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Hermite-Hadamard Inequalities for L(j)-convex Functions and S(j)-convex Functions

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Abstract

In this article, Hermite-Hadamard Inequalities for L(j)-convex functions are analyzed. S(j)-convex functions which is founded upon $\mathbb{B}^{-1}-$convexity concept, are defined and for this functions, Hermite-Hadamard Inequalities are investigated. On some special domains, concrete form of inequalities are denoted.

Keywords: Hermite-Hadamard inequalities, L(j)-convex functions, S(j)-convex functions, abstract convexity.

2010 MSC: 26B25, 26D15, 26D07, 52A40, 52A20.

1 Introduction

Integral inequalities have played an important role in the development of all branches of mathematics. Also, Hermite-Hadamard inequalities are one of the integral inequalities. Recently, Hermite-Hadamard inequalities and their applications have attracted considerable interest. Hence the Hermite-Hadamard inequalities have been studied for varied families of functions which are obtained by many authors. (e.g. [1], [5], [6], etc.)

In this paper, we examine Hermite-Hadamard Type Inequalities for L(j)-convex functions. L(j)-convex functions are founded upon the $\mathbb{B}$-convexity concept in $\mathbb{R}^n_+$ [2] (Section 3). In section 4, S(j)-convex functions which is related to $\mathbb{B}^{-1}-$convexity concept are defined. After, for this family of functions, Hermite-Hadamard Type Inequalities are analyzed (Section 5). Additionally, different examples about both cases are discussed and studied.

2 L(j)-convex Functions

The sets which are given the following forms, are discussed to define the L(j)-convex functions [2]. For all $z \in \mathbb{R}^n_{++}$

$$N_0(z) = \left\{x \in \mathbb{R}^n_{++} : 0 < x_i \leq z_i, \quad i = \overline{1,n}\right\}$$
$$N_j(z) = \left\{x \in \mathbb{R}^n_{++} : z_j \leq x_j \quad \text{and} \quad x_i z_j \leq z_i x_j, \forall i = \overline{1,n}\right\}, j = \overline{1,n}.$$

$N_0(z)$ is closed, convex and radiant set, $N_j(z)\ \left(j = \overline{1,n}\right)$ are closed, convex and co-radiant sets [4].

Using these sets, $(n+1)$ relations are defined as follows ([2]): for $x, y \in \mathbb{R}^n_{++}$

$$x \preceq_0 y \Leftrightarrow x \in N_0(y)$$
$$x \preceq_j y \Leftrightarrow y \in N_j(x), \qquad j = \overline{1,n}.$$

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$\preceq_j, j=\overline{0,n}$ are partial order relation on $\mathbb{R}^n_{++}$ (see [4]).

We can write Minkowski functions according to $N_j(y)$ $\left(y \in \mathbb{R}^n_{++},\ j=\overline{0,n}\right)$ sets and $\preceq_j$ order relations. For $y \in \mathbb{R}^n_{++}$, $N_0(y)$ is radiant set and $\preceq_0$ is coordinate-wise order relation hence Minkowski gauge is

$$\mu_{N_0(y)}(x) := \inf\{\alpha>0 : x \in \alpha N_0(y)\} = \inf\{\alpha>0 : x \preceq_0 \alpha y\}.$$

Let us show this function with $l_{0,y}$, namely

$$l_{0,y}(x) := \mu_{N_0(y)}(x), \qquad x \in \mathbb{R}^n_{++}.$$

For $j=\overline{1,n}$ and $y \in \mathbb{R}^n_{++}$, the sets $N_j(y)$ are co-radiant, thus Minkowski co-gauges are defined by

$$\upsilon_{N_j(y)}(x) := \sup\{\alpha : x \in \alpha N_j(y)\} = \sup\left\{\alpha : \alpha y \preceq_j x\right\}$$

we denote these functions with $l_{j,y}$, namely

$$l_{j,y}(x) := \upsilon_{N_j(y)}(x), \qquad x \in \mathbb{R}^n_{++}.$$

Remark 2.1. *Let $y \in \mathbb{R}^n_{++}$ and $j=\overline{1,n}$. Then the sets $N_j(y)$ coincides with the intersection of the cone*

$$V_j(y) = \left\{x \in \mathbb{R}^n_+ : \frac{x_i}{y_i} \le \frac{x_j}{y_j} \quad \left(i=\overline{1,n}\right)\right\}$$

and the half-space

$$H_j(y) = \{x \in \mathbb{R}^n : x_j \ge y_j\}.$$

Using the cone $V_j(y)$, $l_{j,y}$ can be shown another form. If $x \in V_j(y)$, then

$$l_{j,y}(x) = \sup\left\{\alpha : \alpha y \preceq_j x\right\} = \sup\{\alpha : \alpha y_j \le x_j\} = \frac{x_j}{y_j}.$$

If $x \notin V_j(y)$, then for all $\alpha>0$ the inequality $\alpha y \preceq_j x$ does not hold therefore $l_{j,y}(x)=0$. Consequently,

$$l_{j,y}(x) = \begin{cases} \frac{x_j}{y_j}, & x \in V_j(y) \\ 0, & x \notin V_j(y) \end{cases}.$$

For $j=\overline{0,n}$, let us analyze the convexity with respect to the family of functions $L(j)=\left\{l_{j,y} : y \in \mathbb{R}^n_{++}\right\}$.

Definition 2.1. *Let $j=\overline{0,n}$. A function $f:\mathbb{R}^n_{++} \to \mathbb{R}_{+\infty} = \mathbb{R}\cup\{+\infty\}$ is an IPH(j) function if f is positively homogeneous of degree one and increasing according to order relation $\preceq_j$.*

Theorem 2.1. *For all $j=\overline{0,n}$ and $y \in \mathbb{R}^n_{++}$, $l_{j,y}$ functions are IPH(j) functions.*

Proof. For $j=0$

$$\begin{aligned} l_{0,y}(\lambda x) &= \inf\{\alpha>0 : \lambda x \in \alpha N_0(y)\} = \inf\{\alpha>0 : \lambda x \preceq_0 \alpha y\} \\ &= \inf\left\{\alpha>0 : x \preceq_0 \frac{\alpha}{\lambda} y\right\} = \lambda \inf\{\alpha'>0 : x \preceq_0 \alpha' y\} = \lambda l_{0,y}(x). \end{aligned}$$

For $j=\overline{1,n}$

$$\begin{aligned} l_{j,y}(\lambda x) &= \sup\{\alpha : \lambda x \in \alpha N_j(y)\} = \sup\left\{\alpha : \alpha y \preceq_j \lambda x\right\} \\ &= \sup\left\{\alpha : \frac{\alpha}{\lambda} y \preceq_j x\right\} = \lambda \sup\left\{\alpha' : \alpha' y \preceq_j x\right\} = \lambda l_{j,y}(x). \end{aligned}$$

Namely, $l_{j,y}$ $\left(j=\overline{0,n}\right)$ are positively homogeneous of degree one.

Now, let us prove that the functions $l_{j,y}$ $\left(j=\overline{0,n}\right)$ are increasing. Let $j=0$. If $x_1 \preceq_0 x_2$, then $\{\alpha>0 : x_2 \preceq_0 \alpha y\} \subset \{\alpha>0 : x_1 \preceq_0 \alpha y\}$ and hence $l_{0,y}(x_1) \le l_{0,y}(x_2)$. For $j=\overline{1,n}$, if $x_1 \preceq_j x_2$, then $\left\{\alpha>0 : \alpha y \preceq_j x_1\right\} \subset \left\{\alpha>0 : \alpha y \preceq_j x_2\right\}$ and thus $l_{j,y}(x_1) \le l_{j,y}(x_2)$. □

Following theorem can be proved using Corollary 2.6 in [2].

Theorem 2.2. *The function $f:\mathbb{R}^n_{++} \to \mathbb{R}_{+\infty}$ is L(j)-convex function $\left(j=\overline{0,n}\right)$ if and only if f is IPH(j) function.*

Moreover, some important properties of IPH(j) functions are given, in [2].

3 Hermite-Hadamard Type Inequalities for L(j)-convex Functions

We begin with the following theorem which has an important role in Hermite-Hadamard Type Inequalities for L(j)-convex functions [2].

Theorem 3.3. *For $j = \overline{1,n}$ and $p : \mathbb{R}^n_{++} \to \mathbb{R}_{+\infty}$, the following statements are equivalent:*
(i) p is an IPH(j) function.
(ii) $p(x) \geq \lambda p(y)$ for all $\forall x, y \in \mathbb{R}^n_{++}$ and $\lambda > 0$ such that $\lambda y \preceq_j x$.
(iii) $p(x) \geq l_{j,y}(x)\, p(y)$ for all $\forall x, y \in \mathbb{R}^n_{++}$.

We can obtain Hermite-Hadamard Type Inequalities for L(j)-convex functions as a corollary of the above theorem.

Corollary 3.1. *Let $D \subset \mathbb{R}^n_{++}$, $p : D \to \mathbb{R}_{+\infty}$ be a L(j)-convex function and integrable function on D. Then, for all $y \in D$, we have*

$$p(y) \int_D l_{j,y}(x)\, dx \leq \int_D p(x)\, dx. \tag{3.1}$$

Let us investigate Hermite-Hadamard Type Inequalities via $Q(D)$ sets given in [6].

Let $D \subset \mathbb{R}^n_{++}$ be bounded and hold condition of $cl\,(intD) = D$. We denote by $Q(D)$ the sets of all $x^* \in D$ such that

$$\frac{1}{A(D)} \int_D l_{j,x^*}(x)\, dx = 1 \tag{3.2}$$

where $A(D) = \int_D dx$

Theorem 3.4. *Let p be L(j)-convex function defined on D and integrable on D. If $Q(D)$ is nonempty, then one has the inequality:*

$$\sup_{x^* \in Q(D)} p(x^*) \leq \frac{1}{A(D)} \int_D p(x)\, dx \tag{3.3}$$

Proof. If $p(x^*) = +\infty$, then by using $p(x) \geq l_{j,y}(x)\, p(y)$, it can be shown that p cannot be integrable. It conflicts integrable of p. So $p(x^*) < +\infty$. From Theorem 3.3 (iii), for all $x \in D$

$$p(x) \geq l_{j,x^*}(x)\, p(x^*).$$

Since $x^* \in Q(D)$, by (3.2)

$$\begin{aligned} p(x^*) &= p(x^*) \frac{1}{A(D)} \int_D l_{j,x^*}(x)\, dx \\ &= \frac{1}{A(D)} \int_D p(x^*)\, l_{j,x^*}(x)\, dx \leq \frac{1}{A(D)} \int_D p(x)\, dx. \end{aligned}$$

□

Remark 3.2. *As it is clear that, for each $x^* \in Q(D)$, inequality*

$$p(x^*) \leq \frac{1}{A(D)} \int_D p(x)\, dx \tag{3.4}$$

is hold. If we get $p(x) = l_{j,x^*}(x)$, (3.4) is an equality.

Let p be a L(j)-convex function defined on $D \subset \mathbb{R}^n_{++}$ and be integrable on D. For all $x, y \in D$, the inequality

$$p(x) \geq l_{j,y}(x)\, p(y)$$

is hold. Hence,

$$p(y) \leq \varphi_{j,x}(y)\, p(x) \tag{3.5}$$

where

$$\varphi_{j,x}(y) = \frac{1}{l_{j,y}(x)} = \begin{cases} \frac{y_j}{x_j}, & x \in V_j(y) \\ \infty, & x \notin V_j(y) \end{cases} = \begin{cases} \frac{y_j}{x_j}, & y \notin intV_j(x) \\ \infty, & y \in intV_j(x) \end{cases}.$$

The following theorem can be proved, using the inequality (3.5).

Theorem 3.5. *Let $D \subset \mathbb{R}^n_{++}$, $p : D \to \mathbb{R}_{+\infty}$ be an integrable, L(j)-convex function and $D \cap intV_j(y) = \emptyset$. Then, the following inequality holds:*

$$\int_D p(x)\,dx \le p(y)\int_D \varphi_{j,y}(x)\,dx \tag{3.6}$$

for all $y \in D$.

Examples:

On some special domains of $\mathbb{R}^2_{++}$, Hermite-Hadamard Type Inequalities for L(j)-convex functions have been implied with concrete form.

Firstly, for $D \subset \mathbb{R}^2_{++}$ and every $y \in D$, let us derive computation formula of the integral $\int_D l_{j,y}(x)\,dx$.

Let $D \subset \mathbb{R}^2_{++}$ and $y = (y_1, y_2) \in D$. Then, on $\mathbb{R}^2_{++}$

$$V_1(y) = \left\{x \in \mathbb{R}^2_{++} : \frac{x_2}{y_2} \le \frac{x_1}{y_1}\right\}, \quad V_2(y) = \left\{x \in \mathbb{R}^2_{++} : \frac{x_1}{y_1} \le \frac{x_2}{y_2}\right\}$$

and

$$l_{1,y}(x) = \begin{cases} \frac{x_1}{y_1}, & x \in V_1(y) \\ 0, & x \notin V_1(y) \end{cases}, \quad l_{2,y}(x) = \begin{cases} \frac{x_2}{y_2}, & x \in V_2(y) \\ 0, & x \notin V_2(y) \end{cases}.$$

Let $V_j^c(y)\,(j = 1, 2)$ be the complement of $V_j(y)\,(j = 1, 2)$. Therefore, with the above assumptions, we can separate the region D into two regions: $D_j(y) = D \bigcap V_j(y)$ and $D \setminus D_j(y) = D \bigcap V_j^c(y)$. Thus, we have

$$\begin{aligned}\int_D l_{j,y}(x)\,dx &= \int_{D_j(y)} l_{j,y}(x)\,dx + \int_{D\setminus D_j(y)} l_{j,y}(x)\,dx \\ &= \int_{D_j(y)} \frac{x_j}{y_j}dx + \int_{D\setminus D_j(y)} 0dx = \frac{1}{y_j}\int_{D_j(y)} x_j dx.\end{aligned}$$

Example 3.1. *Consider the triangle D defined as*

$$D = \left\{(x_1, x_2) \in \mathbb{R}^2_{++} : 0 < x_1 \le a, 0 < x_2 \le vx_1\right\}.$$

For $y \in D$, $D_j(y)$ would be as follows:

$$\begin{aligned}D_1(y) &= \left\{x \in D : 0 < x_1 \le a,\ 0 < x_2 \le \frac{y_2}{y_1}x_1\right\} \\ D_2(y) &= \left\{x \in D : 0 < x_1 \le a,\ \frac{y_2}{y_1}x_1 < x_2 \le vx_1\right\}.\end{aligned}$$

For $j = 1$; we deduce that:

$$\int_D l_{1,y}(x)\,dx = \frac{1}{y_1}\int_{D_1(y)} x_1 dx = \frac{1}{y_1}\int_0^a \int_0^{\frac{y_2 x_1}{y_1}} x_1 dx_2 dx_1 = \frac{y_2}{y_1^2}\frac{a^3}{3}.$$

Hence, for the given region D, the inequality (3.1) will be as follows:

$$p(y_1, y_2) \le \frac{3y_1^2}{a^3 y_2}\int_D p(x_1, x_2)\,dx_1 dx_2.$$

For $j = 2$; we have

$$\begin{aligned}\int_D l_{2,y}(x)dx &= \frac{1}{y_2}\int_{D_2(y)} x_2 dx = \frac{1}{y_2}\int_0^a \int_{\frac{y_2 x_1}{y_1}}^{vx_1} x_2 dx_2 dx_1 \\ &= \frac{1}{2y_2}\int_0^a \left[v^2 - \left(\frac{y_2}{y_1}\right)^2\right] x_1^2 dx_1 = \frac{v^2 y_1^2 - y_2^2}{2y_2 y_1^2}\frac{a^3}{3}.\end{aligned}$$

Then, for the same region D, the inequality (3.1) is as follows:

$$p(y_1, y_2) \le \frac{6y_1^2 y_2}{a^3(v^2 y_1^2 - y_2^2)}\int_D p(x_1, x_2)\,dx_1 dx_2.$$

Let's derive the set $Q(D)$ for the given triangular domain D. Since $A(D) = \frac{va^2}{2}$, $y^* \in D$ is element of $Q(D)$ if and only if, for $j = 1$;

$$\frac{2}{va^2}\frac{y_2^*}{(y_1^*)^2}\frac{a^3}{3} = 1 \Leftrightarrow y_2^* = \frac{3v\left(y_1^*\right)^2}{2a}$$

for $j = 2$;

$$\frac{2}{va^2}\frac{\left(v^2\left(y_1^*\right)^2 - \left(y_2^*\right)^2\right)a^3}{6\left(y_1^*\right)^2 y_2^*} = 1 \Leftrightarrow y_1^* = \left(\frac{a\left(y_2^*\right)^2}{av^2 - 3y_2^* v}\right)^{\frac{1}{2}}.$$

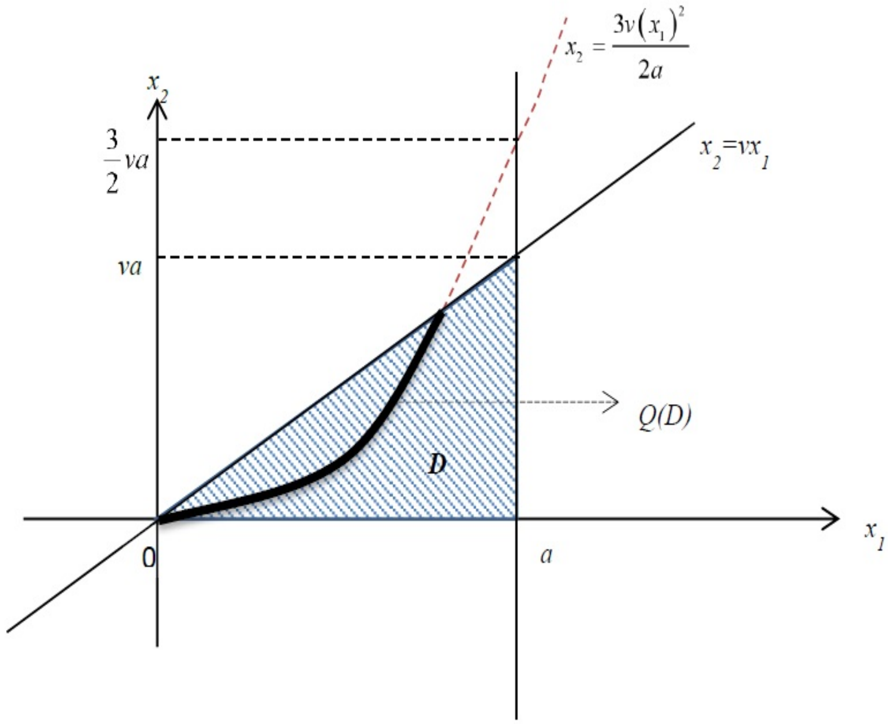

Figure 1. In case of $j = 1$, the set $Q(D)$ for triangular domain D

Example 3.2. *Let the triangular region D be as follows:*

$$D = \left\{(x_1, x_2) \in \mathbb{R}_{++}^2 : \frac{x_1}{a} + \frac{x_2}{b} \le 1\right\}.$$

In this region, for $y \in D$, the sets $D_j(y)$ $(j = 1, 2)$ are as following forms:

$$D_1(y) = \left\{x \in D : 0 < x_2 \le \frac{aby_2}{ay_2 + by_1}, \quad \frac{y_1}{y_2}x_2 \le x_1 \le a - \frac{a}{b}x_2\right\}$$
$$D_2(y) = \left\{x \in D : 0 < x_1 \le \frac{aby_1}{ay_2 + by_1}, \quad \frac{y_2}{y_1}x_1 \le x_2 \le b - \frac{b}{a}x_1\right\}.$$

If $j = 1$, then we have

$$\int_D l_{1,y}(x)\,dx = \frac{1}{y_1}\int_{D_1(y)} x_1 dx = \frac{1}{y_1}\int_0^{\frac{aby_2}{ay_2+by_1}}\int_{\frac{y_1x_2}{y_2}}^{a-\frac{ax_2}{b}} x_1 dx_1 dx_2$$
$$= \frac{1}{2y_1}\int_0^{\frac{aby_2}{ay_2+by_1}}\left[\left(a - \frac{a}{b}\right)^2 - \left(\frac{y_1}{y_2}\right)^2\right] x_2^2 dx_2 = \frac{a^3by_2\left[(ab-a)^2 y_2^2 - b^2y_1^2\right]}{6y_1\left(ay_2 + by_1\right)^3}.$$

For $j=2$; we get

$$\int_D l_{2,y}(x)\,dx = \frac{1}{y_2}\int_{D_2(y)} x_2 dx = \frac{1}{y_2}\int_0^{\frac{aby_1}{ay_2+by_1}} \int_{\frac{y_2x_1}{y_1}}^{b-\frac{bx_1}{a}} x_2 dx_2 dx_1$$
$$=\frac{1}{2y_2}\int_0^{\frac{aby_1}{ay_2+by_1}} \left[\left(b-\frac{b}{a}\right)^2 - \left(\frac{y_2}{y_1}\right)^2\right] x_1^2 dx_1 = \frac{b^3 a y_1 \left[(ba-b)^2 y_1^2 - a^2 y_2^2\right]}{6y_2\left(ay_2+by_1\right)^3}.$$

Thereby, in D, to $j=1$; the inequality is

$$p(y_1,y_2) \le \frac{6y_1\left(ay_2+by_1\right)^3}{a^3 b y_2\left[(ab-a)^2 y_2^2 - b^2 y_1^2\right]}\int_D p(x_1,x_2)\,dx_1 dx_2$$

for $j=2$; the inequality (3.1) is

$$p(y_1,y_2) \le \frac{6y_2\left(ay_2+by_1\right)^3}{b^3 a y_1\left[(ba-b)^2 y_1^2 - a^2 y_2^2\right]}\int_D p(x_1,x_2)\,dx_1 dx_2.$$

Let us construct $Q(D)$ for the given region D. Since $A(D)=\frac{ab}{2}$, if we get $j=1$, then we obtain

$$y^* \in Q(D) \Leftrightarrow \frac{a^2 y_2^*\left[(ab-a)^2 (y_2^*)^2 - b^2 (y_1^*)^2\right]}{3y_1^*\left(ay_2^*+by_1^*\right)^3} = 1$$

also, if we get $j=2$, then we have

$$y^* \in Q(D) \Leftrightarrow \frac{b^2 y_1^*\left[(ba-b)^2 (y_1^*)^2 - a^2 (y_2^*)^2\right]}{3y_2^*\left(ay_2^*+by_1^*\right)^3} = 1.$$

Example 3.3. *Now, let us get a rectangular region D which is defined as follows:*

$$D=\left\{(x_1,x_2)\in\mathbb{R}^2_{++} : x_1 \le a\,,\ x_2 \le b\right\}.$$

In this type region, it can be two cases: For $y \in D$
1) $\frac{y_2}{y_1} \le \frac{b}{a}$
2) $\frac{y_2}{y_1} \ge \frac{b}{a}$
1) *Let $\frac{y_2}{y_1} \le \frac{b}{a}$. Under this condition, the sets $D_j(y)$ will be:*

$$D_1(y) = \left\{x \in D : 0 < x_1 \le a,\quad 0 < x_2 \le \tfrac{y_2}{y_1}x_1\right\}$$
$$D_2(y) = \left\{x \in D : 0 < x_1 \le a,\quad \tfrac{y_2}{y_1}x_1 < x_2 \le b\right\}.$$

Hence, for $j=1$; we have

$$\int_D l_{1,y}(x)\,dx = \int_{D_1(y)} l_{1,y}(x)\,dx = \frac{1}{y_1}\int_{D_1(y)} x_1 dx$$
$$= \frac{1}{y_1}\int_0^a \int_0^{\frac{y_2x_1}{y_1}} x_1 dx_2 dx_1 = \frac{1}{y_1}\int_0^a \left(\frac{y_2}{y_1}\right) x_1^2 dx_1 = \frac{a^3 y_2}{3y_1^2}$$

for $j=2$; we obtain

$$\int_D l_{2,y}(x)\,dx = \frac{1}{y_2}\int_{D_2(y)} x_2 dx = \frac{1}{y_2}\int_0^a \int_{\frac{y_2x_1}{y_1}}^b x_2 dx_2 dx_1$$
$$= \frac{1}{2y_2}\int_0^a \left[b^2 - \left(\frac{y_2}{y_1}\right)^2 x_1^2\right] dx_1 = \frac{3y_1^2 b^2 a - y_2^2 a^3}{6y_1^2 y_2}.$$

By taking into account these, (3.1) becomes following inequalities: for $j = 1$;

$$p(y_1, y_2) \le \frac{3y_1^2}{a^3 y_2} \int_D p(x_1, x_2)\, dx_1 dx_2$$

for $j = 2$;

$$p(y_1, y_2) \le \frac{6y_1^2 y_2}{3y_1^2 b^2 a - y_2^2 a^3} \int_D p(x_1, x_2)\, dx_1 dx_2.$$

Let us derive the set $Q(D)$. Since $A(D) = ab$, then while $j = 1$;

$$y^* \in Q(D) \Leftrightarrow y_2^* = \frac{3b\,(y_1^*)^2}{a^2}$$

while $j = 2$;

$$y^* \in Q(D) \Leftrightarrow y_1^* = \left(\frac{(y_2^*)^2\, a^4 b}{3b^3 a^2 - 6y_2^*}\right)^{\frac{1}{2}}.$$

2) Now, let us consider the second case. Namely, let $\frac{y_2}{y_1} \ge \frac{b}{a}$. Therefore, we have that

$$D_1(y) = \left\{x \in D : \tfrac{y_1}{y_2} x_2 \le x_1 \le a, \quad 0 < x_2 \le b\right\}$$
$$D_2(y) = \left\{x \in D : 0 < x_1 \le \tfrac{y_1}{y_2} x_2, \quad 0 < x_2 \le b\right\}.$$

To $j = 1$; we have

$$\int_D l_{1,y}(x)\, dx = \frac{1}{y_1} \int_{D_1(y)} x_1 dx = \frac{1}{y_1} \int_0^b \int_{\frac{y_1 x_2}{y_2}}^a x_1 dx_1 dx_2$$
$$= \frac{1}{2y_1} \int_0^b \left(a^2 - \left(\frac{y_1}{y_2}\right)^2 x_2^2\right) dx_2 = \frac{3y_2^2 a^2 b - b^3 y_1^2}{6y_1 y_2^2}.$$

Thereby, in this case, the inequality (3.1) is

$$p(y_1, y_2) \le \frac{6y_1 y_2^2}{3y_2^2 a^2 b - b^3 y_1^2} \int_D p(x_1, x_2)\, dx_1 dx_2.$$

In case $j = 2$, we get

$$\int_D l_{2,y}(x)\, dx = \int_{D_2(y)} l_{2,y}(x)\, dx = \frac{1}{y_2} \int_{D_2(y)} x_2 dx$$
$$= \frac{1}{y_2} \int_0^b \int_0^{\frac{y_1 x_2}{y_2}} x_2 dx_1 dx_2 = \frac{1}{y_2} \int_0^b \frac{y_1}{y_2} x_2^2 dx_2 = \frac{b^3 y_1}{3y_2^2}.$$

Thus, the inequality (3.1) will be as follows:

$$p(y_1, y_2) \le \frac{3y_2^2}{b^3 y_1} \int_D p(x_1, x_2)\, dx_1 dx_2.$$

By taking into account both cases, $Q(D)$ becomes as follows: for $j = 1$;

$$Q(D) = \left\{y^* \in D : \frac{y_2^*}{y_1^*} \le \frac{b}{a}, \ \ y_2^* = \frac{3b\,(y_1^*)^2}{a^2}\right\} \bigcup$$
$$\left\{y^* \in D : \frac{y_2^*}{y_1^*} \ge \frac{b}{a}, \ \ y_2^* = \left(\frac{b^2\,(y_1^*)^2}{3a^2 - 6y_1^* a}\right)^{\frac{1}{2}}\right\}$$

for $j = 2$;

$$Q(D) = \left\{y^* \in D : \frac{y_2^*}{y_1^*} \le \frac{b}{a}, \ \ y_1^* = \left(\frac{(y_2^*)^2\, a^4 b}{3b^3 a^2 - 6y_2^*}\right)^{\frac{1}{2}}\right\} \bigcup$$
$$\left\{y^* \in D : \frac{y_2^*}{y_1^*} \ge \frac{b}{a}, \ \ y_1^* = \frac{3a\,(y_2^*)^2}{b^2}\right\}.$$

Example 3.4. *We shall now consider the case where the set D is part of the disk defined as*

$$D=\left\{(x_1,x_2)\in\mathbb{R}^2_{++}:x_1^2+x_2^2\le r^2\right\}.$$

For $y\in D$, the set $D_1(y)$ is combination of

$$D_1^*(y)=\left\{x\in D:0<x_1\le\frac{ry_1}{\sqrt{y_1^2+y_2^2}},0<x_2\le\frac{y_2}{y_1}x_1\right\}$$

and

$$D_1^{**}(y)=\left\{x\in D:\frac{ry_1}{\sqrt{y_1^2+y_2^2}}\le x_1\le r\,,0<x_2\le\sqrt{r^2-x_1^2}\right\}.$$

Namely, $D_1(y)=D_1^*(y)\bigcup D_1^{**}(y)$. The set $D_2(y)$ will be as follows:

$$D_2(y)=\left\{x\in D:0<x_1\le\frac{ry_1}{\sqrt{y_1^2+y_2^2}},\frac{y_2}{y_1}x_1\le x_2\le\sqrt{r^2-x_1^2}\right\}.$$

To $j=1$; we have

$$\begin{aligned}\int_D l_{1,y}(x)\,dx&=\frac{1}{y_1}\int_{D_1(y)}x_1dx=\frac{1}{y_1}\int_{D_1^*(y)}x_1dx+\frac{1}{y_1}\int_{D_1^{**}(y)}x_1dx\\&=\frac{1}{y_1}\int_0^{\frac{ry_1}{\sqrt{y_1^2+y_2^2}}}\int_0^{\frac{y_2}{y_1}x_1}x_1dx_2dx_1+\frac{1}{y_1}\int_{\frac{ry_1}{\sqrt{y_1^2+y_2^2}}}^{r}\int_0^{\sqrt{r^2-x_1^2}}x_1dx_2dx_1\\&=\frac{1}{y_1}\int_0^{\frac{ry_1}{\sqrt{y_1^2+y_2^2}}}\left(\frac{y_2}{y_1}x_1^2\right)dx_1+\frac{1}{y_1}\int_{\frac{ry_1}{\sqrt{y_1^2+y_2^2}}}^{r}x_1\sqrt{r^2-x_1^2}dx_1=\frac{r^3y_2}{3y_1\sqrt{y_1^2+y_2^2}}.\end{aligned}$$

In this case, for the given region D, the inequality (3.1) will be following form:

$$p(y_1,y_2)\le\frac{3y_1\sqrt{y_1^2+y_2^2}}{r^3y_2}\int_D p(x_1,x_2)\,dx_1dx_2.$$

To $j=2$; we obtain that

$$\begin{aligned}\int_D l_{2,y}(x)\,dx&=\frac{1}{y_2}\int_{D_2(y)}x_2dx=\frac{1}{y_2}\int_0^{\frac{ry_1}{\sqrt{y_1^2+y_2^2}}}\int_{\frac{y_2}{y_1}x_1}^{\sqrt{r^2-x_1^2}}x_2dx_2dx_1\\&=\frac{1}{2y_2}\int_0^{\frac{ry_1}{\sqrt{y_1^2+y_2^2}}}\left(r^2-\left(1+\frac{y_2^2}{y_1^2}\right)x_1^2\right)dx_1=\frac{r^3y_1}{3y_2\sqrt{y_1^2+y_2^2}}\end{aligned}$$

and by using the equality above, the inequality (3.1) will be as follows:

$$p(y_1,y_2)\le\frac{3y_2\sqrt{y_1^2+y_2^2}}{r^3y_1}\int_D p(x_1,x_2)\,dx_1dx_2.$$

Since $A(D)=\frac{\pi r^2}{4}$, let us give the conditions for becoming elements of the set $Q(D)$. For $j=1$; we have

$$y^*\in Q(D)\Leftrightarrow\frac{4r\left(y_2^*\right)^2}{3\pi\left(y_1^*\right)^2\left(\left(y_1^*\right)^2+\left(y_2^*\right)^2\right)^{\frac{1}{2}}}=1.$$

For $j=2$; we get

$$y^*\in Q(D)\Leftrightarrow\frac{4r\left(y_1^*\right)^2}{3\pi\left(y_2^*\right)^2\left(\left(y_1^*\right)^2+\left(y_2^*\right)^2\right)^{\frac{1}{2}}}=1.$$

Remark 3.3. *From Theorem 3.5, the right hand side of Hermite-Hadamard Inequalities can be also analyzed for concrete domains. But, in this case, $D\cap intV_j(y)=\emptyset$ is required because of integrability of the function $\varphi_{j,y}$ on D.*

Example 3.5. *As in the Example 3.1, we discuss the triangle*

$$D = \left\{ (x_1, x_2) \in \mathbb{R}^2_{++} : 0 < x_1 \leq a, 0 < x_2 \leq v x_1 \right\}.$$

If $j = 1$, then $D \cap intV_1(y) \neq \emptyset$ for $\forall y \in \mathbb{R}^2_{++}$. Thus, from Theorem 3.5, the right hand side of Hermite-Hadamard Inequalities for L(1)-convex functions is not obtained.

Let $j = 2$. It is obvious that $D \cap intV_2(y) = \emptyset \Leftrightarrow y_2 \geq v y_1$. From Theorem 3.5, we have

$$\int_D p(x_1, x_2)\, dx_1 dx_2 \leq p(y_1, y_2) \int_D \frac{x_2}{y_2} dx_1 dx_2.$$

Since

$$\int_D \frac{x_2}{y_2} dx_1 dx_2 = \frac{1}{y_2} \int_0^a \int_0^{v x_1} x_2 dx_2 dx_1 = \frac{v^2}{2y_2} \int_0^a x_1^2 dx_1 = \frac{a^3 v^2}{6 y_2}.$$

for all $y \in D$ which satisfy the condition $y_2 \geq v y_1$ (namely, y on the long side of the triangle) and all p that are L(2)-convex, integrable on D, the inequality

$$\int_D p(x_1, x_2)\, dx_1 dx_2 \leq \frac{v^2 a^3}{6 y_2} p(y_1, y_2)$$

is hold, or since $A(D)$ is area of triangular domain, we obtain the inequality

$$\frac{1}{A(D)} \int_D p(x_1, x_2)\, dx_1 dx_2 \leq \frac{va}{3 y_2} p(y_1, y_2).$$

4 S(j)-convex Functions

Firstly, let us recall the definition of $\mathbb{B}^{-1}-$convex set [3]:

Definition 4.2. *A subset M of $\mathbb{R}^n_{++}$ is $\mathbb{B}^{-1}-$convex if for all $x_1, x_2 \in M$ and all $t \in [1, \infty)$ one has $t x_1 \wedge x_2 \in M$.*

Here, $\wedge$ is the greatest lower bound of x_1, x_2, that is,

$$x_1 \wedge x_2 = \left(\min\{x_{1,1}, x_{2,1}\}, ..., \min\{x_{1,n}, x_{2,n}\} \right).$$

For every $z \in \mathbb{R}^n_{++}$, $\mathbb{R}^n_{++}$ can be written as the combination of $(n+1)$-parts which are given with the following forms:

$$M_0(z) = \left\{ x \in \mathbb{R}^n_{++} : z_i \leq x_i, \quad i = \overline{1,n} \right\}$$
$$M_j(z) = \left\{ x \in \mathbb{R}^n_{++} : x_j \leq z_j \quad \text{and} \quad x_j z_i \leq z_j x_i, \forall i = \overline{1,n} \right\}.$$

The sets $M_j(z)\left(j = \overline{0,n}\right)$ are closed and convex sets. The following theorem gives construction of the sets $M_j(z)\left(j = \overline{0,n}\right)$.

Theorem 4.6. *$M_0(z)$ is co-radiant, $\mathbb{B}^{-1}-$convex set and $M_j(z)\left(j = \overline{1,n}\right)$ are radiant, $\mathbb{B}^{-1}-$convex sets.*

Proof. Let us show that $M_0(z)$ is co-radiant, namely $x \in M_0(z)$, $\lambda \geq 1 \Rightarrow \lambda x \in M_0(z)$. Since $x \in M_0(z)$, then $z_i \leq x_i \left(i = \overline{1,n}\right)$. $\lambda \geq 1$, so $z_i \leq x_i \leq \lambda x_i \left(i = \overline{1,n}\right)$. Consequently, we have $\lambda x \in M_0(z)$.

Now, let us prove that $M_0(z)$ is $\mathbb{B}^{-1}-$convex. Let $x, y \in M_0(z)$, $t \in [1, \infty)$. Hence, for $\forall i = \overline{1,n}$, we have $z_i \leq x_i$ and $z_i \leq y_i$. By using these inequalities; since $z_i \leq x_i \leq t x_i$ and $z_i \leq y_i$, we obtain $z_i \leq t x_i \wedge y_i$, $i = \overline{1,n}$. We have shown that $tx \wedge y \in M_0(z)$.

And now, we have to see that $M_j(z)\left(j = \overline{1,n}\right)$ are radiant. Let $x \in M_j(z)$ and $0 < \lambda \leq 1$. Since $x \in M_j(z)$, we have $x_j \leq z_j$ and $x_j z_i \leq z_j x_i$, $i = \overline{1,n}$. $0 < \lambda \leq 1$ so that $\lambda x_j \leq x_j \leq z_j$ then $\lambda x_j \leq z_j$. Also, $\lambda > 0$, hence we can derive $\lambda x_j z_i \leq z_j \lambda x_i$, $i = \overline{1,n}$. By taking into account both cases, $\lambda x \in M_j(z)$.

Finally, let us show that $M_j(z)$ are $\mathbb{B}^{-1}-$convex. Let $x, y \in M_j(z)$, $t \in [1, \infty)$.

$$x \in M_j(z) \Leftrightarrow x_j \leq z_j \quad \text{and} \quad x_j z_i \leq z_j x_i, \ i = \overline{1,n}$$
$$y \in M_j(z) \Leftrightarrow y_j \leq z_j \quad \text{and} \quad y_j z_i \leq z_j y_i, \ i = \overline{1,n}.$$

There are two possible cases: for $t \in [1, \infty)$

I) it can be $t x_j \leq z_j$. In this case, from $y_j \leq z_j$, we obtain $t x_j \wedge y_j \leq z_j$.

II) let $tx_j > z_j$. Again, since $y_j \leq z_j$, we have $tx_j \wedge y_j \leq z_j$. Hence, we deduce that $tx_j \wedge y_j \leq z_j$.
In second part, for $z \in \mathbb{R}^n_{++}$

$$(tx_j \wedge y_j)\, z_i = tx_j z_i \wedge y_j z_i \leq tx_i z_j \wedge y_i z_j = (tx_i \wedge y_i)\, z_j .$$

Thus, we have shown that $tx \wedge y \in M_j(z)$. □

The $(n+1)$-relations according to $M_j(z)\ \left(j = \overline{0,n}\right)$ can be given by

$$\begin{aligned} &x \preccurlyeq_0 y \Leftrightarrow y \in M_0(x) \\ &x \preccurlyeq_j y \Leftrightarrow x \in M_j(y), \qquad j = \overline{1,n}. \end{aligned}$$

Let us see that $\preccurlyeq_j,\ \left(j = \overline{0,n}\right)$ are partial order relations.

Theorem 4.7. $\preccurlyeq_j,\ \left(j = \overline{0,n}\right)$ *are partial order relations.*

Proof. Let $j = 0$. $\preccurlyeq_0$ is coordinate-wise order relation, namely,

$$x \preccurlyeq_0 y \Leftrightarrow y - x \in \mathbb{R}^n_{++}.$$

So that $\preccurlyeq_0$ is a partial order relation.
Let $j = \overline{1,n}$.
Firstly, we show that $\preccurlyeq_j\ \left(j = \overline{1,n}\right)$ are reflexivity. For all $x \in \mathbb{R}^n_{++}$ and all $j = \overline{1,n}$, then $x_j \leq x_j$. Also, for all $i = \overline{1,n}$, we have $x_j x_i \leq x_j x_i$. Consequently, $x \preccurlyeq_j x$.
Let us show that $\preccurlyeq_j\ \left(j = \overline{1,n}\right)$ are antisymmetric: Let $x, z \in \mathbb{R}^n_{++}$, $x \preccurlyeq_j z$ and $z \preccurlyeq_j x$. We deduce that

$$\begin{aligned} x \preccurlyeq_j z \ &\Leftrightarrow\ x_j \leq z_j \ \text{ and } \ x_j z_i \leq z_j x_i\ , \qquad i = \overline{1,n} \\ z \preccurlyeq_j x \ &\Leftrightarrow\ z_j \leq x_j \ \text{ and } \ z_j x_i \leq x_j z_i\ , \qquad i = \overline{1,n}. \end{aligned}$$

From the first part, for $j = \overline{1,n}$, we get $x_j = z_j$.
By using this equality and the second part, for all $i = \overline{1,n}$, since

$$\begin{aligned} x_j z_i \leq z_j x_i \Rightarrow z_i \leq x_i \\ z_j x_i \leq x_j z_i \Rightarrow x_i \leq z_i \end{aligned}$$

thus, it is $x_i = z_i$.
Accordingly, we obtain $x = z$.
Now, we have to prove that $\preccurlyeq_j\ \left(j = \overline{1,n}\right)$ are transitive. Let $x, y, z \in \mathbb{R}^n_{++}$ $x \preccurlyeq_j y$ and $y \preccurlyeq_j z$. Hence, we have that

$$\begin{aligned} x \preccurlyeq_j y \ &\Leftrightarrow\ x_j \leq y_j \ \text{ and } \ x_j y_i \leq y_j x_i\ , \qquad i = \overline{1,n} \\ y \preccurlyeq_j z \ &\Leftrightarrow\ y_j \leq z_j \ \text{ and } \ y_j z_i \leq z_j y_i\ , \qquad i = \overline{1,n}. \end{aligned}$$

Since $x_j \leq y_j \leq z_j$, then we obtain

$$x_j \leq z_j. \tag{4.7}$$

Taking into account that the above inequalities hold, we have that

$$\begin{aligned} x_j y_i \leq y_j x_i \ \Rightarrow\ x_j y_i\,(y_j z_i) &\leq y_j x_i\,(y_j z_i) \leq y_j x_i\,(z_j y_i) \\ x_j z_i\,(y_i y_j) &\leq x_i z_j\,(y_j y_i) \\ x_j z_i &\leq z_j x_i. \end{aligned} \tag{4.8}$$

From (4.7) and (4.8), we have $x \preccurlyeq_j z$. The theorem is proved. □

Now, we can write Minkowski functions according to $M_j(z)\ \left(z \in \mathbb{R}^n_{++}, j = \overline{0,n}\right)$ sets and $\preccurlyeq_j$ partial order relations. For $z \in \mathbb{R}^n_{++}$, since that $M_0(z)$ is co-radiant;

$$\upsilon_{M_0(z)}(x) := \sup\{\alpha : x \in \alpha M_0(z)\} = \sup\{\alpha : \alpha z \preccurlyeq_0 x\}$$

then, we denote this function with $s_{0,z}$,

$$s_{0,z}(x) := \upsilon_{M_0(z)}(x)\ , \qquad x \in \mathbb{R}^n_{++}.$$

For $z \in \mathbb{R}^n_{++}$ and $j = \overline{1,n}$; by taking into account that $M_j(z)$ are radiant sets; Minkowski gauge of $M_j(z)$ are

$$\mu_{M_j(z)}(x) := \inf\{\alpha > 0 : x \in \alpha M_j(z)\} = \inf\{\alpha > 0 : x \preccurlyeq_j \alpha z\}.$$

Let us denote this function with the following notation

$$s_{j,z}(x) := \mu_{M_j(z)}(x), \qquad x \in \mathbb{R}^n_{++}.$$

The sets $M_j(z)$ $\left(j = \overline{1,n}, z \in \mathbb{R}^n_{++}\right)$ can be written as the intersection of the cone

$$U_j(z) = \left\{x \in \mathbb{R}^n_{++} : \frac{x_j}{z_j} \le \frac{x_i}{z_i} \;\; i = \overline{1,n}\right\}$$

and the half-space

$$H_j(z) = \{x \in \mathbb{R}^n : x_j \le z_j\}.$$

The functions $s_{j,z}$ can be denoted the following form, if we use the cone $U_j(z)$.

$$s_{j,z}(x) = \begin{cases} \frac{x_j}{z_j}, & x \in U_j(z) \\ \infty, & x \notin U_j(z). \end{cases} \tag{4.9}$$

Let us analyze convexity with respect to the family of functions $S(j) = \{s_{j,z} : z \in \mathbb{R}^n_{++}\}, j = \overline{0,n}$.

Definition 4.3. *Let $j = \overline{0,n}$. A function $f : \mathbb{R}^n_{++} \to \mathbb{R}_{+\infty}$ is an IPH[j] function if f is positively homogeneous of degree one and increasing according to order relation $\preccurlyeq_j$.*

Theorem 4.8. *$\forall j = \overline{0,n}$ and $\forall z \in \mathbb{R}^n_{++}$, $s_{j,z}$ are IPH[j] functions.*

Proof. Let us show that $s_{j,z}$ are positively homogeneous of degree one.

For $j = 0$, we have that

$$\begin{aligned} s_{0,z}(\lambda x) &= \sup\{\alpha : \lambda x \in \alpha M_0(z)\} = \sup\{\alpha : \alpha z \preccurlyeq_0 \lambda x\} \\ &= \sup\left\{\alpha : \alpha z_i \le \lambda x_i, i = \overline{1,n}\right\} = \sup\left\{\lambda\alpha' : \alpha' z_i \le x_i, i = \overline{1,n}\right\} \\ &= \lambda \sup\{\alpha' : \alpha' z \preccurlyeq_0 x\} = \lambda s_{0,z}(x). \end{aligned}$$

For $j = \overline{1,n}$, we get

$$\begin{aligned} s_{j,z}(\lambda x) &= \inf\{\alpha > 0 : \lambda x \in \alpha M_j(z)\} = \inf\{\alpha > 0 : \lambda x \preccurlyeq_j \alpha z\} \\ &= \lambda \inf\{\alpha' > 0 : x \preccurlyeq_j \alpha' z\} = \lambda s_{j,z}(x). \end{aligned}$$

Let us prove that $s_{j,z}$ are increasing according to $\preccurlyeq_j$ $\left(j = \overline{0,n}\right)$.

Let $j = 0$ and $x_1 \preccurlyeq_0 x_2$. Then, we have $\{\alpha : \alpha z \preccurlyeq_0 x_1\} \subset \{\alpha : \alpha z \preccurlyeq_0 x_2\}$. From properties of supremum, we obtain that $s_{0,z}(x_1) \le s_{0,z}(x_2)$.

Let $j = \overline{1,n}$ and $x_1 \preccurlyeq_j x_2$. Hence, we have $\{\alpha > 0 : x_2 \preccurlyeq_j \alpha z\} \subset \{\alpha > 0 : x_1 \preccurlyeq_j \alpha z\}$. Consequently, we obtain $s_{j,z}(x_1) \le s_{j,z}(x_2)$. □

Now, let us give the following theorem which can be easily proved via Corollary 2.6 in [2].

Theorem 4.9. *For $j = \overline{0,n}$, $f : \mathbb{R}^n_{++} \to \mathbb{R}_{+\infty}$ is S(j)-convex function if and only if f is IPH[j] function.*

The following theorem implies some properties of IPH[j] functions.

Theorem 4.10. *Let $j = \overline{1,n}$ and $f : \mathbb{R}^n_{++} \to \mathbb{R}_{+\infty}$ be an IPH[j] function. Then following statements are hold:*

(i) $f(x) \ge 0$ for all $x \in \mathbb{R}^n_{++}$.

(ii) If $f(x^*) = +\infty$ where $x^* \in \mathbb{R}^n_{++}$ then $f(x) = +\infty$ on the set $\left\{x \in \mathbb{R}^n_{++} : \exists\lambda > 0 \;\text{ such that }\; \lambda x^* \preccurlyeq_j x\right\}$.

(iii) If $f(x^*) = 0$ where $x^* \in \mathbb{R}^n_{++}$ then for all $x \in \left\{x \in \mathbb{R}^n_{++} : \exists\lambda > 0,\; x \preccurlyeq_j \lambda x^*\right\}$, $f(x) = 0$.

Proof. (i) Let $x \in \mathbb{R}^n_{++}$. Because $\frac{1}{2}x \preccurlyeq_j x$, we have $\frac{1}{2}f(x) = f\left(\frac{x}{2}\right) \le f(x)$. Therefore $f(x) \ge 0$.

(ii) Let $x \in \mathbb{R}^n_{++}$ be a point such that there exists $\lambda > 0$ with the property $\lambda x^* \preccurlyeq_j x$. Then $f(x) \ge f(\lambda x^*) = \lambda f(x^*) = +\infty$.

(iii) Let $x \in \mathbb{R}^n_{++}$ and let there be $\lambda > 0$ such that $x \preccurlyeq_j \lambda x^*$. Thus, we have that $0 \le f(x) \le f(\lambda x^*) = \lambda f(x^*) = 0$. □

5 Hermite-Hadamard Type Inequalities for S(j)-convex Functions

Let us prove the following theorem which has an important role in Hermite-Hadamard Type Inequalities for S(j)-convex functions.

Theorem 5.11. *For $j = \overline{1,n}$ and $p : \mathbb{R}^n_{++} \to \mathbb{R}_{+\infty}$, the following statements are equivalent:*
(i) p is an IPH[j] function.
(ii) For all $x, z \in \mathbb{R}^n_{++}$ and $\lambda > 0$ such that $x \preccurlyeq_j \lambda z$, we have $p(x) \le \lambda p(z)$.
(iii) For all $x, z \in \mathbb{R}^n_{++}$, we have $p(x) \le s_{j,z}(x)\, p(z)$.

Proof. $i \Rightarrow ii$) Since p is an IPH[j] function, for all $\lambda > 0$, we get $x \preccurlyeq_j \lambda z$. Hence $p(x) \le p(\lambda z) = \lambda p(z)$.

$ii \Rightarrow i$) The monotonicity of p follows from (ii) with $\lambda = 1$. We now show that p is positively homogeneous. Let $x = \lambda z$ with $\lambda > 0$. Then by (ii), we have $p(x) = p(\lambda z) \le \lambda p(z)$. Because $z = \lambda^{-1} x$, we conclude that $p(z) \le \lambda^{-1} p(x)$. Thus $p(\lambda z) = \lambda p(z)$.

$ii \Rightarrow iii$) If $p(z) = 0$, we have $0 \le p(x) \le s_{j,z}(x)\, p(z) = 0$ for all x. Let $p(z) > 0$ and $\lambda > 0$ be a number such that $x \preccurlyeq_j \lambda z$. Applying (ii), we conclude that $\frac{p(x)}{p(z)} \le \lambda$. It follows from the definition of $s_{j,z}$ that $s_{j,z}(x) = \inf\{\lambda > 0 : x \preccurlyeq_j \lambda z\}$, therefore $\frac{p(x)}{p(z)} \le s_{j,z}(x)$.

$iii \Rightarrow ii$) follows directly from the definition of $s_{j,z}$. □

If we use the above theorem, then we can deduce the Hermite-Hadamard Type Inequalities for S(j)-convex functions.

Corollary 5.2. *Let $p : D \to \mathbb{R}_{+\infty}, D \subset \mathbb{R}^n_{++}$ be a S(j)-convex function and integrable function on D where $D \subset U_j(z)$. Then, for all $z \in D$, the following inequality holds:*

$$\int_D p(x)\, dx \le p(z) \int_D s_{j,z}(x)\, dx. \tag{5.10}$$

Proof. It is proven from Theorem 5.11 (iii) and (4.9). □

Let's analyze the inequality (5.10) via sets $Q(D)$.

Let $D \subset \mathbb{R}^n_{++}$ be bounded and satisfy condition $cl\,(intD) = D$. $Q(D)$ consist of all point $x^* \in D$ such that

$$\frac{1}{A(D)} \int_D s_{j,x^*}(x)\, dx = 1,$$

here $A(D) = \int_D dx$.

We can give a theorem about the set $Q(D)$ and Hermite-Hadamard Type Inequalities of S(j)-convex functions.

Theorem 5.12. *Let p be a S(j)-convex function defined and integrable on D. If $Q(D) \neq \emptyset$, then one has the inequality:*

$$\frac{1}{A(D)} \int_D p(x)\, dx \le \inf_{x^* \in Q(D)} p(x^*)$$

Proof. If $p(x^*) = 0$, from $p(x) \le s_{j,x^*}(x)\, p(x^*)$ we have $p(x) = 0$. Thus, let $p(x^*) > 0$. For all $x \in D$,

$$p(x) \le s_{j,x^*}(x)\, p(x^*)$$

is hold. Because $x^* \in Q(D)$, we have

$$\begin{aligned} p(x^*) &= p(x^*) \frac{1}{A(D)} \int_D s_{j,x^*}(x)\, dx \\ &= \frac{1}{A(D)} \int_D p(x^*)\, s_{j,x^*}(x)\, dx \ge \frac{1}{A(D)} \int_D p(x)\, dx. \end{aligned}$$

□

For every $x^* \in Q(D)$, the inequality

$$\frac{1}{A(D)} \int_D p(x)\,dx \le p(x^*) \tag{5.11}$$

is hold. If we take $p(x) = s_{j,x^*}(x)$, the inequality (5.11) will be turn equality.

Let p be a S(j)-convex function defined and integrable on D which is closed, bounded and connected set. For all $x, z \in D$, we have

$$p(x) \le s_{j,z}(x)\,p(z).$$

Hence, below inequality is obtained:

$$p(x)\,\psi_{j,x}(z) \le p(z)$$

where

$$\psi_{j,x}(z) = \frac{1}{s_{j,z}(x)} = \begin{cases} \frac{z_j}{x_j}, & x \in U_j(z) \\ 0, & x \notin U_j(z) \end{cases} = \begin{cases} \frac{z_j}{x_j}, & z \notin intU_j(x) \\ 0, & z \in intU_j(x) \end{cases} \tag{5.12}$$

In this case, we can write second part of the Hermite-Hadamard Type Inequality for S(j)-convex functions.

Theorem 5.13. *Let $D \subset \mathbb{R}^n_{++}$, $p : D \to \mathbb{R}_{+\infty}$ be S(j)-convex and integrable on D. Then, for all $z \in D$, we have the inequality:*

$$p(z) \int_D \psi_{j,z}(x)\,dx \le \int_D p(x)\,dx \tag{5.13}$$

Examples:

On the same domains in previous section, Hermite-Hadamard Inequalities for S(j)-convex functions can be also considered. For example, let us discuss triangular domain in Example 3.1.

Example 5.6. *Let*

$$D = \left\{(x_1, x_2) \in \mathbb{R}^2_{++} : 0 < x_1 \le a,\ \ 0 < x_2 \le vx_1\right\}.$$

$D \subset U_j(z)$ is necessary in order to the inequality (5.10) can be written on this region.

When $j = 1$, for all $z \in \mathbb{R}^2_{++}$ it is $D \not\subset U_1(z)$. Hence, from Corollary 5.2, for S(1)-convex functions, the right part of Hermite-Hadamard Inequalities can not computed on this domain.

Let $j = 2$. This is obvious that $D \subset U_2(z) \Leftrightarrow z_2 \ge vz_1$. From (5.10), we obtain

$$\int_D p(x_1, x_2)\,dx_1dx_2 \le p(z_1, z_2) \int_D \frac{x_2}{z_2} dx_1dx_2.$$

When the right integral is calculated, for all $z \in D$ satisfying the condition $z_2 \ge vz_1$ (thus, z is on the hypotenuse of the triangle) and for all p that is S(2)-convex, integrable on D, we have

$$\frac{1}{A(D)} \int_D p(x_1, x_2)\,dx_1dx_2 \le \frac{va}{3z_2} p(z_1, z_2)$$

where $A(D)$ is area of the triangular domain.

For the same domain, if we apply the Theorem 5.13, then we can estimate the left part of Hermite-Hadamard Inequality.

Let $j = 1$. From (5.13), we have

$$p(z_1, z_2) \int_D \psi_{1,z}(x_1, x_2)\,dx_1dx_2 \le \int_D p(x_1, x_2)\,dx_1dx_2$$

and from (5.12), we obtain

$$\int_D \psi_{1,z}(x_1, x_2)\,dx_1dx_2 = \frac{a^3 z_2}{3z_1^2}.$$

Thereby, the inequality is

$$p(z_1, z_2)\,\frac{a^3 z_2}{3z_1^2} \le \int_D p(x_1, x_2)\,dx_1dx_2.$$

Let $j = 2$. The left part of the Hermite-Hadamard Inequality is

$$p(z_1, z_2) \int_D \psi_{2,z}(x_1, x_2)\,dx_1dx_2 \le \int_D p(x_1, x_2)\,dx_1dx_2.$$

Since, with a simple calculation, we obtain

$$\int_D \psi_{2,z}(x_1, x_2)\, dx_1 dx_2 = \frac{a^3 \left(z_1^2 v^2 - z_2^2\right)}{6 z_1^2 z_2}$$

and from above inequality, we have

$$p(z_1, z_2) \frac{a^3 \left(z_1^2 v^2 - z_2^2\right)}{6 z_1^2 z_2} \le \int_D p(x_1, x_2)\, dx_1 dx_2.$$

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