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Inclusion properties for certain subclasses of analytic functions defined by using the generalized Bessel functions

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Abstract

By making use of the operator B_ν^c defined by the generalized Bessel functions of the first kind, the authors introduce and investigate several new subclasses of starlike, convex, close-to-convex and quasi-convex functions. The authors establish inclusion relationships associated with the aforementioned operator. Some interesting corollaries and consequences of the main inclusion relationships are also considered.

Keywords: Analytic functions; Starlike functions; Convex functions; Close-to-convex functions; Quasi-convex functions; Generalized Bessel functions.

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1 Introduction, Definitions and Preliminaries

Let

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

be the unit disk in the complex z-plane. Also let $\mathcal{A}$ be the class of functions f of the form:

$$f(z) = z + \sum_{n=1}^{\infty} a_{n+1} z^{n+1}, \tag{1.1}$$

which are analytic in $\mathbb{U}$ and satisfy the following normalization condition:

$$f(0) = f'(0) - 1 = 0.$$

Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of all functions which are univalent in $\mathbb{U}$. We denote by $\mathcal{S}^*(\alpha)$, $\mathcal{K}(\alpha)$, $\mathcal{C}(\beta,\alpha)$ and $\mathcal{C}^*(\beta,\alpha)$ the familiar subclasses of $\mathcal{A}$ consisting of functions which are, respectively, starlike of order α in $\mathbb{U}$, convex of order α in $\mathbb{U}$, close-to-convex of order β and type α in $\mathbb{U}$ and quasi-convex of order β and type α in $\mathbb{U}$. Thus, by definition, we have (for details, see [4, 6, 7, 11])

$$\mathcal{S}^*(\alpha) := \left\{ f : f \in \mathcal{A} \quad \text{and} \quad \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (0 \leqq \alpha < 1;\ z \in \mathbb{U}) \right\},$$

$$\mathcal{K}(\alpha) := \left\{ f : f \in \mathcal{A} \quad \text{and} \quad \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (0 \leqq \alpha < 1;\ z \in \mathbb{U}) \right\},$$

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$$\mathcal{C}(\beta,\alpha) := \left\{ f : f \in \mathcal{A},\ g \in \mathcal{S}^*(\alpha) \quad \text{and} \quad \Re\left(\frac{zf'(z)}{g(z)}\right) > \beta \quad (0 \leqq \alpha, \beta < 1;\ z \in \mathbb{U}) \right\}$$

and

$$\mathcal{C}^*(\beta,\alpha) := \left\{ f : f \in \mathcal{A},\ g \in \mathcal{K}(\alpha) \quad \text{and} \quad \Re\left(\frac{(zf'(z))'}{g'(z)}\right) > \beta \quad (0 \leqq \alpha, \beta < 1;\ z \in \mathbb{U} \right\}.$$

It is easily observed from the above definitions that

$$f(z) \in \mathcal{K}(\alpha) \Longleftrightarrow zf'(z) \in \mathcal{S}^*(\alpha)$$

and

$$f(z) \in \mathcal{C}^*(\beta,\alpha) \Longleftrightarrow zf'(z) \in \mathcal{C}(\beta,\alpha).$$

For $f \in \mathcal{A}$ given by (1.1) and $g(z)$ given by $g(z) = z + \sum_{n=1}^{\infty} b_{n+1} z^{n+1}$, the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) := z + \sum_{n=1}^{\infty} a_{n+1} b_{n+1} z^{n+1} =: (g * f)(z) \qquad (z \in \mathbb{U}).$$

The generalized Bessel function of the first kind of order p is defined as a particular solution of the following second-order differential equation (see, for details, [1]):

$$z^2 w''(z) + bzw'(z) + [cz^2 - p^2 + (1-b)p]w(z) = 0 \qquad (b, c, p \in \mathbb{C}) \tag{1.2}$$

and has the familiar representation given by

$$\omega_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-c)^n}{n! \cdot \Gamma(p + n + \frac{b+1}{2})} \left(\frac{z}{2}\right)^{2n+p} \qquad (z \in \mathbb{C}). \tag{1.3}$$

The series in (1.3) permits a unified study of the Bessel, the modified Bessel and the spherical Bessel functions. The following cases are worthy of note here.

1. Taking $b = c = 1$ in (1.3), we obtain the familiar Bessel function of the first kind of order p defined by (see [1, 8, 12])
$$J_p(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \cdot \Gamma(p + n + 1)} \left(\frac{z}{2}\right)^{2n+p} \qquad (z \in \mathbb{C}). \tag{1.4}$$

2. Putting $b = 1$ and $c = -1$ in (1.3), we get the modified Bessel function of the first kind of order p defined by (see [1, 12])
$$I_p(z) = \sum_{n=0}^{\infty} \frac{1}{n! \cdot \Gamma(p + n + 1)} \left(\frac{z}{2}\right)^{2n+p} \qquad (z \in \mathbb{C}). \tag{1.5}$$

3. Letting $b = 2$ and $c = 1$ in (1.3), we have the spherical Bessel function of the first kind of order p defined by (see [1])
$$j_p(z) = \sqrt{\frac{\pi}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \cdot \Gamma(p + n + 3/2)} \left(\frac{z}{2}\right)^{2n+p} \qquad (z \in \mathbb{C}). \tag{1.6}$$

Recently, Deniz *et al.* [3] considered the function $\varphi_{p,b,c}(z)$ defined, in terms of the generalized Bessel function $\omega_{p,b,c}(z)$, by

$$\begin{aligned} \varphi_{p,b,c}(z) &= 2^p\, \Gamma\left(p + \frac{b+1}{2}\right) z^{1-p/2}\, \omega_{p,b,c}(\sqrt{z}) \\ &= z + \sum_{n=1}^{\infty} \frac{(-c)^n}{4^n \cdot (\nu)_n} \frac{z^{n+1}}{n!} \qquad \left(\nu = p + \frac{b+1}{2} \notin \mathbb{Z}_0^- := \{0, -1, -2, \cdots\}\right), \end{aligned} \tag{1.7}$$

where $(\lambda)_n$ denotes the Pochhammer symbol (or the shifted factorial) defined by

$$(\lambda)_0 = 1 \qquad \text{and} \qquad (\lambda)_n = \lambda(\lambda+1)(\lambda+2)\cdots(\lambda+n-1) \quad (n \in \mathbb{N} := \{1, 2, 3, \cdots\}).$$

Subsequently, by using the function $\varphi_{p,b,c}(z)$, Deniz [2] introduced the operator B_ν^c as follows:

$$B_\nu^c f(z) = \varphi_{p,b,c}(z) * f(z) = z + \sum_{n=1}^{\infty} \frac{(-c)^n a_{n+1}}{4^n \cdot (\nu)_n} \frac{z^{n+1}}{n!} \qquad (z \in \mathbb{C}). \tag{1.8}$$

It is easy to verify from (1.8) that

$$z\left(B_{\nu+1}^c f(z)\right)' = \nu B_\nu^c f(z) - (\nu - 1) B_{\nu+1}^c f(z), \tag{1.9}$$

where

$$\nu = p + \frac{b+1}{2} \notin \mathbb{Z}_0^-.$$

In fact, the operator B_ν^c given by (1.8) provides an elementary transform of the generalized hypergeometric function, that is, we have

$$B_\nu^c f(z) = z\,{}_0F_1\left(\text{---}\,; \nu; -\frac{c}{4}\,z\right) * f(z)$$

and

$$\varphi_{\nu,c}\left(-\frac{c}{4}\,z\right) = z\,{}_0F_1(\text{---}\,; \nu; z).$$

In the present article, we investigate various inclusion relationships for each of the following subclasses of the normalized analytic function class $\mathcal{A}$, which are defined by means of the generalized Bessel function of the first kind (see also [9] and [10] for inclusion relationships for various other function classes). Indeed, for $c \in \mathbb{C}$, $\nu \in \mathbb{R} \setminus \mathbb{Z}_0^-$ and $0 \leqq \alpha < 1$, we write

$$\mathcal{S}_{\nu,c}^*(\alpha) := \{f : f \in \mathcal{A} \quad \text{and} \quad B_\nu^c f(z) \in \mathcal{S}^*(\alpha) \quad (z \in \mathbb{U})\},$$

$$\mathcal{K}_{\nu,c}(\alpha) := \{f : f \in \mathcal{A} \quad \text{and} \quad B_\nu^c f(z) \in \mathcal{K}(\alpha) \quad (z \in \mathbb{U})\},$$

$$\mathcal{C}_{\nu,c}(\beta, \alpha) := \{f : f \in \mathcal{A} \quad \text{and} \quad B_\nu^c f(z) \in \mathcal{C}(\beta, \alpha) \quad (z \in \mathbb{U})\}$$

and

$$\mathcal{C}_{\nu,c}^*(\beta, \alpha) := \{f : f \in \mathcal{A} \quad \text{and} \quad B_\nu^c f(z) \in \mathcal{C}^*(\beta, \alpha) \quad (z \in \mathbb{U})\}.$$

We also note that

$$f(z) \in \mathcal{K}_{\nu,c}(\alpha) \Longleftrightarrow zf'(z) \in \mathcal{S}_{\nu,c}^*(\alpha) \tag{1.10}$$

and

$$f(z) \in \mathcal{C}_{\nu,c}^*(\beta, \alpha) \Longleftrightarrow zf'(z) \in \mathcal{C}_{\nu,c}(\beta, \alpha). \tag{1.11}$$

In our investigation of the inclusion relationships involving the function classes $\mathcal{S}_{\nu,c}^*(\alpha)$, $\mathcal{K}_{\nu,c}(\alpha)$, $\mathcal{C}_{\nu,c}(\beta, \alpha)$ and $\mathcal{C}_{\nu,c}^*(\beta, \alpha)$ given by the above definitions, we shall make use of the following Miller-Mocanu lemma.

Lemma 1.1. *(see Miller and Mocanu [5]) Let $\Theta(u, v)$ be a complex-valued function, such that*

$$\Theta : \mathbb{D} \to \mathbb{C} \qquad (\mathbb{D} \subset \mathbb{C} \times \mathbb{C}),$$

$\mathbb{C}$ being the complex plane. Also let

$$u = u_1 + iu_2 \quad \textit{and} \quad v = v_1 + iv_2.$$

Suppose that the function $\Theta(u, v)$ satisfies each of the following conditions:

(i) $\Theta(u, v)$ *is continuous in* $\mathbb{D}$;

(ii) $(1, 0) \in \mathbb{D}$ *and* $\Re\big(\Theta(1, 0)\big) > 0$;

(iii) $\Re\big(\Theta(iu_2, v_1)\big) \leqq 0$ *for all* $(iu_2, v_1) \in \mathbb{D}$ *such that* $v_1 \leqq -\frac{1}{2}(1 + u_2^2)$.

Let

$$\phi(z) = 1 + p_1 z + p_2 z^2 + \cdots \tag{1.12}$$

be analytic (*regular*) *in* $\mathbb{U}$ *such that*

$$\phi(z) \not\equiv 1 \quad \textit{and} \quad (\phi(z), z\phi'(z)) \in \mathbb{D} \qquad (z \in \mathbb{U}).$$

If

$$\Re\Big(\Theta\big(\phi(z), z\phi'(z)\big)\Big) > 0 \qquad (z \in \mathbb{U}),$$

then

$$\Re\big(\phi(z)\big) > 0 \qquad (z \in \mathbb{U}).$$

2 Inclusion Relationships

Our first set of inclusion relationships is given by Theorem 2.1 below.

Theorem 2.1. *Let* $f \in \mathcal{A}$, $c \in \mathbb{C}$, $\nu \in \mathbb{R} \setminus \mathbb{Z}_0^-$ *and* $\alpha + \nu > 1 \quad (0 \leqq \alpha < 1)$. *Then*

$$f \in \mathcal{S}^*_{\nu,c}(\alpha) \Longrightarrow f \in \mathcal{S}^*_{\nu+1,c}(\alpha)$$

or, *equivalently*,

$$\mathcal{S}^*_{\nu,c}(\alpha) \subset \mathcal{S}^*_{\nu+1,c}(\alpha).$$

Proof. Let $f \in \mathcal{S}^*_{\nu,c}(\alpha)$ and set

$$\frac{z\big(B^c_{\nu+1}f(z)\big)'}{B^c_{\nu+1}f(z)} - \alpha = (1-\alpha)\phi(z), \tag{2.13}$$

where $\phi(z)$ is given by (1.12). From (1.9) we get

$$\nu \frac{B^c_{\nu}f(z)}{B^c_{\nu+1}f(z)} = \frac{z\big(B^c_{\nu+1}f(z)\big)'}{B^c_{\nu+1}f(z)} + (\nu - 1). \tag{2.14}$$

By combining (2.13) and (2.14), we obtain

$$\frac{B^c_{\nu}f(z)}{B^c_{\nu+1}f(z)} = \frac{1}{\nu}\big[(1-\alpha)\phi(z) + \alpha + \nu - 1\big]. \tag{2.15}$$

Now, by applying the logarithmic differentiation on both sides of (2.15) and multiplying the resulting equation by z, we have

$$\frac{z\big(B^c_{\nu}f(z)\big)'}{B^c_{\nu}f(z)} = \frac{z\big(B^c_{\nu+1}f(z)\big)'}{B^c_{\nu+1}f(z)} + \frac{(1-\alpha)z\phi'(z)}{(1-\alpha)\phi(z) + \alpha + \nu - 1},$$

which, in view of (2.13), yields

$$\frac{z\big(B^c_{\nu}f(z)\big)'}{B^c_{\nu}f(z)} - \alpha = (1-\alpha)\phi(z) + \frac{(1-\alpha)z\phi'(z)}{(1-\alpha)\phi(z) + \alpha + \nu - 1}. \tag{2.16}$$

Upon taking

$$u = \phi(z) = u_1 + iu_2 \quad \text{and} \quad \nu = z\phi'(z) = v_1 + iv_2,$$

if we define the function $\Theta(u, v)$ by

$$\Theta(u, v) = (1-\alpha)u + \frac{(1-\alpha)v}{(1-\alpha)u + \alpha + \nu - 1},$$

then we observe that $\Theta(u, v)$ is continuous in

$$\mathbb{D} = \left(\mathbb{C} \setminus \left\{\frac{\alpha + \nu - 1}{\alpha - 1}\right\}\right) \times \mathbb{C}$$

and $(1,0) \in \mathbb{D}$, with $\Re\big(\Theta(1,0)\big) > 0$. Also, for all $(iu_2, v_1) \in \mathbb{D}$ such that $v_1 \leqq -\frac{1}{2}(1+u_2^2)$, we have

$$\begin{aligned}\Re\big(\Theta(iu_2, v_1)\big) &= \Re\left(\frac{(1-\alpha)v_1}{(1-\alpha)iu_2 + \alpha + \nu - 1}\right)\\ &= \frac{(1-\alpha)(\alpha+\nu-1)v_1}{(\alpha+\nu-1)^2 + (1-\alpha)^2 u_2^2}\\ &\leqq \frac{-1}{2} \cdot \frac{(1-\alpha)(\alpha+\nu-1)(1+u_2^2)}{(\alpha+\nu-1)^2 + (1-\alpha)^2 u_2^2}\\ &< 0,\end{aligned}$$

which shows that $\Theta(u,v)$ satisfies the hypotheses of the above Miller-Mocanu Lemma. Therefore, we have

$$\Re\big(\phi(z)\big) > 0 \qquad (z \in \mathbb{U}).$$

Thus, by making use of (2.13) and (2.16), we find that $f \in \mathcal{S}^*_{\nu+1,c}(\alpha)$. This completes the proof of Theorem 2.1. □

Theorem 2.2. *Let $f \in \mathcal{A}$, $c \in \mathbb{C}$, $\nu \in \mathbb{R} \setminus \mathbb{Z}_0^-$ and $\alpha + \nu > 1 \quad (0 \leqq \alpha < 1)$. Then*

$$f \in \mathcal{K}_{\nu,c}(\alpha) \Longrightarrow f \in \mathcal{K}_{\nu+1,c}(\alpha)$$

or, equivalently,

$$\mathcal{K}_{\nu,c}(\alpha) \subset \mathcal{K}_{\nu+1,c}(\alpha).$$

Proof. Applying (1.10) and Theorem 2.1, we observe that

$$\begin{aligned}f \in \mathcal{K}_{\nu,c}(\alpha) &\Longleftrightarrow B_\nu^c f(z) \in \mathcal{K}(\alpha)\\ &\Longleftrightarrow z\left(B_\nu^c f(z)\right)' \in \mathcal{S}^*(\alpha)\\ &\Longleftrightarrow B_\nu^c\left(zf'(z)\right) \in \mathcal{S}^*(\alpha)\\ &\Longleftrightarrow zf'(z) \in \mathcal{S}^*_{\nu,c}(\alpha)\\ &\Longrightarrow zf'(z) \in \mathcal{S}^*_{\nu+1,c}(\alpha)\\ &\Longleftrightarrow B_{\nu+1}^c\left(zf'(z)\right) \in \mathcal{S}^*(\alpha)\\ &\Longleftrightarrow z\left(B_{\nu+1}^c f(z)\right)' \in \mathcal{S}^*(\alpha)\\ &\Longleftrightarrow B_{\nu+1}^c f(z) \in \mathcal{K}(\alpha)\\ &\Longleftrightarrow f \in \mathcal{K}_{\nu+1,c}(\alpha),\end{aligned}$$

which evidently proves Theorem 2.2. □

Theorem 2.3. *Let $f \in \mathcal{A}$, $c \in \mathbb{C}$, $\nu \in \mathbb{R} \setminus \mathbb{Z}_0^-$ and $\alpha + \nu > 1 \quad (0 \leqq \alpha < 1)$. Then*

$$f \in \mathcal{C}_{\nu,c}(\beta,\alpha) \Longrightarrow f \in \mathcal{C}_{\nu+1,c}(\beta,\alpha) \quad (0 \leqq \beta < 1)$$

or, equivalently,

$$\mathcal{C}_{\nu,c}(\beta,\alpha) \subset \mathcal{C}_{\nu+1,c}(\beta,\alpha).$$

Proof. Let $f \in \mathcal{C}_{\nu,c}(\beta,\alpha)$. Then, in view of the definition of the class $\mathcal{C}_{\nu,c}(\beta,\alpha)$, there exists a function $g \in \mathcal{S}^*_{\nu,c}(\alpha)$ such that

$$\Re\left(\frac{z\big(B_\nu^c f(z)\big)'}{B_\nu^c g(z)}\right) > \beta, \quad (0 \leqq \beta < 1;\ z \in \mathbb{U}).$$

We now let

$$\frac{z\big(B_{\nu+1}^c f(z)\big)'}{B_{\nu+1}^c g(z)} - \beta = (1-\beta)\phi(z), \tag{2.17}$$

where the function $\phi(z)$ is given by (1.12). Now, making use of the identity (1.9), we also have

$$\begin{aligned}\frac{z\big(B_\nu^c f(z)\big)'}{B_\nu^c g(z)} &= \frac{B_\nu^c\big(zf'(z)\big)}{B_\nu^c g(z)}\\ &= \frac{z\Big(B_{\nu+1}^c\big(zf'(z)\big)\Big)' + (\nu-1)B_{\nu+1}^c\big(zf'(z)\big)}{z\big(B_{\nu+1}^c g(z)\big)' + (\nu-1)B_{\nu+1}^c g(z)}\\ &= \left(\frac{z\Big(B_{\nu+1}^c\big(zf'(z)\big)\Big)'}{B_{\nu+1}^c g(z)} + (\nu-1)\frac{B_{\nu+1}^c\big(zf'(z)\big)}{B_{\nu+1}^c g(z)}\right)\cdot\left(\frac{z\big(B_{\nu+1}^c g(z)\big)'}{B_{\nu+1}^c g(z)} + \nu - 1\right)^{-1}.\end{aligned} \tag{2.18}$$

By Theorem 2.1, we know that

$$g \in \mathcal{S}_{\nu,c}^*(\alpha) \Longrightarrow g \in \mathcal{S}_{\nu+1,c}^*(\alpha),$$

so that we can set

$$\frac{z\big(B_{\nu+1}^c g(z)\big)'}{B_{\nu+1}^c g(z)} = (1-\alpha)q(z) + \alpha, \tag{2.19}$$

where

$$\Re\big(q(z)\big) > 0 \qquad (z \in \mathbb{U}).$$

Upon substituting from (2.17) and (2.19) into (2.18), we have

$$\frac{z\big(B_\nu^c f(z)\big)'}{B_\nu^c g(z)} = \frac{\left[z\Big(B_{\nu+1}^c\big(zf'(z)\big)\Big)'\right]\cdot\left[B_{\nu+1}^c g(z)\right]^{-1} + (\nu-1)[(1-\beta)\phi(z)+\beta]}{(1-\alpha)q(z)+\alpha+\nu-1}. \tag{2.20}$$

By logarithmically differentiating both sides of (2.17) with respect to z, we have

$$\frac{z\Big(B_{\nu+1}^c\big(zf'(z)\big)\Big)'}{B_{\nu+1}^c g(z)} = (1-\beta)z\phi'(z) + [(1-\alpha)q(z)+\alpha]\cdot[(1-\beta)\phi(z)+\beta],$$

which, in conjunction with (2.20), yields

$$\frac{z\big(B_\nu^c f(z)\big)'}{B_\nu^c g(z)} - \beta = (1-\beta)\phi(z) + \frac{(1-\beta)z\phi'(z)}{(1-\alpha)q(z)+\alpha+\nu-1}. \tag{2.21}$$

The remaining part of our proof of Theorem 2.3 is much akin to that of Theorem 2.1. Therefore, we choose to omit the analogous details involved. □

Theorem 2.4. *Let* $f \in \mathcal{A}$, $c \in \mathbb{C}$, $\nu \in \mathbb{R} \setminus \mathbb{Z}_0^-$ *and* $\alpha + \nu > 1 \quad (0 \leqq \alpha < 1)$. *Then*

$$f \in \mathcal{C}_{\nu,c}^*(\beta,\alpha) \Longrightarrow f \in \mathcal{C}_{\nu+1,c}^*(\beta,\alpha)$$

or, *equivalently*,

$$\mathcal{C}_{\nu,c}^*(\beta,\alpha) \subset \mathcal{C}_{\nu+1,c}^*(\beta,\alpha).$$

Proof. Applying (1.11) and Theorem 2.3, we observe that

$$\begin{aligned} f \in \mathcal{C}_{\nu,c}^*(\alpha) &\Longleftrightarrow B_\nu^c f(z) \in \mathcal{C}^*(\beta,\alpha)\\ &\Longleftrightarrow z\left(B_\nu^c f(z)\right)' \in \mathcal{C}(\beta,\alpha)\\ &\Longleftrightarrow B_\nu^c\left(zf'(z)\right) \in \mathcal{C}(\beta,\alpha)\\ &\Longleftrightarrow zf'(z) \in \mathcal{C}_{\nu,c}(\beta,\alpha)\\ &\Longrightarrow zf'(z) \in \mathcal{C}_{\nu+1,c}(\beta,\alpha)\\ &\Longleftrightarrow B_{\nu+1}^c\left(zf'(z)\right) \in \mathcal{C}(\beta,\alpha)\\ &\Longleftrightarrow z\left(B_{\nu+1}^c f(z)\right)' \in \mathcal{C}(\beta,\alpha)\\ &\Longleftrightarrow B_{\nu+1}^c f(z) \in \mathcal{C}^*(\beta,\alpha)\\ &\Longleftrightarrow f \in \mathcal{C}_{\nu+1,c}^*(\beta,\alpha),\end{aligned}$$

which evidently proves Theorem 2.4. □

3 Remarks and Observations

As already discussed in Section 1, the study of the generalized Bessel function of the first kind permits a unified study of the Bessel, the modified Bessel and the spherical Bessel functions. By specializing the parameters in the operator B_ν^c, we obtain the following new operators associated with the Bessel, the modified Bessel and the spherical Bessel functions (see, for details, [2]):

- Choosing $b = c = 1$ in (1.8), we obtain the operator $\mathcal{J}_p : \mathcal{A} \longrightarrow \mathcal{A}$ associated with the Bessel function, which is defined by

$$\begin{aligned}\mathcal{J}_p f(z) = \varphi_{p,1,1}(z) * f(z) &= \left[2^p \Gamma(p+1) z^{1-p/2}\, J_p(\sqrt{z})\right] * f(z) \\ &= z + \sum_{n=1}^{\infty} \frac{(-1)^n a_{n+1}}{4^n (p+1)_n} \frac{z^{n+1}}{n!}. \end{aligned} \tag{3.22}$$

- Taking $b = 1$ and $c = -1$ in (1.8), we obtain the operator $\mathcal{I}_p : \mathcal{A} \longrightarrow \mathcal{A}$ associated with the modified Bessel function, which is defined by

$$\begin{aligned}\mathcal{I}_p f(z) = \varphi_{p,1,-1}(z) * f(z) &= \left[2^p \Gamma(p+1) z^{1-p/2}\, I_p(\sqrt{z})\right] * f(z) \\ &= z + \sum_{n=1}^{\infty} \frac{a_{n+1}}{4^n (p+1)_n} \frac{z^{n+1}}{n!}. \end{aligned} \tag{3.23}$$

- Letting $b = 2$ and $c = 1$ in (1.8), we obtain the operator $\mathcal{Q}_p : \mathcal{A} \longrightarrow \mathcal{A}$ associated with the spherical Bessel function, which is defined by

$$\begin{aligned}\mathcal{Q}_p f(z) = \varphi_{p,2,1}(z) * f(z) &= \left[\pi^{-1/2} 2^{p+1/2} \Gamma\left(p + \frac{3}{2}\right) z^{1-p/2}\, j_p(\sqrt{z})\right] * f(z) \\ &= z + \sum_{n=1}^{\infty} \frac{(-1)^n a_{n+1}}{4^n (p+3/2)_n} \frac{z^{n+1}}{n!}. \end{aligned} \tag{3.24}$$

Our main results (Theorems 2.1 to 2.4) can thus be applied with a view of deducing the following consequences.

Corollary 3.1. *Let* $f \in \mathcal{A}$, $p \in \mathbb{R} \setminus \mathbb{Z}^-$ *and* $\alpha + p > 0 \quad (0 \leqq \alpha < 1)$. *Then*

$$f \in \mathcal{S}^*_{p+1,1}(\alpha) \Longrightarrow f \in \mathcal{S}^*_{p+2,1}(\alpha)$$

or, equivalently,

$$\mathcal{J}_p f(z) \in \mathcal{S}^*(\alpha) \quad \Longrightarrow \quad f(z) \in \mathcal{S}^*_{p+n,1}(\alpha) \quad (n \in \mathbb{N} \setminus \{1\}).$$

Corollary 3.2. *Let* $f \in \mathcal{A}$, $p \in \mathbb{R} \setminus \mathbb{Z}^-$ *and* $\alpha + p > 0 \quad (0 \leqq \alpha < 1)$. *Then*

$$f \in \mathcal{K}_{p+1,1}(\alpha) \Longrightarrow f \in \mathcal{K}_{p+2,1}(\alpha)$$

or, equivalently,

$$\mathcal{J}_p f(z) \in \mathcal{K}(\alpha) \quad \Longrightarrow \quad f(z) \in \mathcal{K}_{p+n,1}(\alpha) \quad (n \in \mathbb{N} \setminus \{1\}).$$

Corollary 3.3. *Let* $f \in \mathcal{A}$, $p \in \mathbb{R} \setminus \mathbb{Z}^-$ *and* $\alpha + p > 0 \quad (0 \leqq \alpha < 1)$. *Then*

$$f \in \mathcal{C}_{p+1,1}(\beta, \alpha) \Longrightarrow f \in \mathcal{C}_{p+2,1}(\beta, \alpha) \quad (0 \leqq \beta < 1)$$

or, equivalently,

$$\mathcal{J}_p f(z) \in \mathcal{C}(\beta, \alpha) \quad \Longrightarrow \quad f(z) \in \mathcal{C}_{p+n,1}(\beta, \alpha) \quad (n \in \mathbb{N} \setminus \{1\}).$$

Corollary 3.4. *Let* $f \in \mathcal{A}$, $p \in \mathbb{R} \setminus \mathbb{Z}^-$ *and* $\alpha + p > 0 \quad (0 \leqq \alpha < 1)$. *Then*

$$f \in \mathcal{C}^*_{p+1,1}(\beta, \alpha) \Longrightarrow f \in \mathcal{C}^*_{p+2,1}(\beta, \alpha) \quad (0 \leqq \beta < 1)$$

or, equivalently,

$$\mathcal{J}_p f(z) \in \mathcal{C}^*(\beta, \alpha) \quad \Longrightarrow \quad f(z) \in \mathcal{C}^*_{p+n,1}(\beta, \alpha) \quad (n \in \mathbb{N} \setminus \{1\}).$$

Finally, we remark that similar results can be obtained involving the operators $\mathcal{I}_p$ and $\mathcal{Q}_p$ by specializing the parameter in Theorems 2.1 to 2.4. Numerous other applications and consequences of our main results (Theorems 2.1 to 2.4) and their aforementioned consequences (Corollaries 3.1 to 3.4) can indeed be derived similarly.

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