\alpha(t-t_p)[x(t_p^-) + I_p(x(t_p^-))] + \int_{t_p}^t T_\alpha(t-s)f(s,x_s,x(a_1(s)),\ldots,x(a_m(s)))ds & \\ + \int_{t_p}^t T_\alpha(t-s)\sigma(s,x_s,x(a_1(s)),\ldots,x(a_m(s)))dw(s), & t \in (t_p,T]. \end{cases}$$

Let $y(.) : (-\infty, T] \to \mathbb{H}$ be the function defined by

$$y(t) = \begin{cases} \phi(t), & t \in (-\infty,0] \\ 0, & t \in J, \end{cases} \text{ then } y_0 = \phi.$$

For each $z : J \to \mathbb{H}$ with $z\,|_{t_k} \in C(J_k, \mathbb{H}), k = 1, \ldots, p$ and $z(0) = 0$, we denote by $\overline{z}$ the function defined by

$$\overline{z} = \begin{cases} 0, & t \in (-\infty,0] \\ z(t), & t \in J. \end{cases}$$

If $x(\cdot)$ satisfies the system (2.4), then we can decompose $x(\cdot)$ as $x(t)=y(t)+\bar{z}(t)$, which implies $x_t=y_t+\bar{z}_t$ for $t\in J$ and the function $z(\cdot)$ satisfies

$$z(t)=\begin{cases} S_\alpha(t)(\phi(0)-g(y+\bar{z}))+\int_0^t T_\alpha(t-s)f(s,y_s+\bar{z}_s,y(a_1(s))+\bar{z}(a_1(s)),\\ \dots,y(a_m(s))+\bar{z}(a_m(s)))ds+\int_0^t T_\alpha(t-s)\sigma(s,y_s+\bar{z}_s,\\ y(a_1(s))+\bar{z}(a_1(s)),\dots,y(a_m(s))+\bar{z}(a_m(s)))dw(s), & t\in[0,t_1],\\ S_\alpha(t-t_1)[y(t_1^-)+\bar{z}(t_1^-)+I_1(y(t_1^-)+\bar{z}(t_1^-))]\\ +\int_{t_1}^t T_\alpha(t-s)f(s,y_s+\bar{z}_s,y(a_1(s))+\bar{z}(a_1(s)),\dots,y(a_m(s))+\bar{z}(a_m(s)))ds\\ +\int_{t_1}^t T_\alpha(t-s)\sigma(s,y_s+\bar{z}_s,y(a_1(s))+\bar{z}(a_1(s)),\dots,y(a_m(s))+\bar{z}(a_m(s)))dw(s), & t\in(t_1,t_2],\\ \dots\\ S_\alpha(t-t_p)[y(t_p^-)+\bar{z}(t_p^-)+I_p(y(t_p^-)+\bar{z}(t_p^-))]\\ +\int_{t_p}^t T_\alpha(t-s)f(s,y_s+\bar{z}_s,y(a_1(s))+\bar{z}(a_1(s)),\dots,y(a_m(s))+\bar{z}(a_m(s)))ds\\ +\int_{t_p}^t T_\alpha(t-s)\sigma(s,y_s+\bar{z}_s,y(a_1(s))+\bar{z}(a_1(s)),\dots,y(a_m(s))+\bar{z}(a_m(s)))dw(s), & t\in(t_p,T].\end{cases}$$

Set $\mathfrak{B}_h''$, such that $z_0=0$ and for any $z\in\mathfrak{B}_h''$, we have

$$\|z\|_{\mathfrak{B}_h''}=\|z_0\|_{\mathfrak{B}_h}+\sup_{t\in J}(E\|z(t)\|^2)^{\frac{1}{2}}=\sup_{t\in J}(E\|z(t)\|^2)^{\frac{1}{2}}.$$

Thus $(\mathfrak{B}_h'',\|\cdot\|_{\mathfrak{B}_h''})$ is a Banach space. Define an operator $N:\mathfrak{B}_h''\to\mathfrak{B}_h''$ by

$$(Nz)(t)=\begin{cases} S_\alpha(t)(\phi(0)-g(y+\bar{z}))+\int_0^t T_\alpha(t-s)f(s,y_s+\bar{z}_s,y(a_1(s))+\bar{z}(a_1(s)),\\ \dots,y(a_m(s))+\bar{z}(a_m(s)))ds+\int_0^t T_\alpha(t-s)\sigma(s,y_s+\bar{z}_s,\\ y(a_1(s))+\bar{z}(a_1(s)),\dots,y(a_m(s))+\bar{z}(a_m(s)))dw(s), & t\in[0,t_1],\\ S_\alpha(t-t_1)[y(t_1^-)+\bar{z}(t_1^-)+I_1(y(t_1^-)+\bar{z}(t_1^-))]\\ +\int_{t_1}^t T_\alpha(t-s)f(s,y_s+\bar{z}_s,y(a_1(s))+\bar{z}(a_1(s)),\dots,y(a_m(s))+\bar{z}(a_m(s)))ds\\ +\int_{t_1}^t T_\alpha(t-s)\sigma(s,y_s+\bar{z}_s,y(a_1(s))+\bar{z}(a_1(s)),\dots,y(a_m(s))+\bar{z}(a_m(s)))dw(s), & t\in(t_1,t_2],\\ \dots\\ S_\alpha(t-t_p)[y(t_p^-)+\bar{z}(t_p^-)+I_p(y(t_p^-)+\bar{z}(t_p^-))]\\ +\int_{t_p}^t T_\alpha(t-s)f(s,y_s+\bar{z}_s,y(a_1(s))+\bar{z}(a_1(s)),\dots,y(a_m(s))+\bar{z}(a_m(s)))ds\\ +\int_{t_p}^t T_\alpha(t-s)\sigma(s,y_s+\bar{z}_s,y(a_1(s))+\bar{z}(a_1(s)),\dots,y(a_m(s))+\bar{z}(a_m(s)))dw(s), & t\in(t_p,T].\end{cases}$$

In order to prove existence results, it is enough to show that N has a unique fixed point. Let $z,z^*\in\mathfrak{B}_h''$ then for $t\in[0,t_1]$, we have

$$\begin{aligned} E\|(Nz)(t)-(Nz^*)(t)\|_{\mathbb{H}}^2 \leq\ & 3E\|S_\alpha(t)[g(y+\bar{z})-g(y+\bar{z}^*)]\|_{\mathbb{H}}^2\\ &+3E\|\int_0^t T_\alpha(t-s)[f(s,y_s+\bar{z}_s,y(a_1(s))+\bar{z}(a_1(s)),\dots,y(a_m(s))+\bar{z}(a_m(s)))\\ &-f(s,y_s+\bar{z}_s^*,y(a_1(s))+\bar{z}^*(a_1(s)),\dots,y(a_m(s))+\bar{z}^*(a_m(s)))]ds\|_{\mathbb{H}}^2\\ &+3E\|\int_0^t T_\alpha(t-s)[\sigma(s,y_s+\bar{z}_s,y(a_1(s))+\bar{z}(a_1(s)),\dots,y(a_m(s))+\bar{z}(a_m(s)))\\ &-\sigma(s,y_s+\bar{z}_s^*,y(a_1(s))+\bar{z}^*(a_1(s)),\dots,y(a_m(s))+\bar{z}^*(a_m(s)))]dw(s)\|_{\mathbb{H}}^2, \end{aligned}$$

by applying assumptions, we have

$$E\|(Nz)(t)-(Nz^*)(t)\|_{\mathbb{H}}^2 \leq (3\widetilde{M}_S^2L_g+3\widetilde{M}_T^2\frac{T^{2\alpha}}{\alpha^2}L_f(l+m)+3\widetilde{M}_T^2\frac{T^{2\alpha-1}}{2\alpha-1}L_\sigma(l+m))\|z-z^*\|_{\mathfrak{B}_h''}^2.$$

For $t \in (t_1, t_2]$, we have

$$\begin{aligned} E\|(Nz)(t)-(Nz^*)(t)\|^2_{\mathbb{H}} \leq\ & 3E\|S_\alpha(t-t_1)[\bar{z}(t_1^-)-\bar{z}^*(t_1^-)+I_1(y(t_1^-)+\bar{z}(t_1^-))-I_1(y(t_1^-)+\bar{z}^*(t_1^-))]\|^2_{\mathbb{H}} \\ & +3E\|\int_{t_1}^t T_\alpha(t-s)[f(s,y_s+\bar{z}_s,y(a_1(s))+\bar{z}(a_1(s)),\dots,y(a_m(s))+\bar{z}(a_m(s))) \\ & -f(s,y_s+\bar{z}^*_s,y(a_1(s))+\bar{z}^*(a_1(s)),\dots,y(a_m(s))+\bar{z}^*(a_m(s)))]ds\|^2_{\mathbb{H}} \\ & +3E\|\int_{t_1}^t T_\alpha(t-s)[\sigma(s,y_s+\bar{z}_s,y(a_1(s))+\bar{z}(a_1(s)),\dots,y(a_m(s))+\bar{z}(a_m(s))) \\ & -\sigma(s,y_s+\bar{z}^*_s,y(a_1(s))+\bar{z}^*(a_1(s)),\dots,y(a_m(s))+\bar{z}^*(a_m(s)))]dw(s)\|^2_{\mathbb{H}}. \end{aligned}$$

by applying assumptions, we obtain

$$E\|(Nz)(t)-(Nz^*)(t)\|^2_{\mathbb{H}} \leq (3\widetilde{M}_S^2(1+L_1)+3\widetilde{M}_T^2\frac{T^{2\alpha}}{\alpha^2}L_f(l+m)+3\widetilde{M}_T^2\frac{T^{2\alpha-1}}{2\alpha-1}L_\sigma(l+m))\|z-z^*\|^2_{\mathfrak{B}''_h}.$$

Similarly, for $t \in (t_p, T]$, we have

$$\begin{aligned} E\|(Nz)(t)-(Nz^*)(t)\|^2_{\mathbb{H}} \leq\ & 3E\|S_\alpha(t-t_p)[\bar{z}(t_p^-)-\bar{z}^*(t_p^-)+I_p(y(t_p^-)+\bar{z}(t_p^-))-I_p(y(t_p^-)+\bar{z}^*(t_p^-))]\|^2_{\mathbb{H}} \\ & +3E\|\int_{t_p}^t T_\alpha(t-s)[f(s,y_s+\bar{z}_s,y(a_1(s))+\bar{z}(a_1(s)),\dots,y(a_n(s))+\bar{z}(a_n(s))) \\ & -f(s,y_s+\bar{z}^*_s,y(a_1(s))+\bar{z}^*(a_1(s)),\dots,y(a_m(s))+\bar{z}^*(a_m(s)))]ds\|^2_{\mathbb{H}} \\ & +3E\|\int_{t_1}^t T_\alpha(t-s)[\sigma(s,y_s+\bar{z}_s,y(a_1(s))+\bar{z}(a_1(s)),\dots,y(a_m(s))+\bar{z}(a_m(s))) \\ & -\sigma(s,y_s+\bar{z}^*_s,y(a_1(s))+\bar{z}^*(a_1(s)),\dots,y(a_m(s))+\bar{z}^*(a_m(s)))]dw(s)\|^2_{\mathbb{H}}, \end{aligned}$$

by applying assumptions, we have

$$E\|(Nz)(t)-(Nz^*)(t)\|^2_{\mathbb{H}} \leq (3\widetilde{M}_S^2(1+L_p)+3\widetilde{M}_T^2\frac{T^{2\alpha}}{\alpha^2}L_f(l+m)+3\widetilde{M}_T^2\frac{T^{2\alpha-1}}{2\alpha-1}L_\sigma(l+m))\|z-z^*\|^2_{\mathfrak{B}''_h}.$$

Thus for all $t \in [0, T]$, we estimate

$$\begin{aligned} E\|(Nz)(t)-(Nz^*)(t)\|^2_{\mathbb{H}} & \leq \left\{3\widetilde{M}_S^2(1+L)+3\widetilde{M}_T^2\frac{T^{2\alpha}}{\alpha^2}L_f(l+m)\ +3\widetilde{M}_T^2\frac{T^{2\alpha-1}}{2\alpha-1}L_\sigma(l+m)\right\}\|z-z^*\|^2_{\mathfrak{B}''_h}, \\ & \leq \Theta\|z-z^*\|^2_{\mathfrak{B}''_h}. \end{aligned}$$

Since $\Theta < 1$ as in the Theorem 3.2, therefore N is a contraction. Hence N has a unique fixed point by Banach contraction principle. This completes the proof of the theorem. □

The second result is proved by using the Schauder fixed point theorem. For this we take the following assumptions

(H4) There exist a constants $M_1 > 0$, such that $E\|g(x)\|^2_{\mathbb{H}} \leq M_1$.

(H5) The functions $I_k : \mathbb{H} \to \mathbb{H}$ are continuous and there exists $M_2 > 0$, such that $E\|I_k(x)\|^2_{\mathbb{H}} \leq M_2$.

(H6) $f, \sigma : J \times \mathfrak{B}_h \times \mathbb{H}^m \to \mathbb{H}$ are continuous and there exits constants M_3, M_4, such that

$$E\|f(t,\varphi,x_1,x_2,\dots,x_m)\|^2_{\mathbb{H}} \leq M_3,\ E\|\sigma(t,\varphi,x_1,x_2,\dots,x_m)\|^2_{\mathbb{H}} \leq M_4.$$

Theorem 3.3. Let the assumptions (H3)-(H6) are satisfied then the impulsive stochastic differential equation (1.1)-(1.3) has at least one mild solution.

Proof. let us consider the space $B_r = \{y \in \mathfrak{B}''_h : \|y\| \leq r\}$. It is obvious that B_r is closed convex and bounded subset of $\mathfrak{B}''_h$. Consider the operator $N : B_r \to B_r$ defined by

$$(Nz)(t) = \begin{cases} S_\alpha(t)(\phi(0) - g(y+\bar{z})) + \int_0^t T_\alpha(t-s)f(s, y_s + \bar{z}_s, y(a_1(s)) + \bar{z}(a_1(s)), \\ \dots, y(a_m(s)) + \bar{z}(a_m(s)))ds + \int_0^t T_\alpha(t-s)\sigma(s, y_s + \bar{z}_s, \\ y(a_1(s)) + \bar{z}(a_1(s)), \dots, y(a_m(s)) + \bar{z}(a_m(s)))dw(s), & t \in [0, t_1], \\ S_\alpha(t-t_1)[y(t_1^-) + \bar{z}(t_1^-) + I_1(y(t_1^-) + \bar{z}(t_1^-))] \\ + \int_{t_1}^t T_\alpha(t-s)f(s, y_s + \bar{z}_s, y(a_1(s)) + \bar{z}(a_1(s)), \dots, y(a_m(s)) + \bar{z}(a_m(s)))ds \\ + \int_{t_1}^t T_\alpha(t-s)\sigma(s, y_s + \bar{z}_s, y(a_1(s)) + \bar{z}(a_1(s)), \dots, y(a_m(s)) + \bar{z}(a_m(s)))dw(s), & t \in (t_1, t_2], \\ \dots \\ S_\alpha(t-t_p)[y(t_p^-) + \bar{z}(t_p^-) + I_p(y(t_p^-) + \bar{z}(t_p^-))] \\ + \int_{t_p}^t T_\alpha(t-s)f(s, y_s + \bar{z}_s, y(a_1(s)) + \bar{z}(a_1(s)), \dots, y(a_m(s)) + \bar{z}(a_m(s)))ds \\ + \int_{t_p}^t T_\alpha(t-s)\sigma(s, y_s + \bar{z}_s, y(a_1(s)) + \bar{z}(a_1(s)), \dots, y(a_m(s)) + \bar{z}(a_m(s)))dw(s), & t \in (t_p, T]. \end{cases}$$

First we shall show that N is continuous, for this let $\{z^n\}_{n=1}^\infty$ be a sequence in B_r such that $\lim z^n \to z \in B_r$. When $t \in [0, t_1]$, we have

$$\begin{aligned} E\|(Nz^n)(t) - (Nz)(t)\|_{\mathbb{H}}^2 \leq\ & 3E\|S_\alpha(t)[g(y+\bar{z}^n) - g(y+\bar{z})\|_{\mathbb{H}}^2 \\ & +3E\|\int_0^t T_\alpha(t-s)[f(s, y_s + \bar{z}_s^n, y(a_1(s)) + \bar{z}^n(a_1(s)), \dots, y(a_m(s)) + \bar{z}^n(a_m(s))) \\ & -f(s, y_s + \bar{z}_s, y(a_1(s)) + \bar{z}(a_1(s)), \dots, y(a_m(s)) + \bar{z}(a_m(s)))]ds\|_{\mathbb{H}}^2 \\ & +3E\|\int_0^t T_\alpha(t-s)[\sigma(s, y_s + \bar{z}_s^n, y(a_1(s)) + \bar{z}^n(a_1(s)), \dots, y(a_m(s)) + \bar{z}^n(a_m(s))) \\ & -\sigma(s, y_s + \bar{z}_s, y(a_1(s)) + \bar{z}(a_1(s)), \dots, y(a_m(s)) + \bar{z}(a_m(s)))]dw(s)\|_{\mathbb{H}}^2. \end{aligned}$$

Then for $t \in (t_i, t_{i+1}]$, where $i = 1, 2, \dots, p$, then we have

$$\begin{aligned} E\|(Nz^n)(t) - (Nz)(t)\|_{\mathbb{H}}^2 \leq\ & 3E\|S_\alpha(t-t_i)[\bar{z}^n(t_i^-) - \bar{z}(t_i^-) + I_i(y(t_i^-) + \bar{z}^n(t_i^-)) - I_i(y(t_i^-) + \bar{z}(t_i^-))]\|_{\mathbb{H}}^2 \\ & +3E\|\int_{t_i}^t T_\alpha(t-s)[f(s, y_s + \bar{z}_s^n, y(a_1(s)) + \bar{z}^n(a_1(s)), \dots, y(a_m(s)) + \bar{z}^n(a_m(s))) \\ & -f(s, y_s + \bar{z}_s, y(a_1(s)) + \bar{z}(a_1(s)), \dots, y(a_m(s)) + \bar{z}(a_m(s)))]ds\|_{\mathbb{H}}^2 \\ & +3E\|\int_{t_i}^t T_\alpha(t-s)[\sigma(s, y_s + \bar{z}_s^n, y(a_1(s)) + \bar{z}^n(a_1(s)), \dots, y(a_m(s)) + \bar{z}^n(a_m(s))) \\ & -\sigma(s, y_s + \bar{z}_s, y(a_1(s)) + \bar{z}(a_1(s)), \dots, y(a_m(s)) + \bar{z}(a_m(s)))]dw(s)\|_{\mathbb{H}}^2. \end{aligned}$$

Since the functions f, σ, g and $I_i, i = 1, 2, \dots, p$, are continuous, hence $\lim_{n\to\infty} E\|(Nz^n)(t) - (Nz)(t)\|_{\mathbb{H}}^2 \to 0$. This implies that the mapping N is continuous on B_r.

Now we show that N maps bounded set into bounded sets in B_r. Let $z \in B_r$ then we have $E\|(Nz)(t)\|_{\mathbb{H}}^2 \leq \hat{M}$, for $t \in (t_i, t_{i+1}], i = 0, 1, 2, \dots, p$. Then for $t \in [0, t_1]$, we have

$$\begin{aligned} E\|(Nz)(t)\|_{\mathbb{H}}^2 \leq\ & 3E\|S_\alpha(t)[\phi(0) + g(y+\bar{z})]\|_{\mathbb{H}}^2 \\ & +3E\|\int_0^t T_\alpha(t-s)[f(s, y_s + \bar{z}_s, y(a_1(s)) + \bar{z}(a_1(s)), \dots, y(a_m(s)) + \bar{z}(a_m(s)))]ds\|_{\mathbb{H}}^2 \\ & +3E\|\int_0^t T_\alpha(t-s)[\sigma(s, y_s + \bar{z}_s, y(a_1(s)) + \bar{z}(a_1(s)), \dots, y(a_m(s)) + \bar{z}(a_m(s)))]dw(s)\|_{\mathbb{H}}^2, \\ & \leq 3\widetilde{M}_S^2[r + M_1] + 3\widetilde{M}_T^2 \frac{T^{2\alpha}}{\alpha^2} M_3 + 3\widetilde{M}_T^2 \frac{T^{2\alpha-1}}{2\alpha-1} M_4. \end{aligned}$$

For $t \in (t_i, t_{i+1}], i = 1, 2, \ldots, p$, then we have

$$\begin{aligned} E\|(Nz)(t)\|_{\mathbb{H}}^2 &\leq 3E\|S_\alpha(t-t_i)[y(t_i^-) + \bar{z}(t_i^-) + I_i(y(t_i^-) + \bar{z}(t_i^-))]\|_{\mathbb{H}}^2 \\ &+3E\|\int_{t_i}^t T_\alpha(t-s)[f(s, y_s + \bar{z}_s, y(a_1(s)) + \bar{z}(a_1(s)), \ldots, y(a_m(s)) + \bar{z}(a_m(s)))ds\|_{\mathbb{H}}^2 \\ &+3E\|\int_{t_i}^t T_\alpha(t-s)[\sigma(s, y_s + \bar{z}_s, y(a_1(s)) + \bar{z}(a_1(s)), \ldots, y(a_m(s)) + \bar{z}(a_m(s)))]dw(s)\|_{\mathbb{H}}^2 \\ &\leq 3\widetilde{M}_S^2[r + M_2] + 3\widetilde{M}_T^2 \frac{T^{2\alpha}}{\alpha^2} M_3 + 3\widetilde{M}_T^2 \frac{T^{2\alpha-1}}{2\alpha-1} M_4 = \hat{M}. \end{aligned}$$

It proves that N maps bounded set into bounded sets in B_r for all sub interval $t \in (t_i, t_{i+1}], i = 1, 2, \ldots, p$. Finally, we show that N maps bounded set into equi-continuous sets in B_r. let $l_1, l_2 \in (t_i, t_{i+1}], t_i \leq l_1 < l_2 \leq t_{i+1}, i = 0, 1, 2, \ldots, p, z \in B_r$, we obtain for $t \in [0, t_1]$

$$\begin{aligned} E\|(Nz)(l_2) - (Nz)(l_1)\|_{\mathbb{H}}^2 &\leq 3E\|[S_\alpha(l_2) - S_\alpha(l_1)][\phi_0 + g(y + \bar{z})]\|_{\mathbb{H}}^2 \\ &+3E\|\int_0^t [T_\alpha(l_2 - s) - T_\alpha(l_1 - s)][f(s, y_s + \bar{z}_s, y(a_1(s)) + \bar{z}(a_1(s)), \ldots, \\ &y(a_m(s)) + \bar{z}(a_m(s)))]ds\|_{\mathbb{H}}^2 + 3E\|\int_0^t [T_\alpha(l_2 - s) - T_\alpha(l_1 - s)] \\ &\times[\sigma(s, y_s + \bar{z}_s, y(a_1(s)) + \bar{z}(a_1(s)), \ldots, y(a_m(s)) + \bar{z}(a_m(s)))]dw(s)\|_{\mathbb{H}}^2, \\ &\leq 3[M_1 + r]E\|[S_\alpha(l_2) - S_\alpha(l_1)]\|_{\mathbb{H}}^2 + 3M_3E\|\int_0^t [T_\alpha(l_2 - s) - T_\alpha(l_1 - s)]\|_{\mathbb{H}}^2 \\ &+3M_4E\|\int_0^t [T_\alpha(l_2 - s) - T_\alpha(l_1 - s)]\|_{\mathbb{H}}^2. \end{aligned}$$

For $t \in (t_i, t_{i+1}], i = 1, 2, \ldots, p$, we have

$$\begin{aligned} E\|(Nz)(l_2) - (Nz)(l_1)\|_{\mathbb{H}}^2 &\leq 3E\|[S_\alpha(l_2 - t_i) - S_\alpha(l_1 - t_i)][y(t_i^-) + \bar{z}(t_i^-) + I_i(y(t_i^-) + \bar{z}(t_i^-))]\|_{\mathbb{H}}^2 \\ &+3E\|\int_{t_i}^t [T_\alpha(l_2 - s) - T_\alpha(l_1 - s)][f(s, y_s + \bar{z}_s, y(a_1(s)) + \bar{z}(a_1(s)), \ldots, \\ &y(a_m(s)) + \bar{z}(a_m(s)))]ds\|_{\mathbb{H}}^2 + 3E\|\int_{t_i}^t [T_\alpha(l_2 - s) - T_\alpha(l_1 - s)] \\ &\times[\sigma(s, y_s + \bar{z}_s, y(a_1(s)) + \bar{z}(a_1(s)), \ldots, y(a_m(s)) + \bar{z}(a_m(s)))]dw(s)\|_{\mathbb{H}}^2, \\ &\leq 3[M_2 + r]E\|[S_\alpha(l_2 - t_i) - S_\alpha(l_1 - t_i)]\|_{\mathbb{H}}^2 + 3M_3E\|\int_{t_i}^t [T_\alpha(l_2 - s) - T_\alpha(l_1 - s)]\|_{\mathbb{H}}^2 \\ &+3M_4E\|\int_{t_i}^t [T_\alpha(l_2 - s) - T_\alpha(l_1 - s)]\|_{\mathbb{H}}^2. \end{aligned}$$

Since $T_\alpha(t)$ and $S_\alpha(t)$ are strongly continuous its implies that $\lim_{l_2 \to l_1} \|[S_\alpha(l_2 - t_i) - S_\alpha(l_1 - t_i)]\|_{\mathbb{H}}^2 = 0$ and $\lim_{l_2 \to l_1} \|[T_\alpha(l_2 - t_i) - T_\alpha(l_1 - t_i)]\|_{\mathbb{H}}^2 = 0$ This implies that N is equi-continuous on all subintervals $(t_i, t_{i+1}], i = 1, 2, \ldots, p$. Thus by Arzela -Ascoli theorem, it follows that N is a compact operator. Hence N is completely continuous operator. Therefore, by Schauder fixed point theorem, the operator N has a fixed point, which in turns implies that (1.1)-(1.3) has at least one solution on $[0, T]$. This completes the proof of the theorem. □

4 Example

Example 4.1. *Consider the following nonlocal impulsive fractional partial differential equation of the form*

$$\begin{aligned}
\frac{\partial^q}{\partial t^q}u(t,x) &= \frac{\partial^2}{\partial y^2}u(t,x)+\frac{1}{25}\int_{-\infty}^{t}H(t,x,s-t)Q_1(u(s,x),u(a_1(s),\dots,u(a_m(s)))ds \\
&\quad+\left[\frac{1}{25}\int_{-\infty}^{t}V(t,x,s-t)Q_2(u(s,x),u(a_1(s),\dots,u(a_m(s)))ds\right]\frac{dw(t)}{dt}, \qquad (4.5)\\
u(t,0) &= u(t,\pi)=0,\ \ t\geq 0, \qquad (4.6)\\
u(t,x)+\sum_{k=1}^{m}c_k u(x,t_k) &= \phi(t,x), t\in(-\infty,0], x\in[0,\pi], \qquad (4.7)\\
\Delta u(t_i)(x) &= \frac{1}{9}\int_{-\infty}^{t_i}q_i(t_i-s)u(s,x)ds, x\in[0,\pi], \qquad (4.8)
\end{aligned}$$

where $\frac{\partial^q}{\partial t^q}$ *is Caputo's fractional derivative of order* $0<q<1, 0<t_1<t_2<\cdots<t_n\leq T$ *are prefixed numbers,* $\phi\in\mathfrak{B}_h$. *Let* $\mathbb{H}=L^2[0,\pi]$ *and define the operator* $A: D(A)\subset\mathbb{H}\to\mathbb{H}$ *by* $A\omega=\omega''$ *with the domain* $D(A):=\{\omega\in X:\omega,\omega' \text{are absolutely continuous}, \omega''\in\mathbb{H}, \omega(0)=0=\omega(\pi)\}$. *Then*

$A\omega=\sum_{n=1}^{\infty}n^2(\omega,\omega_n)\omega_n, \omega\in D(A)$, *where* $\omega_n(x)=\sqrt{\frac{2}{\pi}}sin(nx), n\in N$ *is the orthogonal set of eigenvectors of A. It is well known that A is the infinitesimal generator of an analytic semigroup* $(T(t))_{t\geq 0}$ *in* $\mathbb{H}$ *and is given by*

$$T(t)\omega=\sum_{n=1}^{\infty}e^{-n^2t}(\omega,\omega_n)\omega_n, \textit{ for all } \omega\in\mathbb{H}, \textit{ and every } t>0.$$

The subordination principle of solution operator (Theorem 3.1 in [3]) implies that A is the infinitesimal generator of a solution operator $\{S_\alpha(t)\}_{t\geq 0}$. *Since* $S_\alpha(t)$ *is strongly continuous on* $[0,\infty)$, *by uniformly bounded theorem, there exists a constant* $M>0$, *such that* $\|S_\alpha(t)\|_{L(\mathbb{H})}\leq M$, *for* $t\in[0,T]$. *Let* $h(s)=e^{2s}, s<0$ *then* $l=\int_{-\infty}^{0}h(s)ds=\frac{1}{2}$ *and define*

$$\|\phi\|_{\mathfrak{B}_h}=\int_{-\infty}^{0}h(s)\sup_{\theta\in[s,0]}\|\phi(\theta)\|_{L^2}ds.$$

Hence for $(t,\phi)\in[0,T]\times\mathfrak{B}_h$, *where* $\phi(\theta)(x)=\phi(\theta,x), (\theta,x)\in(-\infty,0]\times[0,\pi]$. *Set* $u(t)(x)=u(t,x)$,

$$\begin{aligned}
f(t,\phi,u(a_1(t)),\dots,u(a_m(t))))(x) &= \frac{1}{25}\int_{-\infty}^{0}H(t,x,\theta)Q_1(\phi(\theta,u(a_1(t)),\dots,u(a_m(t)))(x))d\theta,\\
\sigma(t,\phi,u(a_1(t)),\dots,u(a_m(t))))(x) &= \frac{1}{25}\int_{-\infty}^{0}V(t,x,\theta)Q_2(\phi(\theta,u(a_1(t)),\dots,u(a_m(t)))(x))d\theta,\\
I_i(\phi)(x) &= \frac{1}{9}\int_{-\infty}^{0}q_i(-\theta)\phi(\theta)(x)d\theta,\\
g(x) &= \sum_{k=1}^{m}c_k u(x,t_k).
\end{aligned}$$

Then with these settings the equations (4.5)-(4.8) can be written in the abstract form of equations (1.1)-(1.3). Further we have here $L_f=\frac{1}{25}, L_\sigma=\frac{1}{25}, L=\frac{1}{9}, T=1, l=\frac{1}{2}, \widetilde{M}_T=1, \widetilde{M}_S=\frac{1}{5}$ *and* $m=2$. *In this formulation of the problem we can verify the assumptions of Theorem (3.2). We get the value of condition in Theorem (3.2) as* $\Theta=.73<1$. *This implies that there exists a unique mild solution u on* $[0,1]$.

Example 4.2. *Here we consider the following non-trivial problem*

$$\begin{aligned}
\frac{\partial^q}{\partial t^q}u(t,x) &= \frac{\partial^2}{\partial y^2}u(t,x)+\frac{e^{-t}}{25+e^t}\int_{-\infty}^{t}H(t,x,s-t)[Q_1(u(s,x),u(a_1(s),\dots,u(a_m(s)))+\frac{t}{7}]ds\\
&\quad+\frac{e^{-t}}{25+e^t}\int_{-\infty}^{t}V(t,x,s-t)[Q_2(u(s,x),u(a_1(s),\dots,u(a_m(s)))+\frac{t}{7}]dw(s) \qquad (4.9)\\
u(t,x) &= u(t,\pi)=0, t\geq 0, \qquad (4.10)\\
u(t,x)+\sum_{k=1}^{m}c_k u(x,t_k) &= \phi(t,x),\ t\in(-\infty,0],\ x\in[0,\pi], \qquad (4.11)\\
\Delta u\big|_{t=\frac{1}{2}^-} &= \sin(\frac{1}{9}\|u(\frac{1}{2}^-,x)\|),\ 0\leq t\leq 1, 0\leq x\leq\pi, \qquad (4.12)
\end{aligned}$$

where $q \in (0,1)$. In the perspective of Example 1 we set

$$f(t,\phi,u(a_1(t)),\ldots,u(a_m(t))))(x) = \frac{e^{-t}}{25+e^t}\int_{-\infty}^{0} H(t,x,\theta)[Q_1(\phi(\theta,u(a_1(t)),\ldots,u(a_m(t)))(x)) + \frac{t}{7}]d\theta,$$

$$\sigma(t,\phi,u(a_1(t)),\ldots,u(a_m(t))))(x) = \frac{e^{-t}}{25+e^t}\int_{-\infty}^{0} V(t,x,\theta)[Q_2(\phi(\theta,u(a_1(t)),\ldots,u(a_m(t)))(x)) + \frac{t}{7}]d\theta.$$

Then with these settings the equations (4.9)-(4.12) can be written in the abstract form of equations (1.1)-(1.3). Hence the our problem (4.9)-(4.12) have a unique mild solution on $[0,1]$.

References

[1] D. Araya and C. Lizama, Almost automorphic mild solutions to fractional differential equations, *Nonlinear Anal. TMA* , 69(11)(2009), 3692-3705.

[2] P. Balasubramaniam, J. Y. Park and A. V. A. Kumar, Existence of solutions for semilinear neutral stochastic functional differential equations with nonlocal conditions, *Nonlinear Anal.*, 71(2009), 1049-1058.

[3] E. Bazhlekova, *Fractional Evolution Equations in Banach Spaces*, in: University Press Facilities, Eindhoven University of Technology, 2001.

[4] D. Bhuguna, Existence, uniqueness and regularity of solution to semilinear nonlocal functional differential problems, *Nonlinear Anal.*, 57 (2004), 1021-1028.

[5] A. Chauhan and J. Dabas, Existence of mild solutions for impulsive fractional-order semilinear evolution equations with nonlocal conditions, *Elect. J. of Diff. Equ.*, 2011(107)(2011), 1-10.

[6] A. Chauhan and J. Dabas, Local and global existence of mild solution to an impulsive fractional functional integro-differential equation with nonlocal condition, *Commun Nonli. Sci Numer Simu.*, 19(2014), 821-829.

[7] A. Chauhan, J. Dabas and M. Kumar, Integral boundary-value problem for impulsive fractional functional differential equations with infinite delay, *Elect. J. of Diff. Equ.*, 2012(229)(2012), 1-13.

[8] J. Dabas and A. Chauhan, Existence and uniqueness of mild solution for an impulsive neutral fractional integro-differential equation with infinite delay, *Math. and Comp. Model.*, 57(2013), 754-763.

[9] J. Dabas and G. R. Gautam, Impulsive neutral fractional integro-differential equations with state dependent delays and integral conditions, *Elect. J. of Diff. Equ.*, 2013(273)(2013), 1-13.

[10] K. Deng, Exponential decay of solutions of semilinear parabolic equations with nonlocal initial conditions, *J. of Math. anal. and appl.*, 179(1993), 630-637.

[11] G. R. Gautam and J. Dabas, Mild solution for fractional functional integro-differential equation with not instantaneous impulse, *Malaya J. of Matematik*, 2(3)(2014), 428-437.

[12] M. Haase, The functional calculus for sectorial operators, *Operator Theory, Advances and Applications*, vol. 169, Birkhauser-Verlag, Basel, 2006.

[13] D. N. Keck and M. A. McKibben, Functional integro-differential stochastic evolution equations in hilbert spaces, *J. of Appl. Math. and Sto. Anal.*, 16(2)(2003), 141-161.

[14] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science B.V, Amsterdam, 2006.

[15] V. Lakshmikantham, Theory of fractional differential equations, *Nonlinear Anal.*, 69(2008), 3337-3343.

[16] C. Li, J. Sun and R. Sun, Stability analysis of a class of stochastic differential delay equations with nonlinear impulsive effects, *J. of the Franklin Institute*, 347(2010), 1186-1198.

[17] A. Lin, Y. Ren and N. Xia, On neutral impulsive stochastic integro-differential equations with infinite delays via fractional operators, *Math. and Comp. Model.*, 51(2010), 413-424.

[18] S. Longa, L. Teng and D. Xu, Global attracting set and stability of stochastic neutral partial functional differential equations with impulses, *Statistics and Probability Letters*, 82(2012), 1699-1709.

[19] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, USA, 1993.

[20] Y. Ren and D. D. Sun, Second-order neutral stochastic evolution equations with infinite delay under carathodory conditions, *Journal of Optimization Theory and Applications*, 147(2010), 569-582.

[21] Y. Ren, Q. Zhou and L. Chen, Existence, uniqueness and stability of mild solutions for time-dependent stochastic evolution equations with Poisson jumps and infinite delay, *J. of Optim. Theo. and Appli.*, 149(2011), 315-331.

[22] R. Sakthivel, P. Revathi and Y. Ren, Existence of solutions for nonlinear fractional stochastic differential equations, *Nonlilear Anal.*, 81(2013), 70-86.

[23] L. Shen, J. Shi and J. Sun, Complete controllability of impulsive stochastic integro-differential systems, *Automatica*, 46(2010), 1068-1073.

[24] L. Shen and J. Sun, Approximate controllability of stochastic impulsive functional systems with infinite delay, *Automatica*, 48(2012), 2705-2709.

[25] R. Subalakshmi and K. Balachandran, Approximate controllability of nonlinear stochastic impulsive intergrodifferential systems in Hilbert spaces, *Chaos Solitons Fractals*, 42(2009), 2035-2046.

[26] E. Zeidler, *Nonlinear Functional Analysis and Its Applications,* Fixed-Point Theorems, Springer, New York (1986).

Received: October 24, 2014; *Accepted*: March 09, 2015

UNIVERSITY PRESS

Website: http://www.malayajournal.org/