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On Quasi Weak Commutative Near-rings-II

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Abstract

A right near-ring N is called weak Commutative,(Definition 9.4 Pilz [9]) if xyz = xzy for every x,y,z ε N. A right near-ring N is called pseudo commutative (Definition 2.1, S.Uma and others [10]) if xyz = zyx for all x,y,z ε N. A right near-ring N is called quasi weak commutative near-ring if xyz = yxz for every x,y,z ε N [4]. In [4], we have obtained some interesting results of quasi-weak commutative near-rings. In this paper we obtain some more results of quasi weak commutative near-rings.

Keywords: Quasi-weak commutative near-ring, Boolean-like near-ring.

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1 Introduction

Through out this paper, N denotes a right near-ring (N,+,.) with atleast two elements.For any non-empty subset A of N,we denote A - {0} = A*.The following definitions and results are well known.

Definition:1.1

An element a ε N is said to be
1.Idempotent if a^2 = a.
2.Nilpotent, if there exists a positive integer k such that a^k = 0.

Result: 1.2 (Theorem 1.62 Pilz [9])

Each near-ring N is isomorphic to a subdirect product of subdirectly irreducible near-rings.

Definition: 1.3

A near-ring N is said to be zero symmetric if ab = 0 implies ba = 0,where a,b ε N.

Result: 1.4

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If N is zero symmetric, then
Every left ideal A of N is an N-subgroup of N.
Every ideal I of N satisfies the condition NIN $\subseteq$ I. (i.e) every ideal is an N-subgroup. N* I* N* $\subseteq$ I*.

Result: 1.5

Let N be a near-ring. Then the following are true.
If A is an ideal of N and B is any subset of N,then (A:B) = $\{n\varepsilon$ N such that nB $\subseteq A\}$ is always a left ideal.
If A is an ideal of N and B is an N-subgroup,then (A : B) is an ideal.
In particular if A and B are ideals of a zero-symmetric near-ring, then
(A : B) is an ideal.

Result: 1.6

1. Let N be a regular near-ring, a ε N and a = axa,then ax,xa are idempotents and so the set of idempotent elements of N is non-empty.
2. axN = aN and Nxa = Na.
3. N is S and S'near-rings.

Result: 1.7 (Lemma 4 Dheena [1])

Let N be a zero-symmetric reduced near-ring. For any a,b ε N and for any idempotent element e ε N, abe = aeb.

Result: 1.8 (Gratzer [6] and Fain [3])

A near-ring N is sub-directly irreducible if and only if the intersection of all non-zero ideals of N is not zero.

Result: 1.9 (Gratzer [6])

Each simple near-ring is sub directly irreducible.

Result: 1.10 (Pilz [9])

A non-zero symmetric near-ring N has IFP if and only if (O : S) is an ideal for any subset S of N.

Result: 1.11 (Oswald [8])

An N-subgroup A of N is essential if A$\cap$ B = $\{0\}$,where B is any N subgroup of N,implies B = $\{0\}$.

Definition: 1.12

A near-ring N is said to be reduced if N has no non-zero nilpotent elements.

Definition: 1.13

A near-ring N is said to be an integral near-ring,if N has no non-zero divisors.

Lemma: 1.14

Let N be a near-ring.If for all a ε N,$a^2 = 0 \Rightarrow a = 0$,then N has no non-zero nilpotent elements.

Definition: 1.15

Let N be a near-ring. N is said to satisfy intersection of factors property (IFP) if ab = o anb = 0 for all n ε N, where a,b ε N.

Definition: 1.16

1. An ideal I of N is called a prime ideal if for all ideals A,B of N, AB is subset of I ⇒ A is subset of I or B is subset of I.
2. I is called a semi-prime ideal if for all ideals A of N, A^2 is subset of I implies A is subset of I.
3. I is called a completely semi-prime-ideal,if for any x ε N, x2 ε I ⇒ x ε I.
4. A completely prime ideal,if for any x,y ε N, xy ε I ⇒ x ε I or y ε I.
5. N is said to have strong IFP,if for all ideals I of N, ab ε I implies anb ε I for all n ε N.

Result: 1.17 (Proposition 2.4[10])

Let N be a Pseudo commutative near-ring. Then every idempotent element is central.

Result: 1.18[4]

Let N be a regular quasi weak commutative near-ring. Then
1. A = $\sqrt{A}$, for every N sub-group A of N
2. N is reduced
3. N has (*IFP)

Result: 1.19[4]

Let N be a regular quasi weak commutative near-ring. Then every N sub group is an ideal N = Na = Na^2 = aN = aNa for all a ε N

Result: 1.20[4]

Let N be a quasi weak commutative near-ring. For every ideal I of N, (I:S) is an ideal of N where S is any subset of N.

Result: 1.21[4]

Every quasi weak commutative near-ring N is isomorphic to a sub-direct product of Sub-directly irreducible quasi weak commutative near-rings.

2. Main Results:

Lemma: 2.1

Let N be a regular quasi weak commutative near-ring.
Then
(i) P∩Q = PQ for any two N-subgroups P,Q of N.
(ii) P = P^2 for every N-sub group(ideal) P of N.
(iii) If P is a proper N-subgroup of N,then each element of P is a zero divisor.
(iv) Na Nb = Na ∩ Nb = Nab for all a,b ε N.
(v) Every N-subgroup of N is essential if N is integral.

Proof:

(i) Let P and Q be two N-subgroups of N.

Then by Result1.19[4] they are ideals.

Hence PQ $\subset$ P and PQ $\subseteq$ Q.So PQ $\subseteq$ P$\cap$Q.

Let a ε P $\cap$ Q.Since N is regular,there exists b ε N such that

a = aba = (ab) a ε (PN)Q $\subseteq$ PQ.

Hence P$\cap$Q = PQ.This completes (i).

(ii) Taking Q = P in (i) we get P = P^2.

(iii) Let P be a proper N-subgroup of N.

Let 0 $\neq$ a ε P.Now by(ii) Na = $(Na)^2$ = NaNa.

Therefore for every n ε N,there exists x,y ε N such that na = xaya.

Hence (n-xay)a = 0.If a is not a zero divisor,then n-xay = 0.

(i.e) n = xay ε NPN $\subseteq$P.

Hence N = P, contradicting P is a proper ideal of N.So a is a zero divisor of N. This proves (iii).

(iv) Since Na and Nb are N-subgroups,

Na $\cap$ Nb = Na Nb. (by(i))

Since Na $\subseteq$ N, Na $\cap$ N = Na = Na$\cap$ Na =Na Na

$\subseteq$ Na N = Na N.

and Na is an ideal implies Na N = (Na)N $\subseteq$ Na

= Na $\cap$ N.

Therefore Na = Na $\cap$ N = Na N.

This implies that Nab = (Na)b = (Na N)b = Na Nb = Na $\cap$ Nb.

This proves (iv).

(v) Let P be a non-zero N-subgroup of N.

Suppose there exists an N-subgroup Q of N such that P$\cap$Q = $\{0\}$.

Then by (i) PQ = $\{0\}$ and since N is an integral near-ring Q = $\{0\}$.

This proves (v).

Theorem:2.2

Let N be a regular quasi weak commutative near-ring and P be a proper N-subgroup of N.Then the following are equivalent

(i) P is a prime ideal.

(ii) P is a completely prime ideal.

(iii) P is a primary ideal.

(iv) P is a maximal ideal.

Proof:

(i)$\Rightarrow$ (ii)

Let P be a proper N-subgroup of N.

Assume P is prime.Let ab ε P.

By Lemma 2.1(iv)

Na Nb = Nab $\subseteq$ NP $\subseteq$ P.

Also by Result1.19[4],Na and Nb are ideals of N.

Since P is prime, Na Nb $\subseteq$ P implies Na $\subseteq$ P (or) Nb $\subseteq$ P.

Since N is regular,there exists x,y ε N such that a = axa and b = byb.

If $Na \subseteq P$,then $a = axa \, \varepsilon \, Na \subseteq P$ or if $Nb \subseteq P$,then $b = byb \, \varepsilon \, Nb \subseteq P$.

(i.e) $a \varepsilon P$ or $b \varepsilon P$ and hence P is completely prime.

(ii) $\Rightarrow$ (i) is obvious.

(ii)$\Rightarrow$ (iii)

Let a,bε N.By Lemma 2.1(iv) $Nab = Na \cap Nb$.

Since $Na \cap Nb = Nb \cap Na$, $Nab = Nba$ for all a,b ε N.

Hence for all a,b,c ε N.

Nabc = Nacb = Nbca = Nbac = Ncab = Ncba.

Suppose abc ε P and ab $\notin$ P,by (ii) cεP.

Again suppose abc ε P and ac $\notin$ P.

Since N is regular,acb ε Nacb $\subseteq$ NP $\subseteq$ P.

Thus acb = (ac)b εP implies bεP (by(ii)).

Continuing in the sameway, we can easily prove that if abcεP and if the product of any two of a,b,c doesnot belong to P,then the third belongs to P:

This proves (iii).

(iii)$\Rightarrow$ (i)

Let abεP and a $\notin$ P.

Since N is regular a = axa for some xε N.

We shall first prove that xa $\notin$ P.

Suppose xa ε P, then a = axa = a(xa) ε NP $\subseteq$ P,which is a contradiction.

Therefore xa $\notin$ P.

Also x(ab) ε NP $\subseteq$P.Thus xab ε P and xa $\notin$ P.

As P is a primary ideal of N,bk εP for some integer k.Now bk εP

implies b$\varepsilon\sqrt{P}$ P.But by Result1.18[4] $\sqrt{P}$ = P.So bε P.

This proves (ii).

(i) $\Rightarrow$ (iv)

Let J be an ideal of N such that $P \subseteq J \subseteq N$.

Suppose P = J,there is nothing to prove.

So,assume $P \subset J$.We shall prove that J = N.

Let a ε J\ P.Since N is regular there exists x ε N such that a = axa.

Then a = (xa)a = xa^2 (quasi weak commutative).

So, for all n ε N, na = nxa^2 and this implies (n - nxa) a = 0.

Since N has $I \subset P$,we get n - nxa) ya = 0 for all y ε N.

Consequently, N(n-nxa) Na = N0 =$\{0\}$.

If b = (n-nxa) then Na Nb = Nab = $\{0\} \subseteq P$.

Since P is a prime ideal and Na and Nb are ideals in N, $Na \subseteq P$ or $Nb \subset P$.

If $Na \subseteq P$, then a = axa ε P which is a contradiction.

Hence $Nb \subseteq P \subseteq J$.

Since N is regular,there exists y ε N such that b = byb, (i.e) b = (by)b ε Nb$\subseteq$ J.

(i.e) b = n-nxa ε J.Since a ε J, nxa ε nJ $\subseteq$ J. (By Lemma 1.4)

Therefore nε J.Hence J = N.So P is maximal.

(v) $\Rightarrow$ (i) is obvious.

This completes the proof of the theorem.

Theorem:2.3

Any quasi-weak commutative near-ring N with left identity is commutative.

Proof:

Let a,b ε N and e ε N be the identity.

Then ab = abe = bae (quasi weak commutative).

= ba

Hence N is commutative.

Theorem : 2.4

Let N be a subdirectly irreducible quasi weak commutative near-ring.

Then either N is simple with each non-zero idempotent element is an identity or the intersection of the non-zero ideals of N has no non-zero idempotents.

Proof:

Let N be a subdirectly irreducible quasi weak commutative near-ring.

Suppose that N is simple.

Let e ε N be a non-zero idempotent element.

Then by Result1.8[4] N has IFP.By Theorem1.20 [4], (0:e) is an ideal.

Since e $\notin$ (0:e) and N is simple, we get (0:e) = $\{0\}$.

Hence (ene - en)e = ene^2 - ene = ene - ene = 0 for all nε N.

This implies (ene - en) ε (0: e) = $\{0\}$.

Hence ene - en = 0.

(i.e) ene = en $\cdots\cdots$ (1)

Also since N is quasi weak commutative,

ene = nee = ne^2 = ne $\cdots\cdots$ (2)

(1) and (2) gives ne = en $\cdots\cdots$ (3)

Also (ne - n)e = ne^2 - ne = ne - ne = 0 for all n ε N.

This implies ne - n = 0 $\cdots\cdots$ (4)

(3) and (4) gives

ne = en = n. Hence e is an identity of N.

Suppose N is not simple.

Let I be the intersection of non-zero ideals of N.Since N is subdirectly irreducible, we have I $\neq \{0\}$.

Suppose that I contains a non-zero idempotent e.

We claim that e is a right identity.

If not,there exists nεN such that ne $\neq$ n.

Hence ne - n $\neq$ 0.Since (ne - n)e = 0.

We have ne - n ε (0:e) and hence (0:e) is a non-zero ideal of N.

Therefore I $\subseteq$ (0:e).Hence eεI $\subseteq$ (0:e)

(i.e) e ε (0:e).This contradiction leads to conclude that e is a right identity of N. Hence for all nεN, n = ne ε NI $\subseteq$ I.

This implies that I = N,again a contradiction.Hence the intersection of the non-zero ideals of N has no non-zero idempotents.

This proves the theorem.

Theorem:2.5

Let N be a regular quasi weak commutative near-ring.

Then the following are equivalent

(i) N is subdirectly irreducible.

(ii) Non-zero idempotents of N are not zero divisors.

(iii) N is simple.

Proof:

(i) $\Rightarrow$ (ii)

Let J be the set of all non-zero idempotents in N which are zero divisor too.We shall prove that J is empty.If J is not empty, let I = $\cap \{(0 : e)/e\varepsilon J\}$.

Since N is sub-directly irreducible, I $\neq$ 0 by Result1.8([6],[3])

Let 0 $\neq$ a εI.

Since N is regular,there exists an element bεN such that a = aba $\cdots\cdots$ (1)

Also ab,ba are idempotents.Since 0 $\neq$ a εI, ae = 0 for all e εJ $\cdots\cdots$ (2)

Then (ae)b = 0.

Since N is zero symmetric b(ae) = 0.

(i.e) (ba)e = 0.Hence ba is a zero divisor and so ba εJ.

So by (2) a(ba) = 0.

This is a contradiction as a $\neq$ 0.Hence J is empty.

(ii) $\Rightarrow$ (iii)

Let I be a non-zero ideal of N and 0$\neq$ x εI.

Since N is regular,there exists yεN such that x = xyx $\cdots\cdots$ (3)

Also yx is an idempotent element of N.

Therefore for every nεN, nx = nxyx.

(i.e) (n-nxy)x = 0.Since N has IFP, (n-nyx)yx = 0.By (ii) n-nxy = 0

(i.e) for every nεN, n = nxy ε NIN $\subset$ I.

Thus N $\subseteq$ I.This proves that N has no non-trivial ideal of N.

So N is Simple.

(iv) $\Rightarrow$ (i)

This follows from the Result 1.9.

Corollary:2.6

Let N be a regular quasi weak commutative near-ring.Then N is subdirectly irreducible if and only if N is a field.

Proof: By theorem 2.4 and 2.5 every non-zero idempotent is an identity.

Since N is regular,

a = aba for some b ε N $\cdots\cdots$ (1)

a = (ba)a

That is inverse exists for every a ε N.

Hence N is a field.The converse is obvious.

Theorem:2.7

Let N be a regular quasi weak commutative near-ring.Then N is isomorphic to a subdirect product of fields.

Proof:

By Result1.21[4] N is isomorphic to a subdirect product of subdirectly irreducible quasi weak commutative near-rings N_k's, each N_k is regular and quasi weak commutative. Then the proof follows from the above corollary.

Corollary:2.8

Let N be a regular quasi weak commutative near-ring.Then N has no non-zero zero divisors if and only if N is a field.

Proof:

Follows from the theorem.

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