Malaya J. Mat. 3(3)(2015) 303–311

Malaya Journal of Matematik

MJM
an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

On semi-invariant submanifolds of a nearly trans-hyperbolic Sasakian manifold

Shamsur Rahman[a,*] and Arjumand Ahmad [b]

[a,b]Department of Mathematics, Maulana Azad National Urdu University & Polytechnic Darbhanga (Centre),Bihar 846001, India.

Abstract

Semi-invariant submanifold of a trans Sasakian manifold has been studies. In the present paper we study semi invariant submanifolds of a nearly trans hyperbolic Sasakian manifold. Nejenhuis tensor in a nearly trans hyperbolic Sasakian manifold is calculated. Integrability conditions for some distributions on a semi invariant submanifold of a nearly trans hyperbolic Sasakian manifold are investigated.

Keywords: Semi-invariant submanifolds, nearly trans hyperbolic Sasakian manifold, Gauss and Weingarten equations, integrability conditions, distributions.

2010 MSC: 53D12,53C05.

1 Introduction

The study of geometry of semi invariant submanifold of a Sasakian manifold has been studied by Bejancu [1] and Bejancu and Papaghuic [4]. After that a number of authors have studied these submanifolds ([3],[5],[12]). Latter on, Oubina [8] introduced a new class of almost contact Riemannian manifold known as trans Sasakian manifold. Upadhyay and Dube [13] have studied almost contact hyperbolic (f, g, η, ξ)-structure. Shahid studied on semi invariant submanifolds of a nearly Sasakian manifold [14]. Matsumoto, Shahid, and Mihai [10] have also worked on semi invariant submanifolds of certain almost contact manifolds. Joshi and Dube [15] studied on Semi-invariant submanifold of an almost r-contact hyperbolic metric manifold. Gill and Dube have worked on CR submanifolds of trans-hyperbolic Sasakian manifolds [7].

2 Preliminaries

Nearly trans hyperbolic Sasakian Manifolds: Let $\bar{M}$ be an n dimensional almost hyperbolic contact metric manifold with the almost hyperbolic contact metric structure (ϕ, ξ, η, g) where a tensor ϕ of type $(1, 1)$, a vector field ξ, called structure vector field and η, the dual 1-form of is a 1-form ξ satisfying the following

$$\phi^2 X = X - \eta(X)\xi, \quad g(X, \xi) = \eta(X), \tag{2.1}$$

$$\phi(\xi) = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = -1 \tag{2.2}$$

$$g(\phi X, \phi Y) = -g(X, Y) - \eta(X)\eta(Y) \tag{2.3}$$

for any X,Y tangents to $\bar{M}$ [6]. In this case

$$g(\phi X, Y) = -g(X, \phi Y) \tag{2.4}$$

*Corresponding author.
E-mail address: shamsur@rediffmail.com (Shamsur Rahman), aarrjjuummaanndd@gmail.com(Arjumand Ahmad).

An almost hyperbolic contact metric structure (ϕ, ξ, η, g) on $\bar{M}$ is called trans-hyperbolic Sasakian [7] if and only if

$$(\bar{\nabla}_X\phi)Y = \alpha[g(X,Y)\xi - \eta(Y)\phi X] + \beta[g(\phi X, Y)\xi - \eta(Y)\phi X] \tag{2.5}$$

for all X,Y tangents to $\bar{M}$ and α, β are functions on $\bar{M}$. On a trans-hyperbolic Sasakian manifold M, we have

$$\bar{\nabla}_X\xi = -\alpha(\phi X) + \beta[X - \eta(X)\xi] \tag{2.6}$$

a Riemannian metric g and Riemannian connection $\bar{\nabla}$. Further, an almost contact metric manifold $\bar{M}$ on (ϕ, ξ, η, g) is called nearly trans-hyperbolic Sasakian if [9]

$$(\bar{\nabla}_X\phi)Y + (\bar{\nabla}_Y\phi)X = \alpha[2g(X,Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y] - \beta[\eta(X)\phi Y + \eta(Y)\phi X] \tag{2.7}$$

Semi-invariant submanifolds: Let M be a submanifold of a Riemannian manifold $\bar{M}$ endowed with a Riemannian metric g. Then Gauss and Wiengarten formulae are given respectively by

$$\bar{\nabla}_XY = \nabla_XY + h(X,Y) \quad (X, Y\epsilon TM) \tag{2.8}$$
$$\bar{\nabla}_XN = -A_NX + \nabla^{\perp}_XN \quad (N\epsilon T^{\perp}M) \tag{2.9}$$

where $\bar{\nabla}$, ∇ and $\nabla^{\perp}$ are respectively the Riemannian, induced Riemannian and induced normal connections in $\bar{M}$, M and the normal bundle of $T^{\perp}M$ of M respectively, and h is the second fundamental form related to A by

$$g(h(X,Y), N) = g(A_NX, Y) \tag{2.10}$$

Moreover, if ϕ is a $(1,1)$ tensor field on $\bar{M}$, for $X\epsilon TM$ and $N\epsilon T^{\perp}M$ we have

$$(\bar{\nabla}_X\phi)Y = ((\nabla_XP)Y - A_{FY}X - th(X,Y)) + ((\nabla_XF)Y + h(X, PY) - fh(X,Y)) \tag{2.11}$$
$$(\bar{\nabla}_X\phi)N = ((\nabla_Xt)Y - A_{fN}X - PA_NX)) + ((\nabla_Xf)N + h(X, tN) - FA_NX)) \tag{2.12}$$

where

$$\phi X \equiv PX + FX \quad (PX\epsilon TM, FX\epsilon T^{\perp}M) \tag{2.13}$$
$$\phi N \equiv tN + fN \quad (tN\epsilon TM, fN\epsilon T^{\perp}M) \tag{2.14}$$
$$(\nabla_XP)Y \equiv \nabla_XPY - P\nabla_XY, \quad (\nabla_XF)Y \equiv \nabla^{\perp}_XFY - F\nabla_XY$$
$$(\nabla_Xt)N \equiv \nabla_XtN - t\nabla^{\perp}_XN, \quad (\nabla_Xf)N \equiv \nabla^{\perp}_XfN - f\nabla^{\perp}_XN$$

The submanifold M is known to be totally geodesic in $\bar{M}$ if $h = 0$, minimal in $\bar{M}$ if $H = trace(h)/dim(M) = 0$, and totally umbilical in $\bar{M}$ if $h(X,Y) = g(X,Y)H$.

For a distribution D on M, M is said to be D-totally geodesic if for all $X, Y\epsilon D$ we have $h(X,Y) = 0$. If for all $X, Y\epsilon D$ we have $h(X,Y) = g(X,Y)K$ for some normal vector K, then M is called D-totally umbilical. For two distributions D and ε defined on M, M is said to be (D, ε)-mixed totally geodesic if for all $X\epsilon D$ and $Y\epsilon\varepsilon$ we have $h(X,Y) = 0$.

Let D and ε be two distributions defined on a manifold M. We say that D is ε -parallel if for all $X\epsilon\varepsilon$ and $Y\epsilon D$ we have $\nabla_XY\epsilon D$. If D is D-parallel then it is called autoparallel. D is called X-parallel for some $X\epsilon TM$ if for all $Y\epsilon D$ we have $\nabla_XY\epsilon D$. D is said to be parallel if for all $X\epsilon TM$ and $Y\epsilon D, \nabla_XY\epsilon D$.

If a distribution D on M is autoparallel, then it is clearly integrable, and by Gauss formula D is totally geodesic in M. If D is parallel then the orthogonal complementary distribution $D^{\perp}$ is also parallel, which implies that D is parallel if and only if $D^{\perp}$ is parallel. In this case M is locally the product of the leaves of D and $D^{\perp}$.

Let M be a submanifold of an almost contact metric manifold. If $\xi\epsilon TM$ then we write $TM = \{\xi\} \oplus \{\xi\}^{\perp}$, where (ξ) is the distribution spanned by ξ and $\{\xi\}^{\perp}$ is the complementary orthogonal distribution of $\{\xi\}$ in M. Then one gets

$$P\xi = 0 = F\xi, \quad \eta oP = 0 = \eta oF, \tag{2.15}$$
$$P^2 + tF = -I + \eta \otimes \xi, \quad FP + fF = 0, \tag{2.16}$$
$$f^2 + Ft = -I, \quad tf + Pt = 0 \tag{2.17}$$

A submanifold M of an almost contact metric manifold $\bar{M}$ with $\xi\epsilon TM$ is called a semi-invariant submanifold (Bejancu, [1]) of $\bar{M}$ if there exists two differentiable distributions D^1 and D^0 on M such that

(1) $TM = D^1 \oplus D^0 \oplus \{\xi\}$,

(2) the distribution D^1 is invariant by ϕ, that is, $\phi(D^1) = D^1$ and

(3) the distribution D^0 is anti-invariant by ϕ, that is, $\phi(D^0) \subseteq T^{\perp}M$.

For $X\epsilon TM$ we can write

$$X = D^1X + D^0X + \eta(X)\xi \tag{2.18}$$

where D^1 and D^0 are the projection operators of TM on D^1 and D^0, respectively. A semi-invariant submanifold of an almost contact metric manifold becomes an invariant submanifold ([2], [11]) (resp. anti-invariant submanifold ([2], [11]) if $D^0 = \{0\}$ (resp. $D^1 = \{0\}$).

3 The Nijenhuis tensor

A hyperbolic contact metric manifold is said to be normal ([6]) if the torsion tensor N^1 vanishes:

$$N^1 \equiv [\phi, \phi] + d\eta \otimes \xi = 0 \tag{3.19}$$

where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ and d denotes the exterior derivatives operatoer. In this section we obtain expression for Nijenhuis tensor $[\phi, \phi]$ of the structure tensor field ϕ given by

$$[\phi, \phi](X, Y) = ((\bar{\nabla}_{\phi X}\phi)Y - (\bar{\nabla}_{\phi Y}\phi)X) - \phi((\bar{\nabla}_X\phi)Y - (\bar{\nabla}_Y\phi)X) \tag{3.20}$$

in a nearly trans hyperbolic Sasakian manifold. First, we need the following lemma.

Lemma 3.1. *In an almost hyperbolic contact metric manifold we have*

$$(\bar{\nabla}_Y\phi)\phi X = -\phi(\bar{\nabla}_Y\phi)X - ((\bar{\nabla}_Y\eta)X)\xi - \eta(X)\bar{\nabla}_Y\xi \tag{3.21}$$

Proof. For $X, Y\epsilon T\bar{M}$, we have

$$\begin{aligned}(\bar{\nabla}_Y\phi)\phi X &= -\phi^2\bar{\nabla}_Y X - \phi(\bar{\nabla}_Y\phi)X + \bar{\nabla}_Y X - ((\bar{\nabla}_Y\eta)X)\xi - \eta(\bar{\nabla}_Y X)\xi - \eta(X)\bar{\nabla}_Y\xi \\ &= -\bar{\nabla}_Y X + \eta(\bar{\nabla}_Y X)\xi - \phi(\bar{\nabla}_Y\phi)X + \bar{\nabla}_Y X - ((\bar{\nabla}_Y\eta)X)\xi - \eta(\bar{\nabla}_Y X)\xi - \eta(X)\bar{\nabla}_Y\xi\end{aligned}$$

which gives the equation (3.21). □

Now, we prove the following theorem

Theorem 3.1. *In a nearly trans-hyperbolic Sasakian manifold the Nijenhuis tensor $[\phi, \phi]$ of ϕ is given by*

$$\begin{aligned}[\phi, \phi](X, Y) = 4\phi(\bar{\nabla}_Y\phi)X + 2d\eta(X, Y)\xi + \eta(X)\bar{\nabla}_Y\xi - \eta(Y)\bar{\nabla}_X\xi \\ +4\alpha g(\phi X, Y)\xi + (\alpha + \beta)\eta(Y)\phi^2 X + 3(\alpha + \beta)\eta(X)\phi^2 Y\end{aligned} \tag{3.22}$$

Proof. Using Lemma 3.1 and $\eta o\phi = 0$ in (2.7) we get

$$(\bar{\nabla}_{\phi X}\phi)Y = \phi(\bar{\nabla}_Y\phi)X + ((\bar{\nabla}_Y\eta)X)\xi + \eta(X)\bar{\nabla}_Y\xi + 2\alpha g(\phi X, Y)\xi - (\alpha + \beta)\eta(Y)\phi^2 X \tag{3.23}$$

Thus

$$\begin{aligned}
[\phi,\phi](X,Y) &= ((\bar{\nabla}_{\phi X}\phi)Y - (\bar{\nabla}_{\phi Y}\phi)X) - \phi((\bar{\nabla}_X\phi)Y - (\bar{\nabla}_Y\phi)X) \\
&= 2\phi(\bar{\nabla}_Y\phi)X - 2\phi(\bar{\nabla}_X\phi)Y + [((\bar{\nabla}_X\eta)Y)\xi - ((\bar{\nabla}_Y\eta)X)\xi] + \eta(X)\bar{\nabla}_Y\xi \\
&\quad -\eta(Y)\bar{\nabla}_X\xi + 4\alpha g(\phi X,Y)\xi - (\alpha+\beta)[\eta(Y)\phi^2X - \eta(X)\phi^2Y] \\
&= 4\phi(\bar{\nabla}_Y\phi)X - 2\phi[\alpha(2g(X,Y)\xi \\
&\quad -\eta(Y)\phi X - \eta(X)\phi Y - \beta(\eta(X)\phi Y + \eta(Y)\phi X)] \\
&\quad +2d\eta(X,Y)\xi + \eta(X)\bar{\nabla}_Y\xi - \eta(Y)\bar{\nabla}_X\xi \\
&\quad +4\alpha g(\phi X,Y)\xi - (\alpha+\beta)[\eta(Y)\phi^2X - \eta(X)\phi^2Y] \\
&= 4\phi(\bar{\nabla}_Y\phi)X + 2\alpha\eta(Y)\phi^2X + 2\alpha\eta(X)\phi^2Y - \beta[\eta(X)\phi Y + \eta(Y)\phi X] \\
&\quad +2d\eta(X,Y)\xi + \eta(X)\bar{\nabla}_Y\xi - \eta(Y)\bar{\nabla}_X\xi \\
&\quad +4\alpha g(\phi X,Y)\xi - (\alpha+\beta)[\eta(Y)\phi^2X - \eta(X)\phi^2Y] \\
&= 4\phi(\bar{\nabla}_Y\phi)X + 2(\alpha+\beta)\eta(Y)\phi^2X + 2(\alpha+\beta)\eta(X)\phi^2Y + 2d\eta(X,Y)\xi \\
&\quad +\eta(X)\bar{\nabla}_Y\xi - \eta(Y)\bar{\nabla}_X\xi + 4\alpha g(\phi X,Y)\xi \\
&\quad -(\alpha+\beta)\eta(Y)\phi^2X + (\alpha+\beta)\eta(X)\phi^2Y] \\
[\phi,\phi](X,Y) &= 4\phi(\bar{\nabla}_Y\phi)X + 2d\eta(X,Y)\xi + \eta(X)\bar{\nabla}_Y\xi - \eta(Y)\bar{\nabla}_X\xi \\
&\quad +4\alpha g(\phi X,Y)\xi + (\alpha+\beta)\eta(Y)\phi^2X + 3(\alpha+\beta)\eta(X)\phi^2Y
\end{aligned}$$

which implies the equation (3.22). From Equation (3.22), we get

$$\eta(N^1(X,Y)) = 3d\eta(X,Y) - 4\alpha g(X,\phi Y) \tag{3.24}$$

In particular, if X and Y are perpendicular to ξ, then (3.22) gives

$$[\phi,\phi](X,Y) = 4\phi(\bar{\nabla}_Y\phi)X - 2(\eta[X,Y])\xi \tag{3.25}$$

□

4 Some basic results

Let M be a submanifold of a nearly trans-hyperbolic Sasakian manifold. Using (2.11), (2.13) in (2.7) for $X, Y \epsilon TM$, we get

$$\begin{aligned}
&(\nabla_X P)Y + (\nabla_Y P)X - A_{FY}X - A_{FX}Y - 2th(X,Y) + (\nabla_X F)Y \\
&\quad +(\nabla_Y F)X + h(X,PY) + h(Y,PX) - 2fh(X,Y) \\
&= \alpha[2g(X,Y)\xi - \eta(Y)PX - \eta(Y)FX - \eta(X)PY - \eta(X)FY] \\
&\quad -\beta[\eta(X)PY + \eta(X)FY + \eta(Y)PX + \eta(Y)FX]
\end{aligned} \tag{4.26}$$

Consequently, we have

Proposition 4.1. *Let M be a submanifold of a nearly trans-hyperbolic Sasakian manifold. Then for all $X, Y \epsilon TM$ we have*

$$\begin{aligned}
&(\nabla_X P)Y + (\nabla_Y P)X - A_{FY}X - A_{FX}Y - 2th(X,Y) \\
&= 2\alpha g(X,Y)\xi - (\alpha+\beta)(\eta(Y)PX + \eta(X)PY)
\end{aligned} \tag{4.27}$$

and

$$\begin{aligned}
&(\nabla_X F)Y + (\nabla_Y F)X + h(X,PY) + h(Y,PX) - 2fh(X,Y) \\
&= -(\alpha+\beta)[\eta(X)FY + \eta(Y)FX]
\end{aligned} \tag{4.28}$$

for all $X, Y \epsilon TM$.

Now we state the following proposition.

Proposition 4.2. *Let M be a submanifold of a nearly trans-hyperbolic Sasakian manifold. Then*

$$\begin{aligned}\bar{\nabla}_X\phi Y+\bar{\nabla}_Y\phi X-\phi[X,Y] &= 2((\nabla_X P)Y-A_{FY}X-th(X,Y)) \\ &+2((\nabla_X F)Y+h(X,PY)-fh(X,Y))+2\alpha g(X,Y)\xi) \\ &-(\alpha+\beta)(\eta(Y)PX+\eta(X)PY)-(\alpha+\beta)(\eta(Y)FX+\eta(X)FY)\end{aligned} \tag{4.29}$$

Consequently,

$$\begin{aligned}P[X,Y] &= A_{FY}X+A_{FX}Y+2th(X,Y)-2\alpha g(X,Y)\xi \\ &-(\alpha+\beta)(\eta(Y)PX+\eta(X)PY-\nabla_X PY-\nabla_Y PX+2P\nabla_X Y\end{aligned} \tag{4.30}$$

$$\begin{aligned}F[X,Y] &= -\nabla^{\perp}_X FY-\nabla^{\perp}_Y FY-h(X,PY)-h(Y,PY)+2fh(X,Y) \\ &-(\alpha+\beta)(\eta(Y)FX+\eta(X)FY)+2F\nabla_X Y\end{aligned} \tag{4.31}$$

The proof is straightforward and hence omitted.

Proposition 4.3. *Let M be a semi invariant submanifold of a nearly trans-hyperbolic Sasakian manifold. Then (P,ξ,η,g) is a nearly trans-hyperbolic Sasakian structure on the distribution $D^1\oplus\{\xi\}$ if $th(X,Y)=0$ for all $X,Y\epsilon D^1\oplus\{\xi\}$.*

Proof. From $D^1\oplus\{\xi\}=ker(F)$ and (2.16) we have $P^2=I-\eta\otimes\xi$ on $D^1\oplus\{\xi\}$. We also get $P\xi=0,\eta(\xi)=2,\eta\circ P=0$.Using $D^1\oplus\{\xi\}=ker(F)$ and $th(X,Y)=0$ in 4.27 we get

$$(\nabla_X P)Y+(\nabla_Y P)X=2\alpha g(X,Y)\xi-(\alpha+\beta)(\eta(Y)PX+\eta(X)PY), \tag{4.32}$$

for all X, Y $\epsilon D^1\oplus\{\xi\}$. □

This completes the proof.

Theorem 4.2. *Let M be a semi invariant submanifold of a nearly trans-hyperbolic Sasakian manifold. We have (i) if $D^0\oplus\{\xi\}$ is autoparallel then*

$$A_{FX}Y+A_{FY}X+2th(X,Y)=0,\quad \forall X,Y\epsilon D^0\oplus\{\xi\} \tag{4.33}$$

(ii) if $D^1\oplus\{\xi\}$ is autoparallel then

$$h(X,PY)+h(PX,Y)=2fh(X,Y)\quad \forall X,Y\epsilon D^1\oplus\{\xi\}. \tag{4.34}$$

Proof. In view of (4.27) and autoparallelness of $D^0\oplus\{\xi\}$ we get (1), while in view of (4.28) and appropriateness of $D^1\oplus\{\xi\}$ we get (ii). In view of Proposition 4.3 and Theorem 4.2(ii), we get □

Theorem 4.3. *Let M be a submanifold of a nearly trans-hyperbolic Sasakian manifold with $\xi\epsilon TM$. If M is invariant then M is nearly trans-hyperbolic Sasakian. Moreover*

$$h(X,PY)+h(PX,Y)-2fh(X,Y)=0,\quad X,Y\epsilon TM.$$

5 Integrability Conditions

Integrability of the distribution $D^1\oplus\{\xi\}$: We begin with a lemma

Lemma 5.2. *Let M be a semi-invariant submanifold of a nearly trans-hyperbolic Sasakian manifold. For $X,Y\epsilon D^1\oplus\{\xi\}$ we get*

$$F[X,Y]=-h(X,PY)-h(PX,Y)+2F\nabla_X Y+2fh(X,Y) \tag{5.35}$$

or equivalently

$$-h(X,PX)+F\nabla_X X+fh(X,X)=0 \tag{5.36}$$

Proof. Equation (5.1) follows from $D^1 \oplus \{\xi\} = ker(F)$ and (4.6). Equivalence of (5.1) and (5.2) is obvious. In view of (5.1) and $D^1 \oplus \{\xi\} = ker(F)$ we can state the following theorem. □

Theorem 5.4. *The distribution $D^1 \oplus \{\xi\}$ on a semi-invariant submanifold of a nearly trans-hyperbolic Sasakian manifold is integrable if and only if*

$$h(X, PY) + h(PX, Y) = 2(F\nabla_X Y + fh(X, Y)) \tag{5.37}$$

Now, we need the following

Definition 5.1. *([16]) Let M be a Riemannian manifold with the Riemannian connection ∇ . A distribution D on M will be called nearly autoparallel if for all $X, Y \epsilon D$ we have $(\nabla_X Y + \nabla_Y X)\epsilon D$ or equivalently $\nabla_X X \epsilon D$.*

Thus, we have the following flow chart ([16]):
Parallel ⇒ Autoparallel ⇒ Nearly autoparallel,
Parallel ⇒ Integrable,
Autoparallel ⇒ Integrable, and
Nearly autoparallel + Integrable ⇒ Autoparallel.

Theorem 5.5. *Let M be a semi-invariant submanifold of a nearly trans-hyperbolic Sasakian manifold. Then the following four statements*

(a) the distribution $D^1 \oplus \{\xi\}$ is autoparallel,
(b) $h(X, PY) + h(PX, Y) = 2fh(X, Y), \quad X, Y \epsilon D^1 \oplus \{\xi\}$,
(c) $h(X, PX) = fh(X, X), \quad X \epsilon D^1 \oplus \{\xi\}$,
(d) the distribution $D^1 \oplus \{\xi\}$ is nearly autoparallel,

are related by (a) ⇒ (b) ⇔ (c) ⇒ (d). In particular, if $D^1 \oplus \{\xi\}$ is integrable then the above four statements are equivalent.

The proof is similar to that Theorem 4.4 of [16].
Let $X, Y \epsilon D^1 \oplus \{\xi\}$. Using (2.1) and (2.13) in (3.19) and we get

$$\begin{aligned} N^{(1)}(X, Y) = [\phi X, \phi Y] - P[\phi X, Y] - F[\phi X, Y] - P[X, \phi Y] \\ -F[X, \phi Y] + [X, Y] + \eta([X, Y])\xi + 2d\eta \otimes \xi \end{aligned} \tag{5.38}$$

On the other hand from equation (3.23) we have

$$(\bar{\nabla}_{\phi X}\phi)Y = \phi(\bar{\nabla}_Y\phi)X + ((\bar{\nabla}_Y\eta)X)\xi + \eta(X)\bar{\nabla}_Y\xi + 2\alpha g(\phi X, Y)\xi - (\alpha + \beta)\eta(Y)\phi^2 X$$

which implies that

$$\begin{aligned} (\bar{\nabla}_{\phi X}\phi)Y - (\bar{\nabla}_{\phi Y}\phi)X = \phi((\bar{\nabla}_Y\phi)X - (\bar{\nabla}_X\phi)Y) + 2d\eta(X, Y)\xi + \eta(X)U^1\nabla_Y\xi \\ +\eta(X)U^0\nabla_Y\xi + \eta(X)h(Y, \xi) - \eta(Y)U^1\nabla_X\xi - \eta(Y)U^0\nabla_X\xi \\ -\eta(Y)h(X, \xi) - (\alpha + \beta)(\eta(Y)\phi^2 X - \eta(X)\phi^2 Y) \end{aligned} \tag{5.39}$$

Next we easily can get

$$\begin{aligned} \phi(\bar{\nabla}_Y\phi)X = \phi(\bar{\nabla}_Y\phi X) - \phi^2(\bar{\nabla}_Y X) \\ = \phi(\nabla_Y\phi X + h(Y, \phi X)) - (\bar{\nabla}_Y X + \eta\bar{\nabla}_Y X)\xi \end{aligned} \tag{5.40}$$

so that

$$\begin{aligned} \phi((\bar{\nabla}_Y\phi)X - (\bar{\nabla}_X\phi)Y) = (\nabla_Y\phi X - \nabla_X\phi Y) + [X, Y] - \eta([X, Y])\xi \\ +F(\nabla_Y\phi X - \nabla_X\phi Y) + \phi(h(Y, \phi X) - h(X, \phi Y)) \end{aligned} \tag{5.41}$$

In view of (5.39) and (5.41) we get

$$\begin{aligned} N^{(1)}(X, Y) = 4d\eta \otimes \xi + 2[X, Y] - 2\eta([X, Y])\xi + 2P[\nabla_Y\phi X - \nabla_X\phi Y] \\ +2F[\nabla_Y\phi X - \nabla_X\phi Y] + 2\phi(h(Y, \phi X) - h(X, \phi Y)) + \eta(X)U^1\nabla_Y\xi \\ +\eta(X)U^0\nabla_Y\xi + \eta(X)h(Y, \xi) - \eta(Y)U^1\nabla_X\xi - \eta(Y)U^0\nabla_X\xi \\ -\eta(Y)h(X, \xi) - (\alpha + \beta)(\eta(Y)\phi^2 X - \eta(X)\phi^2 Y) \end{aligned} \tag{5.42}$$

Theorem 5.6. *The distribution $D^1 \oplus \{\xi\}$ is integrable on a semi-invariant submanifold M of a nearly trans-hyperbolic Sasakian manifold if and only if for all $X, Y \epsilon D^1 \oplus \{\xi\}$*

$$N^1(X,Y) \epsilon D^1 \oplus (\xi) \tag{5.43}$$

$$2(h(Y,\phi X) - h(X,\phi Y)) = -\eta(X)(\phi U^0 \nabla_Y \xi + fh(Y,\xi)) + \eta(Y)(\phi U^0 \nabla_X \xi + fh(X,\xi)) \tag{5.44}$$

Proof. Let $X, Y \epsilon D^1 \oplus \{\xi\}$. If $D^1 \oplus \{\xi\}$ is integrable, then (5.43) is true and from (5.42) we get

$$0 = 2F(\nabla_Y \phi X - \nabla_X \phi Y) + 2\phi(h(Y,\phi X) - h(X,\phi Y) + \eta(X)U^0 \nabla_Y \xi$$
$$+\eta(X)h(Y,\xi) - \eta(Y)U^0 \nabla_X \xi - \eta(Y)h(X,\xi)$$

Applying ϕ to the above equation, we get

$$0 = -2U^0(\nabla_Y \phi X - \nabla_X \phi Y) + 2(h(Y,\phi X) - h(X,\phi Y) + \eta(X)\phi U^0 \nabla_Y \xi$$
$$+\eta(X)th(Y,\xi) + \eta(X)fh(Y,\xi) - \eta(Y)\phi U^0 \nabla_X \xi - \eta(Y)th(X,\xi) - \eta(Y)fh(X,\xi)$$

Hence taking the normal part we get (5.44).

Conversely, let (5.43) and (5.44) be true. Using (5.44) in (5.42) we get

$$0 = 2U^0[X,Y] + 2F(\nabla_Y \phi X - \nabla_X \phi Y) + 2\phi(h(Y,\phi X) - h(X,\phi Y) + \eta(X)U^0 \nabla_Y \xi$$
$$+\eta(X)h(Y,\xi) - \eta(Y)U^0 \nabla_X \xi - \eta(Y)h(X,\xi)$$

Applying ϕ to the above equation and using (5.44) we get $\phi U^0[X,Y] = 0$, from which we get $U^0[X,Y] = 0$, and hence $D^1 \oplus \{\xi\}$ is integrable.

If $\bar{M}$ is a trans-hyperbolic Sasakian manifold then for all $X \epsilon D^1 \oplus \{\xi\}$ it is known that $h(X,\xi) = 0$ and $U^0 \nabla_X \xi = 0$. Hence in view of the previous theorem we have □

Corollary 5.1. . *If M is a semi-invariant submanifold of a trans-hyperbolic Sasakian manifold, then the distribution $D^1 \oplus \{\xi\}$ is integrable if and only if for all $X, Y \epsilon D^1 \oplus \{\xi\}$*

$$h(X,\phi Y) = h(Y,\phi X)$$

Integrability of the distribution $D^0 \oplus \{\xi\}$:

Lemma 5.3. *Let M be a semi-invariant submanifold of a nearly trans-hyperbolic Sasakian manifold. Then*

$$3(A_{FX}Y - A_{FY}X) = P[X,Y], \quad X, Y \epsilon D^0 \oplus (\xi) \tag{5.45}$$

Proof. Let $X, Y \epsilon D^0 \oplus \{\xi\}$ and $Z \epsilon TM$. We have

$$-A_{\phi X}Z + \nabla^{\perp}_Z \phi X = \bar{\nabla}_Z \phi X = (\bar{\nabla}_Z \phi)X + \phi(\bar{\nabla}_Z X)$$
$$= -(\bar{\nabla}_X \phi)Z - \eta(X)\phi Z - \eta(Z)\phi X + \phi \nabla_Z X + \phi h(Z,X)$$

so that

$$\phi h(Z,X) = -A_{\phi X}Z + \nabla^{\perp}_Z \phi X + (\bar{\nabla}_X \phi)Z + \eta(X)\phi Z + \eta(Z)\phi X - \phi \nabla_Z X$$

and hence we have

$$g(\phi h(Z,X),Y) = -g(A_{\phi X}Y,Z) - g((\bar{\nabla}_X \phi)Y,Z)$$

On the other hand

$$g(\phi h(Z,X),Y) = -g(h(Z,X),\phi Y) = -g(A_{\phi Y}X,Z)$$

Thus from the above two relations we get

$$g(A_{\phi Y}X,Z) = g(A_{\phi X}Y,Z) + g((\bar{\nabla}_X \phi)Y,Z) \tag{5.46}$$

For $X, Y\epsilon D^0 \oplus \{\xi\}$ we calculate $(\bar{\nabla}_X\phi)Y$ as follows. In view of

$$\bar{\nabla}_X\phi Y - \bar{\nabla}_Y\phi X = A_{\phi X}Y - A_{\phi Y}X + \nabla^{\perp}_X\phi Y - \nabla^{\perp}_Y\phi X$$

and

$$\bar{\nabla}_X\phi Y - \bar{\nabla}_Y\phi X = (\bar{\nabla}_X\phi)Y - (\bar{\nabla}_Y\phi)X + \phi[X, Y]$$

we have

$$(\bar{\nabla}_X\phi)Y - (\bar{\nabla}_Y\phi)X = A_{\phi X}Y - A_{\phi Y}X + \nabla^{\perp}_X\phi Y - \nabla^{\perp}_Y\phi X - \phi[X, Y]$$

which gives

$$(\bar{\nabla}_X\phi)Y = 1/2(A_{\phi X}Y - A_{\phi Y}X + \nabla^{\perp}_X\phi Y - \nabla^{\perp}_Y\phi X \\ -\phi[X, Y] - \eta(Y)\phi X - \eta(X)\phi Y)$$

□

Using this equation in the equation (5.46) we get (5.45).

In view of $D^0 \oplus \{\xi\} = ker(P)$, this lemma leads to the following

Theorem 5.7. *Let M be a semi-invariant submanifold of a nearly trans-hyperbolic Sasakian manifold. Then the distribution $D^0 \oplus \{\xi\}$ is integrable if and only if*

$$A_{FX}Y = A_{FY}X \quad forall X, Y\epsilon D^0 \oplus \{\xi\}$$

Integrability of the distribution D^0: We calculate the torsion tensor $N^1(Y, X)$ for $Y, X\epsilon D^0$. It can be verified that

$$\phi((\bar{\nabla}_X\phi)Y - (\bar{\nabla}_Y\phi)X) = -[X, Y] + \eta([X, Y])\xi + \phi(A_{\phi X}Y - A_{\phi Y}X) + \phi(\nabla^{\perp}_X\phi Y - \nabla^{\perp}_Y\phi X) \tag{5.47}$$

$$(\bar{\nabla}_{\phi X}\phi)Y - (\bar{\nabla}_{\phi Y}\phi)X = [X, Y] - \phi(A_{\phi X}Y - A_{\phi Y}X) - \phi(\nabla^{\perp}_X\phi Y - \nabla^{\perp}_Y\phi X \tag{5.48}$$

Using (5.13), (5.14) and (5.11) we get for $Y, X\epsilon D^0$

$$N^1(Y, X) = -2[X, Y] + 2/3\phi P[X, Y] + 2\phi(\nabla^{\perp}_X\phi Y - \nabla^{\perp}_Y\phi X) \tag{5.49}$$

Theorem 5.8. *The distribution D^0 is integrable on a semi-invariant submanifold M of a nearly trans-hyperbolic Sasakian manifold if and only if*

$$N^{(1)}(Y, X)\epsilon D^0 \oplus \bar{D}^1 \quad X, Y\epsilon D^0 \tag{5.50}$$

$$A_{FX}Y = A_{FY}X \quad X, Y\epsilon D^0 \tag{5.51}$$

Proof. If D^0 is integrable, then in view of (5.48) and (5.49), the relation (5.50) and (5.51) follow easily. Conversely, let $X, Y\epsilon D^0$ and let the relation (5.50) and (5.51) be true. Then in view (5.48), we get $P[X, Y] = 0$ and in view of (5.49), we get

$$0 = g(\xi, N^1(Y, X)) = g(\xi, 2[Y, X]).$$

Thus $[X, Y]\epsilon D^0$. □

Non-integrability of the distribution D^1:

Theorem 5.9. *Let M be a semi-invariant submanifold of a nearly trans-hyperbolic Sasakian manifold with $\alpha \neq 0$. Then the non-zero invariant distribution D^1 is not integrable.*

Proof. If D^1 is integrable then for $X, Y\epsilon D^1$ it follows that $d\eta(X, Y) = 0$ and $[\phi, \phi](X, Y)\epsilon D^1$. Therefore, for $X\epsilon D^1$ in view of (3.24), we get

$$0 = \eta([\phi, \phi](X, PX) + 2d\eta(X, PX)\xi) \\ = \eta(N^1(X, PX) = 4\alpha g(\phi X, PX) = 4\alpha g(PX, PX),$$

which is a contradiction. □

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Received: October 10, 2014; *Accepted*: May 23, 2015

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