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Super Edge-antimagic Graceful labeling of Graphs

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Abstract

For a graph $G = (V, E)$, a bijection g from $V(G) \cup E(G)$ into $\{1, 2, \ldots, |V(G)| + |E(G)|\}$ is called (a, d)-edge-antimagic graceful labeling of G if the edge-weights $w(xy) = |g(x) + g(y) - g(xy)|, xy \in E(G)$, form an arithmetic progression starting from a and having a common difference d. An (a, d)-edge-antimagic graceful labeling is called super (a, d)-edge-antimagic graceful if $g(V(G)) = \{1, 2, \ldots, |V(G)|\}$. Note that the notion of super (a, d)-edge-antimagic graceful graphs is a generalization of the article "G. Marimuthu and M. Balakrishnan, Super edge magic graceful graphs, Inf.Sci.,287(2014)140–151", since super $(a, 0)$-edge-antimagic graceful graph is a super edge magic graceful graph.We study super (a, d)-edge-antimagic graceful properties of certain classes of graphs, including complete graphs and complete bipartite graphs.

Keywords: Edge-antimagic graceful labeling, Super edge-antimagic graceful labeling.

2010 MSC: 34G20.

1 Introduction

We consider finite undirected nontrivial graphs without loops and multiple edges. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of a graph G, respectively. Let $|V(G)| = p$ and $|E(G)| = q$ be the number of vertices and the number of edges of G respectively. General references for graph-theoretic notions are [2, 24].

A labeling of a graph is any map that carries some set of graph elements to numbers. Kotzig and Rosa [15, 16] introduced the concept of edge-magic labeling. For more information on edge-magic and super edge-magic labelings, please see [10].

Hartsfield and Ringel [11] introduced the concept of an antimagic labeling and they defined an antimagic labeling of a (p, q) graph G as a bijection f from $E(G)$ to the set $\{1, 2, \ldots, q\}$ such that the sums of label of the edges incident with each vertex $v \in V(G)$ are distinct. (a, d)-edge-antimagic total labeling was introduced by Simanjuntak, Bertault and Miller in [22]. This labeling is the extension of the notions of edge-magic labeling, see [15, 16].

For a graph $G = (V, E)$, a bijection g from $V(G) \cup E(G)$ into $\{1, 2, \ldots, |V(G)| + |E(G)|\}$ is called a (a, d)-edge-antimagic total labeling of G if the edge-weights $w(xy) = g(x) + g(y) + g(xy), xy \in E(G)$, form an arithmetic progression starting from a and having a common difference d. The $(a, 0)$-edge-antimagic total labelings are usually called edge-magic in the literature (see [8, 9, 15, 16]). An (a, d)-edge antimagic total labeling is called super if the smallest possible labels appear on the vertices.

All cycles and paths have a (a, d)-edge antimagic total labeling for some values of a and d, see [22]. In [1], Baca et al. proved the (a, d)-edge-antimagic properties of certain classes of graphs. Ivanco and Luckanicova [13] described some constructions of super edge-magic total (super $(a, 0)$-edge-antimagic total) labelings for

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disconnected graphs, namely, $nC_k \cup mP_k$ and $K_{1,m} \cup K_{1,n}$. Super (a,d)-edge-antimagic labelings for $P_n \cup P_{n+1}, nP_2 \cup P_n$ and $nP_2 \cup P_{n+2}$ have been described by Sudarsana et al. in [23].

In [7], Dafik et al. proved super edge-antimagicness of a disjoint unionof m copies of C_n. For most recent research in the subject, refer to [3, 14, 17, 19, 20, 21].

We look at a computer network as a connected undirected graph. A network designer may want to know which edges in the network are most important. If these edges are removed from the network, there will be a great decrease in its performance. Such edges are called the most vital edges in a network [5, 6, 12]. However, they are only concerned with the effect of the maximum flow or the shortest path in the network. We can consider the effect of a minimum spanning tree in the network. Suppose that $G = (V, E)$ is a weighted graph with a weight $w(e)$ assigned to every edge e in G. In the weighted graph G, the weight of a spanning tree $T, w(T)$ is defined to be $\sum w(e)$ for all $e \in E(T)$. A spanning tree T in G is called a minimum spanning tree if $w(T) \leq w(T')$ for all spanning trees T' in G. Let $g(G)$ denote the weight of a minimum spanning tree of G if G is connected; otherwise, $g(G) = \infty$. An edge e is called a most vital edge (MVE) in G if $g(G-e) \geq g(G-e')$ for every edge e' of G. We have a question : Is there any possibility to label the vertices and edges of a network G in such a way that every spanning tree of G is minimum and every edge is a most vital edge in G? The answer is 'yes'.

To solve this problem Marimuthu and Balakrishnan [18] introduced an edge magic graceful labeling of a graph.

They presented some properties of super edge magic graceful graphs and proved some classes of graphs are super edge magic graceful.

A (p,q) graph G is called edge magic graceful if there exists a bijection $g : V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ such that $|g(x) + g(y) - g(xy)| = k$, a constant for any edge xy of G. G is said to be super edge magic graceful if $g(V(G)) = \{1, 2, \ldots, p\}$.

An (a,d)-edge-antimagic graceful labeling is defined as a bijective mapping from $V(G) \cup E(G)$ into the set $\{1, 2, 3, \ldots, p+q\}$ so that the set of edge-weights of all edges in G is equal to $\{a, a+d, a+2d, \ldots, a+(q-1)d\}$, for two integers $a \geq 0$ and $d \geq 0$.

An (a,d)-edge-antimagic graceful labeling g is called super (a,d)-edge-antimagic graceful if $g(V(G)) = \{1, 2, \ldots, p\}$ and $g(E(G)) = \{p+1, p+2, \ldots, p+q\}$. A graph G is called (a,d)-edge-antimagic graceful or super (a,d)-edge-antimagic graceful if there exists an (a,d)-edge-antimagic graceful or a super (a,d)-edge-antimagic graceful labeling of G.

Note that the notion of super (a,d)-edge-antimagic graceful graphs is a generalization of the article 'G. Marimuthu and M. Balakrishnan, Super edge magic graceful graphs, Inf.Sci.,287(2014)140–151", since super $(a,0)$-edge-antimagic graceful graph is a super edge magic graceful graph.

In this paper, we study super (a,d)-edge-antimagic graceful properties of certain classes of graphs, including complete graphs and complete bipartite graphs.

2 Complete graphs

Theorem 2.1. *If the complete graph $K_n, n \geq 3$, is super (a,d)-edge-antimagic graceful, then $d \leq 1$.*

Proof. Assume that a one-to-one mapping $f : V(K_n) \cup E(K_n) \to \{1, 2, \ldots, |V(K_n)| + |E(K_n)|\}$ is a super (a,d)-edge-antimagic graceful labeling of complete graph K_n, where the set of edge-weights of all edges in K_n is equal to $\{a, a+d, \ldots, a+(|E(K_n)|-1)d\}$.

The maximum edge-weight $a + (|E(K_n)| - 1)d$ is no more than $\left|1 + (n-1) - \left(\frac{n^2+n}{2} - 1\right)\right|$.

Thus, $a + (|E(K_n)| - 1)d \leq \frac{n^2-n-2}{2}$.

$$a + \left(\frac{n^2-n-2}{2}\right) d \leq \frac{n^2-n-2}{2} \tag{2.1}$$

The minimum edge-weight is $|1+n-(n+1)|=0$.

Therefore,

$$a=0 \tag{2.2}$$

From (1) and (2) we get $0+d\left(\frac{n^2-n-2}{2}\right)\leq\frac{n^2-n-2}{2}$. Hence $d\leq 1$. □

Theorem 2.2. *Every complete graph $K_n, n\geq 3$ is super $(a,1)$-edge-antimagic graceful.*

Proof. For $n\geq 3$, let K_n be the complete graph with $V(K_n)=\{x_i : 1\leq i\leq n\}$ and $E(K_n)=\bigcup_{i=1}^{n-1}\{x_ix_{i+j} : 1\leq j\leq n-i\}$. Construct the one-to-one mapping $f: V(K_n)\cup E(K_n)\rightarrow\left\{1,2,\ldots,\frac{n^2}{2}+\frac{n}{2}\right\}$ as follows:

If $1\leq i\leq n$, then $f(x_i)=i$. If $1\leq j\leq n-1$ and $1\leq i\leq n-j$, then $f(x_ix_{i+j})=nj+i+\sum_{k=1}^{j}(1-k)$. It is a routine procedure to verify that the set of edge-weights consists of the consecutive integers $\left\{0,1,2,\ldots,\frac{n(n-1)}{2}-1\right\}$ which implies that f is a super $(0,1)$-edge-antimagic graceful labeling of K_n. □

An example to illustrate Theorem 2.2 is given in Fig. 1

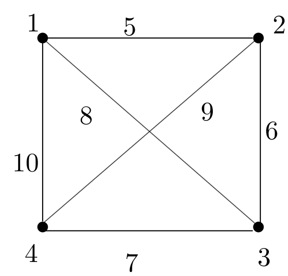

Fig. 1 A $(0,1)$-super edge-antimagic graceful completegraph.

3 Complete bipartite graphs

Let $K_{n,n}$ be the complete bipartite graph with $V(K_{n,n})=\{x_i : 1\leq i\leq n\}\cup\{y_j : 1\leq j\leq n\}$ and $E(K_{n,n})=\{x_iy_j : 1\leq i\leq n \text{ and } 1\leq j\leq n\}$.

Our first result in this section provides an upper bound for the parameter d for a super (a,d)-edge-antimagic graceful labeling of the complete bipartite graph $K_{n,n}$.

Theorem 3.1. *If a complete bipartite graph $K_{n,n}$ $n\geq 2$, is super (a,d)-edge-antimagic graceful, then $d=1$.*

Proof. Let $K_{n,n}, n\geq 2$ be a super (a,d)-edge-antimagic graceful graph with a super (a,d)-edge-antimagic graceful lableing $g: V(K_{n,n})\cup E(K_{n,n})\rightarrow\{1,2,\ldots,2n+n^2\}$ and $W=\{w(xy) : xy\in E(K_{n,n})\}=\{a,a+d,a+2d,\ldots,a+(n^2-1)d\}$ be the set of edge-weights.

The sum of all vertex labels and edge labels used to calculate the edge-weight is equal to

$$\left|n\sum_{i=1}^{n}g(x_i)+n\sum_{j=1}^{n}g(y_j)-\sum_{i=1}^{n}\sum_{j=1}^{n}g(x_iy_j)\right|=\frac{n^4-n^2}{2} \tag{3.3}$$

The sum of edge-weights in the set W is

$$\sum_{xy\in E(K_{n,n})}w(xy)=\frac{n^2}{2}(2a+d(n^2-1)) \tag{3.4}$$

The minimum edge-weight $a=|1+2n-(2n+1)|=0$. Therefore $a=0$.

Combining (3) and (4) we get, $\frac{n^4-n^2}{2} = \frac{n^2}{2}(2a + d(n^2 - 1))$.

Hence $d = 1$ for $n \geq 2$. □

Theorem 3.2. *Every complete bipartite graph $K_{n,n}, n \geq 2$ is super $(a, 1)$-edge-antimagic graceful.*

Proof. Define the bijective function $g : V(K_{n,n}) \cup E(K_{n,n}) \rightarrow \{1, 2, \ldots, |V(K_{n,n})| + |E(K_{n,n})|\}$ of $K_{n,n}$ in the following way:

$$\begin{aligned}
g(x_i) &= i \text{ for } 1 \leq i \leq n \\
g(y_j) &= n + j \text{ for } 1 \leq j \leq n \\
g(x_i y_j) &= (j - i + 2)n + i - 1 + \sum_{k=0}^{j-i}(1 - k) \text{ for } 1 \leq i \leq n \text{ and } i \leq j \leq n \\
g(x_i y_j) &= \frac{n^2 + n}{2} + (i - j + 1)n + j - 1 + \sum_{k=0}^{i-j}(1 - k) \text{ for } 1 \leq j \leq n - 1 \\
&\qquad \text{and } j + 1 \leq i \leq n.
\end{aligned}$$

Let $A = (a_{ij})$ be a square matrix, where $a_{ij} = g(x_i) + g(y_j), 1 \leq i \leq n$ and $1 \leq j \leq n$.

The matrix A is formed from the edge-weights of $K_{n,n}$ under the vertex labeling:

$$A = \begin{bmatrix}
n+2 & n+3 & n+4 & n+5 & \ldots & 2n & 2n+1 \\
n+3 & n+4 & n+5 & n+6 & \ldots & 2n+1 & 2n+2 \\
n+4 & n+5 & n+6 & n+7 & \ldots & 2n+2 & 2n+3 \\
n+5 & n+6 & n+7 & n+8 & \ldots & 2n+3 & 2n+4 \\
\vdots & & & & & & \\
2n & 2n+1 & 2n+2 & 2n+3 & \ldots & 3n-2 & 3n-1 \\
2n+1 & 2n+2 & 2n+3 & 2n+4 & \ldots & 3n-1 & 3n
\end{bmatrix}$$

It is not difficult to see that the labels of the edges $x_i y_j$ form the square matrix $B = (b_{ij})$, where $b_{ij} = g(x_i y_j)$, for $1 \leq i \leq n, 1 \leq j \leq n$ and $t = \frac{n^2+5n}{2}, r = n^2 + 2n$:

$$B = \begin{bmatrix}
2n+1 & 3n+1 & 4n & 5n-2 & \ldots & t-2 & t \\
\frac{n^2+5n}{2}+1 & 2n+2 & 3n+2 & 4n+1 & \ldots & t-4 & t-1 \\
\frac{n^2+7n}{2} & \frac{n^2+5n}{2}+2 & 2n+3 & 3n+3 & \ldots & t-7 & t-3 \\
\frac{n^2+9n}{2}-2 & \frac{n^2+7n}{2}+1 & \frac{n^2+5n}{2}+3 & 2n+4 & \ldots & t-11 & t-6 \\
\vdots & & & & & & \\
r-2 & r-4 & r-7 & r-11 & \ldots & 3n-1 & 4n-1 \\
r & r-1 & r-3 & r-6 & \ldots & n+t-1 & 3n
\end{bmatrix}$$

The vertex labeling and the edge labeling of $K_{n,n}$ combine to give a total labeling where the edge-weights of edges $x_i y_j, 1 \leq i \leq n$ and $1 \leq j \leq n$ are given by the square matrix $C = (c_{ij})$ which is $|A - B|$.

We are setting $p = \frac{n^2+n}{2}$ and $q = n^2$.

$$C = \begin{bmatrix}
n-1 & 2n-2 & 3n-4 & 4n-7 & \ldots & p-2 & p-1 \\
\frac{n^2+3n-4}{2} & n-2 & 2n-3 & 3n-5 & \ldots & p-5 & p-3 \\
\frac{n^2+5n-8}{2} & \frac{n^2+3n-6}{2} & n-3 & 2n-4 & \ldots & p-9 & p-6 \\
\frac{n^2+7n-14}{2} & \frac{n^2+5n-10}{2} & \frac{n^2+3n-8}{2} & n-4 & \ldots & p-14 & p-10 \\
 & & & \vdots & & & \\
q-2 & q-5 & q-9 & q-14 & \ldots & 1 & n \\
q-1 & q-3 & q-6 & q-10 & \ldots & \frac{n^2+n}{2} & 0
\end{bmatrix}$$

We can see that the matrix C is formed from consecutive integers $0, 1, 2, \ldots, n^2 - 1$. This implies that the labeling $g : V(K_{n,n}) \cup E(K_{n,n}) \to \{1, 2, \ldots, n^2 + 2n\}$ is super $(0, 1)$-edge-antimagic graceful.

□

Figure 2 illustrates the proof of the above theorem.

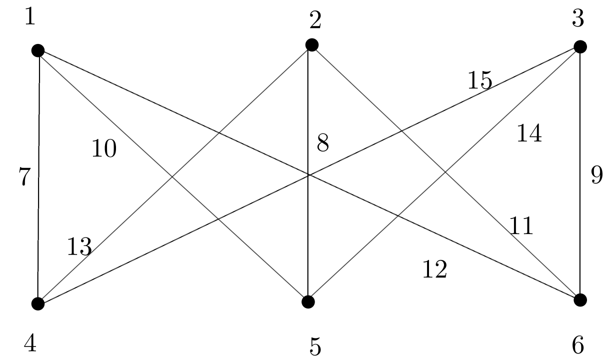

Fig. 2 A (0,1)- super edge-antimagic graceful completebipartite graph.

4 Conclusion

In the foregoing sections we studied super (a, d)-edge-antimagic graceful labeling for complete graphs and complete bipartite graphs. We have shown a bound for the feasible values of the parameter d and observed that for every super (a, d)-edge-antimagic graceful graph,$d < 2$.There are many research avenues on super (a, d)-edge-antimagic gracefullness of graphs.

If a graph G is super (a, d)-edge-antimagic graceful,is the disjoint union of multiple copies of the graph G super (a, d)-edge-antimagic graceful as well?An example of super (a, d)-edge-antimagic graceful disconnected graph is given in Figure 3.

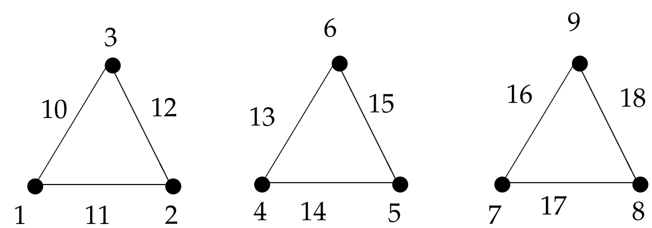

Figure 3. A super edge-antimagic gracefulness of disconnected graph.

To find the solution for the above question, We propose the following open problem.

Open Problem 4.1. *Discuss the super (a, d)-edge-antimagic gracefulness of disconnected graphs.*

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