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Common fixed point theorems in intuitionistic menger spaces using CLR property

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Abstract

We use the notion of CLR property to prove some common fixed point theorems for weakly compatible mappings in intuitionistic Menger spaces. Our theorems generalize and improve theorems of [5], [6], [7], [8], [10], [20] and [28].

Keywords: Common fixed point, intuitionistic Menger space, weakly compatible mappings, CLR property, JCLR property.

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1 Introduction

There have been a number of generalizations of metric spaces. One of such generalization is a probabilistic metric space, briefly, PM-spaces, introduced in 1942 by Menger [21]. In the PM-space, we do not know exactly the distance between two points, but we know probabilities of possible values of this distance. This space was developed by Schweizer and Sklar [26, 27]. Modifying the idea of Kramosil and Michalek [15], George and Veeramani [9] introduced fuzzy metric spaces which are very similar to Menger spaces. Recently, using the idea of intuitionistic fuzzy set, see Atanassovas [2] and [3], which is a generalization of a fuzzy set, see Zadeh [32], Park [24] introduced the notion of intuitionistic fuzzy metric spaces as a generalization of fuzzy metric spaces due to George and Veeramani [9]. Kutukcu et. al [17] introduced the notion of intuitionistic Menger spaces as a generalization of Menger spaces.

Jungck [13] introduced the notion of compatible mappings in metric spaces. Mishra [22] extended the notion of compatibility to probabilistic metric spaces and this condition has been weakened by introducing the notion of weak compatibility by Jungck [14].

Sintunavarat and Kumam [31] introduced the concept of CLR property. Very recentlly, Chauhan et. al [4] introduced the notion of JCLR property. The importance of these properties is that we don't require the closedness of subspaces for the existence of fixed points.

The purpose of this paper is to prove common fixed point theorems for weakly compatible mappings in intuitionistic Menger spaces using these propertirs. Our theorems generalize and improve theorems of [5], [6], [7], [8], [10], [20] and [28].

2 Preliminaries

Definition 2.1 ([26]). *A binary operation $*: [0,1]\times[0,1]\to[0,1]$ is a continuous t-norm if $*$ satisfies the following conditions.*

a) $*$ is commutative and associative,

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b) $*$ *is continuous,*
c) $a * 1 = a$ *for all* $a \in [0, 1]$ *,*
d) $a * bc * d$ *wherever* ac*,* bd *and* $a, b, c, d \in [0, 1]$ *.*

Examples of t-norms are $a * b = \min\{a, b\}$ and $a * b = ab$.

Definition 2.2 ([26])**.** *A binary operation* $\diamondsuit : [0, 1] \times [0, 1] \to [0, 1]$ *is a continuous t-conorm if* $\diamondsuit$ *satisfies the following conditions.*
a) $\diamondsuit$ *is commutative and associative,*
b) $\diamondsuit$ *is continuous,*
c) $a\diamondsuit 0 = a$ *for all* $a \in [0, 1]$ *,*
d) $a\diamondsuit b \geq c\diamondsuit d$ *wherever* $a \geq c$*,* $b \geq d$ *and* $a, b, c, d \in [0, 1]$ *.*

Examples of t-conorms are $a\diamondsuit b = \max\{a, b\}$ and $a\diamondsuit b = \min\{1, a + b\}$.

Remark 2.1. *The concepts of triangular norms (t-norms) and triangular conorms (t-conorms) are known as the axiomatic sketlons that we use for characterizing fuzzy intersection and union respectively. These concepts were originally introduced by Menger [21] in his study of statistical metric spaces.*

Definition 2.3 ([26])**.** *A distance distribution function is a function* $F : \mathbb{R} \to \mathbb{R}_+$ *which is left continuous on* $\mathbb{R}$*, non-decreasing and* $\inf_{t\in\mathbb{R}} F(t) = 0$*,* $\sup_{t\in\mathbb{R}} F(t) = 1$*. We will denote by* D *the family of all distance distribution functions and by* H *a special element of* D *defined by* $H(t) = \begin{cases} 0, & if \quad t0 \\ 1, & if \quad t > 0 \end{cases}$ *.*

If X is a non-empty set, $F : X \times X \to D$ is called a probabilistic distance on X and $F(x, y)$ is usually denoted by F_{xy}.

Definition 2.4 ([17])**.** *A non-distance distribution function is a function* $L : \mathbb{R} \to \mathbb{R}_+$ *which is right continuous on* $\mathbb{R}$*, non-increasing and* $\inf_{t\in\mathbb{R}} L(t) = 1$*,* $\sup_{t\in\mathbb{R}} L(t) = 0$*. We will denote by* E *the family of all non-distance distribution functions and by* G *a special element of* E *defined by* $G(t) = \begin{cases} 1, & if \quad t0 \\ 0, & if \quad t > 0 \end{cases}$ *.*

If X is a non-empty set, $L : X \times X \to E$ is called a probabilistic non-distance on X and $L(x, y)$ is usually denoted by L_{xy}.

Definition 2.5 ([17])**.** *A triplet* (X, F, L) *is said to be an intuitionistic probabilistic metric space if* X *is an arbitrary set,* F *is a probabilistic distance and* L *is a probabilistic non-distance on* X *satisfying the following conditions for all* $x, y, z \in X$ *and* $t, s > 0$
1) $F_{xy}(t) + L_{xy}(t) 1$*,*
2) $F_{xy}(0) = 0$*,*
3) $F_{xy}(t) = 1$ *if and only if* $x = y$*,*
4) $F_{xy}(t) = F_{yx}(t)$*,*
5) If $F_{xy}(t) = 1$ *and* $F_{yz}(s) = 1$*, then* $F_{xz}(t + s) = 1$*,*
6) $L_{xy}(0) = 1$*,*
7) $L_{xy}(t) = 0$ *if and only if* $x = y$*,*
8) $L_{xy}(t) = L_{yx}(t)$*,*
9) If $L_{xy}(t) = 0$ *and* $L_{yz}(s) = 0$*, then* $L_{xz}(t + s) = 0$*.*

Definition 2.6 ([17])**.** *A 5-tuple* $(X, F, L, *, \diamondsuit)$ *is said to be an intuitionistic Menger metric space if* (X, F, L) *is an intuitionistic probabilistic metric space and in addition, the following inequalities hold for all* $x, y, z \in X$ *and* $t, s > 0$*,*
1) $F_{xy}(t) * F_{yz}(s) F_{xz}(t + s)$*,*
2) $L_{xy}(t) \diamondsuit L_{yz}(s) L_{xz}(t + s)$*,*
where $*$ *is a continous t-norm and* $\diamondsuit$ *is a continous t-conorm.*

The functions F_{xy} and L_{xy} denote the degree of nearness and the degree of non-nearness between x and y with respect to t respectively.

Remark 2.2. *In intuitionistic Menger space* $(X, F, L, *, \diamondsuit)$*,* F_{xy} *is non-decreasing and* L_{xy} *is non-increasing for all* $x, y \in X$*.*

Remark 2.3 ([17]). *Every Menger space $(X, F, *)$ is an intuitionistic Menger space of the form $(X, F, 1-F, *, \diamondsuit)$ such that the t-norm $*$ and the t-conorm $\diamondsuit$ are associated, see [19], that is $x \diamondsuit y = 1-(1-x)*(1-y)$ for any $x, y \in X$.*

Remark 2.4. *Kutukcu et al. [17] proved that if the t-norm $*$ and the t-conorm of an intuitionistic Menger space $(X, F, L, *, \diamondsuit)$ satisfy the conditions*

$$\sup_{t\in(0,1)} (t*t) = 1 \text{ and } \inf_{t\in(0,1)} ((1-t)\diamondsuit(1-t)) = 0,$$

then $(X, F, L, *, \diamondsuit)$ is a Hausdorff topological space in the (ϵ, λ) topology, i.e., the family of sets

$$\{U_x(\epsilon, \lambda), \epsilon > 0, \lambda \in (0,1], x \in X\}$$

is a basis of neighborhoods of point x for a Hausdorff topology $\tau_{(F,L)}$, or (ϵ, λ) topology on X, where

$$U_x(\epsilon, \lambda) = \left\{y \in X : F_{xy}(\epsilon) > 1-\lambda \text{ and } L_{xy}(\epsilon) < \lambda\right\}.$$

Example 2.1 ([17]). *Let (X, d) be a metric space. Then the metric d induces a distance distribution function F defined by $F_{xy}(t) = H(t-d(x,y))$ and a non-distance distribution function L defined by $L_{xy}(t) = G(t-d(x,y))$ for all $x, y \in X$ and $t > 0$. Therefore, (X, F, L) is an intuitionistic probabilistic metric space induced by a metric d. If the t-norm $*$ is defined by $a*b = \min\{a,b\}$ and the t-conorm $\diamondsuit$ is defined by $a\diamondsuit b = \min\{1, a+b\}$ for all $a, b \in [0,1]$, then $(X, F, L, *, \diamondsuit)$ is an intuitionistic Menger space.*

Remark 2.5 ([17]). *Note that the above example holds even with the t-norm $a*b = \min\{a,b\}$ and the t-conorm $a\diamondsuit b = \max\{a,b\}$ and hence $(X, F, L, *, \diamondsuit)$ is an intuitionistic Menger space with respect to any t-norm and t-conorm. Also note that, in the above example, t-norm $*$ and t-conorm $\diamondsuit$ are not associated.*

Remark 2.6. *Every an intuitionistic fuzzy metric space $(X, F, L, *, \diamondsuit)$ is an intuitionistic Menger space by considering $F : X \times X \to D$ and $L : X \times X \to E$ defined by $F_{xy}(t) = M(x,y,t)$ and $L_{xy}(t) = N(x,y,t)$ for all $x, y \in X$.*

Throughout this paper, $(X, F, L, *, \diamondsuit)$ is an intuitionistic Menger space with the following conditions:

$$\lim_{t\to+\infty} F_{xy}(t) = 1 \text{ and } \lim_{t\to+\infty} L_{xy}(t) = 0, \text{ for all } x, y \in X \text{ and } t > 0. \tag{2.1}$$

Definition 2.7 ([17]). *Let $(X, F, L, *, \diamondsuit)$ be an intuitionistic Menger space.*

(a) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be convergent to a point $x \in X$, if for each $t > 0$ and $\epsilon \in (0,1)$, there exists a positive integer $n_0 = n_0(t, \epsilon)$ such that for all $n \geq n_0$

$$F_{x_n x}(t) > 1-\epsilon \text{ and } L_{x_n x}(t) < \epsilon.$$

(b) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is called a Cauchy sequence if for all $t > 0$ and $\epsilon \in (0,1)$, there exists a positive integer $n_0 = n_0(t, \epsilon)$ such that for all $n, m \geq n_0$

$$F_{x_n x_m}(t) > 1-\epsilon \text{ and } L_{x_n x_m}(t) < \epsilon.$$

(c) An intuitionistic Menger space in which every Cauchy sequence is convergent is said to be complete.

Remark 2.7 ([17]). *An induced intuitionistic Menger space $(X, F, L, *, \diamondsuit)$ is complete if (X, d) is complete.*

Theorem 2.1 ([17]). *Let $(X, F, L, *, \diamondsuit)$ be an intuitionistic Menger space.*

A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be convergent to a point $x \in X$ if and only if

$$\lim_{n\to+\infty} F_{x_n x}(t) = 1 \text{ and } \lim_{n\to+\infty} L_{x_n x}(t) = 0, \text{ for all } t > 0.$$

A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is called a Cauchy sequence if and only if

$$\lim_{n\to+\infty} F_{x_n x_m}(t) = 1 \text{ and } \lim_{n\to+\infty} L_{x_n x_m}(t) = 0, \text{ for all } t > 0.$$

Lemma 2.1 ([17]). *Let $(X, F, L, *, \diamondsuit)$ be an intuitionistic Menger space and $\{x_n\}$, $\{y_n\}$ be two sequences in X with $x_n \to x$ and $y_n \to y$, respectively. Then*

(a)

$$\lim_{n\to\infty}\inf F_{x_n y_n}(t) \geq F_{xy}(t) \text{ and } \lim_{n\to\infty}\sup L_{x_n y_n}(t) \leq L_{xy}(t) \text{ for all } t > 0.$$

(b) If $t > 0$ is a continuous point of F_{xy} and L_{xy}, then

$$\lim_{n\to\infty} F_{x_n y_n}(t) = F_{xy}(t) \text{ and } \lim_{n\to\infty} L_{x_n y_n}(t) = L_{xy}(t).$$

Lemma 2.2 ([23]). *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in an intuitionistic Menger space with the condition (2.1). If there exists a number $k \in (0,1)$ such that for $x, y \in X$, $t > 0$ and $n = 0, 1, 2,...$*

$$F_{x_{n+2},x_{n+1}}(kt) \geq F_{x_{n+1},x_n}(t) \text{ and } L_{x_{n+2},x_{n+1}}(kt) \leq L_{x_{n+1},x_n}(t),$$

then $\{x_n\}$ is a Cauchy sequence in X.

Lemma 2.3 ([23]). *Let $(X, F, L, *, \diamondsuit)$ be an intuitionistic Menger space. If there exists a number $k \in (0,1)$ such that for all $x, y \in X$ and $t > 0$*

$$F_{xy}(kt) \geq F_{xy}(t) \text{ and } L_{xy}(kt) \leq L_{xy}(t),$$

then $x = y$.

Definition 2.8 ([23]). *Two self-mappings A and S of an intuitionistic Menger space are said to be compatible if*

$$\lim_{n\to+\infty} F_{ASx_n,SAx_n}(t) = 1 \text{ and } \lim_{n\to+\infty} L_{ASx_n,SAx_n}(t) = 0 \text{ for all } t > 0,$$

whenever $\{x_n\} \subset X$ such that

$$\lim_{n\to+\infty} Ax_n = \lim_{n\to+\infty} Sx_n = z \text{ for some } z \in X.$$

Definition 2.9. *Two self-mappings A and S of an intuitionistic Menger space are said to be non-compatible if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to+\infty} F_{Ax_n,z}(t) = \lim_{n\to+\infty} F_{Sx_n,z}(t) = 1$ for some $z \in X$, but for some $t > 0$, either $\lim_{n\to+\infty} F_{ASx_n,SAx_n}(t) \neq 1$ or $\lim_{n\to+\infty} L_{ASx_n,SAx_n}(t) \neq 0$ or one of the limits do not exist.*

Definition 2.10 ([14]). *Two self-mappings A and S of a non-empty set X are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, i.e., if $Ax = Sx$ for some $x \in X$, then $ASx = SAx$.*

Remark 2.8. *Two compatible self-mappings are weakly compatible, however the converse is not true in general, see [30], example 1.*

Definition 2.11 ([1, 25]). *A pair of self-mappings A and S of an intuitionistic Menger space $(X, F, L, *, \diamondsuit)$ is said to be tangential or satisfy the property (E.A) if there exists a sequence $\{x_n\}$ in X such that for some $z \in X$*

$$\lim_{n\to+\infty} F_{Ax_n,z}(t) = \lim_{n\to+\infty} F_{Sx_n,z}(t) = 1 \text{ and } \lim_{n\to+\infty} L_{Ax_n,z}(t) = \lim_{n\to+\infty} L_{Sx_n,z}(t) = 0 \text{ for all } t > 0. \tag{2.2}$$

Remark 2.9. *It is easy to see that two non-ncompatible self-mappings of an intuitionistic Menger space satisfy the property (E.A), but the converse is not true in general.*

Definition 2.12 ([18]). *Two pairs (A, S) and (B, T) of self-mappings of an intuitionistic Menger space $(X, F, L, *, \diamondsuit)$ are said to satisfy the common property E.A, if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that some $z \in X$ and for all $t > 0$*

$$\begin{aligned}\lim_{n\to+\infty} F_{Ax_n,z}(t) &= \lim_{n\to+\infty} F_{Sx_n,z}(t) = \lim_{n\to+\infty} F_{By_n,z}(t) = \lim_{n\to+\infty} F_{Ty_n,z}(t) = 1 \text{ and} \\ \lim_{n\to+\infty} L_{Ax_n,z}(t) &= \lim_{n\to+\infty} L_{Sx_n,z}(t) = \lim_{n\to+\infty} L_{By_n,z}(t) = \lim_{n\to+\infty} L_{Ty_n,z}(t) = 0.\end{aligned}$$

If $B = A$ and $T = S$ in this definition we get the definition of the property (E.A).

Definition 2.13 ([31]). *A pair of self-mappings A and S of an intuitionistic Menger space $(X, F, L, *, \diamondsuit)$ is said to satisfy the common limit range property with respect to the mapping S (briefly CLR_S property), if there exists a sequence $\{x_n\}$ in X such that (2.2) holds, where $z \in S(X)$.*

Now, we give an example of self-mappings A and S satisfying the CLR_S property.

Example 2.2. *Let $(X, F, L, *, \Diamond)$ be an intuitionistic Menger space, where $X = [0, \infty)$, the t-norm $*$ is defined by $a * b = \min\{a, b\}$, the t-conorm $\Diamond$ is defined by $a \Diamond b = \max\{a, b\}$ and*

$$F_{xy}(t) = H(t - |x - y|),\ L_{xy}(t) = G(t - |x - y|)$$

for all $x, y \in X$ and $t > 0$. Define self-mappings A and S on X by: $Ax = x + 4$, $Sx = 5x$. Let a sequence $\left\{x_n = 1 + \frac{1}{n}\right\}_{n \in \mathbb{N}^*}$ in X. Since $\lim_{n\to+\infty} Ax_n = \lim_{n\to+\infty} Sx_n = 5$, then

$$\begin{aligned}\lim_{n\to+\infty} F_{Ax_n,5}(t) &= \lim_{n\to+\infty} F_{Sx_n,5}(t) = 1 \text{ and}\\ \lim_{n\to+\infty} L_{Ax_n,5}(t) &= \lim_{n\to+\infty} L_{Sx_n,5}(t) = 0 \text{ for all } t > 0,\end{aligned}$$

where $5 \in S(X)$. Therefore, the mappings A and S satisfy the CLR_S property.

From this example, it is clear that a pair (A, S) satisfying the property (E.A)with the closedness of the subspace $S(X)$ always verifies the CLR_S property.

Definition 2.14 ([12]). *Two pairs (A, S) and (B, T) of self-mappings of an intuitionistic Menger space $(X, F, L, *, \Diamond)$ are said to satisfy the common limit range property with respect to mappings S and T (briefly, CLR_{ST} property), if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that for all $t > 0$*

$$\begin{aligned}\lim_{n\to+\infty} F_{Ax_n,z}(t) &= \lim_{n\to+\infty} F_{Sx_n,z}(t) = \lim_{n\to+\infty} F_{By_n,z}(t) = \lim_{n\to+\infty} F_{Ty_n,z}(t) = 1 \text{ and}\\ \lim_{n\to+\infty} L_{Ax_n,z}(t) &= \lim_{n\to+\infty} L_{Sx_n,z}(t) = \lim_{n\to+\infty} L_{By_n,z}(t) = \lim_{n\to+\infty} L_{Ty_n,z}(t) = 0,\end{aligned}$$

where $z \in S(X) \cap T(X)$.

Remark 2.10. *If $B = A$ and $T = S$ in this definition we get the definition of CLR_S property.*

Remark 2.11. *The CLR_{ST} property implies the common property (E.A.), but the converse is not true in general, see [5], example 21.*

Proposition 2.1 ([5]). *If the pairs (A, S) and (B, T) satisfy the common property (E.A.) and $S(X)$ and $T(X)$ are closed subsets of X, then the pairs satsfy also the CLR_{ST} property.*

Definition 2.15 ([4]). *Two pairs (A, S) and (B, T) of self-mappings of an intuitionistic Menger space $(X, F, L, *, \Diamond)$ are said to satisfy the joint common limit range property with respect to mappings S and T(briefly $JCLR_{ST}$ property), if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that for all $t > 0$*

$$\begin{aligned}\lim_{n\to+\infty} F_{Ax_n,z}(t) &= \lim_{n\to+\infty} F_{Sx_n,z}(t) = \lim_{n\to+\infty} F_{By_n,z}(t) = \lim_{n\to+\infty} F_{Ty_n,z}(t) = 1 \text{ and}\\ \lim_{n\to+\infty} L_{Ax_n,z}(t) &= \lim_{n\to+\infty} L_{Sx_n,z}(t) = \lim_{n\to+\infty} L_{By_n,z}(t) = \lim_{n\to+\infty} L_{Ty_n,z}(t) = 0,\end{aligned}$$

where $z = Su = Tu$, $u \in X$.

Remark 2.12. *If $B = A$ and $T = S$ in this definition we get the definition of CLR_S property.*

Definition 2.16 ([11]). *Two families of self-mappings $\{A_i\}$ and $\{S_j\}$ are said to be pairwise commuting if*

(1) $A_iA_j = A_jA_i$, $i, j \in \{1, 2, ..., m\}$,
(2) $S_kS_l = S_lS_k$, $k, l \in \{1, 2, ..., n\}$,
(3) $A_iS_k = S_kA_i$, $i \in \{1, 2, ..., m\}, k \in \{1, 2, ..., n\}$.

3 Main results

Lemma 3.4. *Let A, B, S and T be self-mappings of an intuitionistic Menger space $(X, F, L, *, \Diamond)$ satisfying the following conditions.*

1) The pair (A, S) satisfies the CLR_S property or the pair (B, T) satisfies the CLR_T property,
2) $A(X) \subseteq T(X)$ or $B(X) \subseteq S(X)$,

3) $T(X)$ or $S(X)$ is a closed subset of X.

4) $B(y_n)$ converges for every sequence $\{y_n\}$ in X whenever $T(y_n)$ converges or $A(x_n)$ converges for every sequence $\{x_n\}$ in X whenever $S(x_n)$ converges.

$$\begin{aligned} (1+\alpha F_{Sx,Ty}(t))\, F_{Ax,By}(t) &> \alpha \min\{F_{Ax,Sx}(t)\, F_{By,Ty}(t), F_{Sx,By}(t)\, F_{Ax,Ty}(t)\} \\ &+ \min\left\{ \begin{array}{c} F_{Sx,Ty}(t), \sup_{t_1+t_2=\frac{2}{k}t} \min\{F_{Ax,Sx}(t_1), F_{By,Ty}(t_2)\}, \\ \sup_{t_3+t_4=2t} \min\{F_{Sx,By}(t_3), F_{Ax,Ty}(t_4)\} \end{array} \right\} \\ (1+\beta L_{Sx,Ty}(t))\, L_{Ax,By}(t) &< \beta \max\{L_{Ax,Sx}(t)\, L_{By,Ty}(t), L_{Sx,By}(t)\, L_{Ax,Ty}(t)\} \\ &+ \max\left\{ \begin{array}{c} L_{Sx,Ty}(t), \inf_{t_1+t_2=\frac{2}{k}t} \max\{L_{Ax,Sx}(t_1), L_{By,Ty}(t_2)\}, \\ \inf_{t_3+t_4=2t} \max\{L_{Sx,By}(t_3), L_{Ax,Ty}(t_4)\} \end{array} \right\} \end{aligned} \tag{3.1}$$

for all $x, y \in X$, $t > 0$, for some α, $\beta \geq 0$ and $1 \leq k < 2$. Then the pairs (A, S) and (B, T) satisfy the CLR_{ST} property.

Proof. Suppose that the pair (A, S) satisfies the CLR_S property and $T(X)$ is a closed subset of X. Then, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to+\infty} Ax_n = \lim_{n\to+\infty} Sx_n = z, \text{ where } z \in S(X).$$

Since $A(X) \subseteq T(X)$, there exists a sequence $\{y_n\}$ in X such that $Ax_n = Ty_n$. So

$$\lim_{n\to+\infty} Ty_n = \lim_{n\to+\infty} Ax_n = z, \text{ where } z \in S(X) \cap T(X).$$

Thus, $Ax_n \to z$, $Sx_n \to z$ and $Ty_n \to z$. Now, we show that $By_n \to z$.

Let $\lim_{n\to+\infty} F_{By_n,l}(t_0) = 1$ and $\lim_{n\to+\infty} L_{By_n,l}(t_0) = 0$. We assert that $l = z$. Assume that $l \neq z$. We prove that there exists $t_0 > 0$ such that

$$F_{z,l}\left(\frac{2}{k}t_0\right) > F_{z,l}(t_0) \text{ and } L_{z,l}\left(\frac{2}{k}t_0\right) < L_{z,l}(t_0). \tag{3.2}$$

Suppose the contrary. Therefore, for all $t > 0$ we have

$$F_{z,l}\left(\frac{2}{k}t\right) \leq F_{z,l}(t) \text{ and } L_{z,l}\left(\frac{2}{k}t\right) \geq L_{z,l}(t). \tag{3.3}$$

Using repeatedly (3.3), we obtain

$$\begin{aligned} F_{z,l}(t) &\geq F_{z,l}\left(\frac{2}{k}t\right) \geq ... \geq F_{z,l}\left(\left(\frac{2}{k}\right)^n t\right) \to 1 \text{ and} \\ L_{z,l}(t) &\leq L_{z,l}\left(\frac{2}{k}t\right) \leq ... \leq L_{z,l}\left(\left(\frac{2}{k}\right)^n t\right) \to 0; \end{aligned}$$

as $n \to +\infty$, this shows that $F_{z,l}(t) = 1$ and $L_{z,l}(t) = 0$ for all $t > 0$, which contradicts $l \neq z$ and hence (3.2) is proved.

Without loss of generality, we may assume that t_0 in (3.2) is a continuous point of $F_{z,l}$ and L_{zl}. Since every distance distribution function is left-continuous and every a non-distance distribution function is right continuous, (3.2) implies that there exists $\epsilon > 0$ such that (3.2) holds for all $t \in (t_0 - \epsilon, t_0)$. Since $F_{z,l}$ is non-decreasing and $L_{z,l}$ is non-increasing, the set of all discontinuous points of $F_{z,l}$ and $L_{z,l}$ is a countable set at most. Thus, when t_0 is a discontinuous point of $F_{z,l}$ and $L_{z,l}$, we can choose a continuous point t_1 of $F_{z,l}$ and $F_{z,l}$ in $(t_0 - \epsilon, t_0)$ to replace t_0. Using the inequality (3.1) with $x = x_n$, $y = y_n$, we get for some $t_0 > 0$

$$\begin{aligned} (1+\alpha F_{Sx_n,Ty_n}(t_0))\, F_{Ax_n,By_n}(t_0) &> \alpha \min\left\{ \begin{array}{c} F_{Ax_n,Sx_n}(t_0)\, F_{By_n,Ty_n}(t_0), \\ F_{Sx_n,By_n}(t_0)\, F_{Ax_n,Ty_n}(t_0) \end{array} \right\} \\ &+ \min\left\{ \begin{array}{c} F_{Sx_n,Ty_n}(t_0), \\ \min\left\{F_{Ax_n,Sx_n}(\varepsilon), F_{By_n,Ty_n}\left(\frac{2}{k}t_0 - \varepsilon\right)\right\}, \\ \min\{F_{Sx_n,By_n}(2t_0 - \varepsilon), F_{Ax_n,Ty_n}(\varepsilon)\} \end{array} \right\} \end{aligned}$$

and

$$\begin{aligned}(1+\beta L_{Sx_n,Ty_n}(t_0))\, L_{Ax_n,By_n}(t_0) &< \beta \max\left\{\begin{array}{c} L_{Ax_n,Sx_n}(t_0)\, L_{By_n,Ty_n}(t_0), \\ L_{Sx_n,By_n}(t_0)\, L_{Ax_n,Ty_n}(t_0)\end{array}\right\} \\ &\quad + \max\left\{\begin{array}{c} L_{Sx_n,Ty_n}(t_0), \\ \max\left\{L_{Ax_n,Sx_n}(\varepsilon), L_{By_n,Ty_n}\left(\frac{2}{k}t_0-\varepsilon\right)\right\}, \\ \max\left\{L_{Sx_n,By_n}(2t_0-\varepsilon), L_{Ax_n,Ty_n}(\varepsilon)\right\}\end{array}\right\}\end{aligned}$$

for all $\varepsilon \in \left(0, \frac{2}{k}t_0\right)$. Letting $n \to +\infty$, we have

$$\begin{aligned}F_{z,l}(t_0) + \alpha F_{z,l}(t_0) &\geq \alpha F_{z,l}(t_0) + \min\left\{F_{l,z}\left(\frac{2}{k}t_0-\varepsilon\right), F_{l,z}(2t_0-\varepsilon)\right\} \text{ and} \\ L_{z,l}(t_0) &< \max\left\{L_{l,z}\left(\frac{2}{k}t_0-\varepsilon\right), L_{l,z}(2t_0-\varepsilon)\right\}\end{aligned}$$

As $\varepsilon \to 0$, we obtain

$$F_{z,l}(t_0) \geq F_{z,l}\left(\frac{2}{k}t_0\right) \text{ and } L_{z,l}(t_0) \leq L_{z,l}\left(\frac{2}{k}t_0\right)$$

which contradicts (3.2) and so we have $z = l$. Thus, the pairs (A, S) and (B, T) satisfy the CLR_{ST} property. □

Remark 3.13. *The converse of lemma 3.4 is not true in general, see the example 3.3 below.*

Theorem 3.1. *Let A, B, S and T be self-mappings of an intuitionistic Menger space $(X, F, L, *, \diamondsuit)$ satisfying the inequality (3.1) of lemma 3.4. If the pairs (A, S) and (B, T) satisfy the CLR_{ST} property, then (A, S) and (B, T) have coincidence points. Moreover, if (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point in X.*

Proof. Since the pairs (A, S) and (B, T) satisfy the CLR_{ST} property, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to+\infty} Ax_n = \lim_{n\to+\infty} Sx_n = \lim_{n\to+\infty} By_n = \lim_{n\to+\infty} Ty_n = z,$$

where $z \in S(X) \cap T(X)$. Hence, there exist $u, v \in X$ such that $Su = Tv = z$. Now, we show that $Au = Su = z$. As in the proof of lemma 3.4, we can prove that $Au = Su = z$ by putting $x = u$ and $y = y_n$ in the inequality (3.1). Therefore, u is a coincidence point of the pair (A, S).

Now, we assert that $Bv = Tv = z$. If $z \neq Bv$, putting $x = u$ and $y = v$ in the inequality (3.1), we get for some $t_0 > 0$

$$\begin{aligned}(1+\alpha F_{Su,Tv}(t_0))\, F_{Au,Bv}(t_0) &> \alpha F_{Bv,z}(t_0) + \min\left\{F_{Bv,z}\left(\frac{2}{k}t_0-\epsilon\right), F_{Bv,z}(2t_0-\epsilon)\right\} \text{ and} \\ (1+\beta L_{Su,Tv}(t_0))\, L_{Au,Bv}(t_0) &< \max\left\{L_{Bv,z}\left(\frac{2}{k}t_0-\epsilon\right), L_{z,Bv}(2t_0-\epsilon)\right\}\end{aligned}$$

for all $\varepsilon \in \left(0, \frac{2}{k}t_0\right)$. Letting $\varepsilon \to 0$, we have

$$F_{z,Bv}(t_0) \geq F_{z,Bv}\left(\frac{2}{k}t_0\right) \text{ and } L_{z,Bv}(t_0) \leq L_{z,Bv}\left(\frac{2}{k}t_0\right),$$

which contradicts (3.2) and so $Bv = Tv = z$. Therefore, v is a coincidence point of the pair (B, T).

Since the pair (A, S) is weakly compatible and $Au = Su$ we obtain $Az = Sz$. Now, we prove that z is a common fixed point of A and S. If $z \neq Az$, applying the inequality (3.1) with $x = z$ and $y = v$, we get for some $t_0 > 0$

$$(1+\alpha F_{Sz,Tv}(t_0))\, F_{Az,Bv}(t_0) > \alpha (F_{Az,z}(t))^2 + \min\left\{F_{Az,z}(t_0), F_{Az,z}(t_0)\right\}$$

and

$$(1+\beta L_{Sz,Tv}(t_0))\, L_{Az,Bv}(t_0) < \beta (L_{Az,z}(t_0))^2 + \max\left\{L_{Az,z}(t_0), L_{Az,z}(t_0)\right\}.$$

Hence

$$F_{Az,z}(t_0) > F_{Az,z}(t_0) \text{ and } L_{Az,z}(t_0) < L_{Az,z}(t_0),$$

which is impossible and so $Az = z = Sz$, which shows that z is a common fixed point of A and S.

Since the pair (B,T) is weakly compatible, we get $Bz = Tz$. Similarly, we can prove that z is a common fixed point of B and T. Hence, z is a common fixed point of A, B, S and T. The uniqueness of z follows easily by the inequality (3.1). □

Remark 3.14. *Theorem 3.1 improves and generalizes theorem 3.1 of [8].*

Now, we give an example to support our theorem 3.1.

Example 3.3. *Let $(X, F, L, *, \Diamond)$ be an intuitionistic Menger space, where $X = [3, 11[$, $a * b = \min\{a, b\}$ and $a\Diamond b = \max\{a, b\}$ with*

$$F_{xy}(t) = H(t - |x - y|), \quad L_{xy}(t) = G(t - |x - y|)$$

for all $x, y \in X$ and $t > 0$. Define the self-mappings A, B, S and T by

$$Ax = \begin{cases} 3 & x \in \{3\} \cup]5, 11[\\ 10 & x \in]3, 5] \end{cases}, \quad Bx = \begin{cases} 3 & x \in \{3\} \cup]5, 11[\\ 9 & x \in]3, 5] \end{cases}$$

$$Sx = \begin{cases} 3 & \text{if} \quad x = 3 \\ 7 & \text{if} \quad x \in]3, 5] \\ \frac{x+1}{2} & \text{if} \quad x \in]5, 11[\end{cases}, \quad Tx = \begin{cases} 3 & \text{if} \quad x = 3 \\ x + 4 & \text{if} \quad x \in]3, 5] \\ x - 2 & \text{if} \quad x \in]5, 11[\end{cases}.$$

We Take $\{x_n = 3\}$, $\left\{y_n = 5 + \frac{1}{n}\right\}$ or $\left\{x_n = 5 + \frac{1}{n}\right\}$, $\{y_n = 3\}$. Since

$$\lim_{n\to+\infty} Ax_n = \lim_{n\to+\infty} Sx_n = \lim_{n\to+\infty} By_n = \lim_{n\to+\infty} Ty_n = 3 \in S(X) \cap T(X),$$

then, the pairs (A, S) and (B, T) satisfy the property CLR_{ST}. Also,

$$A(X) = \{3, 10\}\,]3, 9[= T(X) \text{ and } B(X) = \{3, 9\}\, (\{7\} \cup]3, 6[) = S(X).$$

Thus, all the conditions of theorem 3.1 are satisfied and 3 is a unique common fixed point of the pairs (A, S) and (B, T).

Remark that all the mappings are even discontinuous at their unique common fixed 3. In this example $S(X)$ and $T(X)$ are not closed subsets of X.

Lemma 3.5. *Let A, B, S and T be self-mappings of an intuitionistic Menger space $(X, F, L, *, \Diamond)$ satisfying the conditions 1,2,3,4 of lemma 3.4 and*

$$\begin{aligned} &(1 + \alpha F_{Sx,Ty}(t))\, F_{Ax,By}(t) > \alpha \min\{F_{Ax,Sx}(t)\, F_{By,Ty}(t), F_{Sx,By}(t)\, F_{Ax,Ty}(t)\} \\ &\quad + \min\left\{ \begin{array}{c} F_{Sx,Ty}(t), \sup_{t_1+t_2=\frac{2}{k}t} \min\{F_{Ax,Sx}(t_1), F_{Sx,By}(t_2)\}, \\ \sup_{t_3+t_4=\frac{2}{k}t} \min\{F_{By,Ty}(t_3), F_{Ax,Ty}(t_4)\} \end{array} \right\} \\ &(1 + \beta L_{Sx,Ty}(t))\, L_{Ax,By}(t) < \beta \max\{L_{Ax,Sx}(t)\, L_{By,Ty}(t), L_{Sx,By}(t)\, L_{Ax,Ty}(t)\} \\ &\quad + \max\left\{ \begin{array}{c} L_{Sx,Ty}(t), \inf_{t_1+t_2=\frac{2}{k}t} \max\{L_{Ax,Sx}(t_1), L_{Sx,By}(t_2)\}, \\ \inf_{t_3+t_4=\frac{2}{k}t} \max\{L_{By,Ty}(t_3), L_{Ax,Ty}(t_4)\} \end{array} \right\} \end{aligned} \tag{3.4}$$

for all $x, y \in X$, $t > 0$ for some α, $\beta \geq 0$ and $1 \leq k < 2$. Then the pairs (A, S) and (B, T) satisfy the CLR_{ST} property.

Proof. As in the proof of lemma 3.4, there exists $t_0 > 0$ such that (3.2) holds. Using the inequality (3.4) with $x = x_n$, $y = y_n$ and letting $n \to +\infty$, we have for all $\varepsilon \in \left(0, \frac{2}{k}t_0\right)$.

$$\begin{aligned} F_{z,l}(t_0) \qquad & \min\left\{1, F_{l,z}\left(\frac{2}{k}t_0 - \varepsilon\right)\right\} = F_{l,z}\left(\frac{2}{k}t_0 - \varepsilon\right), \\ L_{z,l}(t_0) \qquad & \max\left\{0, L_{l,z}\left(\frac{2}{k}t_0 - \varepsilon\right)\right\} = L_{l,z}\left(\frac{2}{k}t_0 - \varepsilon\right) \end{aligned}$$

As $\varepsilon \to 0$, we obtain

$$F_{z,l}(t_0) \geq F_{z,l}\left(\frac{2}{k}t_0\right) \text{ and } L_{z,l}(t_0) \leq L_{z,l}\left(\frac{2}{k}t_0\right)$$

which contradicts (3.2) and so we have $z = l$. Thus, the pairs (A,S) and (B,T) satisfy the CLR_{ST} property. □

Theorem 3.2. *Let A, B, S and T be self-mappings of an intuitionistic Menger space $(X, F, L, *, \Diamond)$ satisfying the inequality (3.4) of lemma 3.5. If the pairs (A,S) and (B,T) satisfy the CLR_{ST} property, then (A,S) and (B,T) have coincidence points. Moreover, if (A,S) and (B,T) are weakly compatible, then A, B, S and T have a unique common fixed point in X.*

Proof. As in the proof of theorem 3.1, there exist $u, v \in X$ such that $Su = Tv = z$. Now, we show that $Au = Su = z$. If $z \neq Au$, putting $x = u$ and $y = y_n$ in the inequality (3.4) and letting $n \to +\infty$, we have for all $\varepsilon \in \left(0, \frac{2}{k}t_0\right)$.

$$F_{Au,z}(t_0)\, F_{Au,z}\left(\frac{2}{k}t_0 - \varepsilon\right) \text{ and } L_{Au,z}(t_0) \leq L_{Au,z}\left(\frac{2}{k}t_0 - \varepsilon\right).$$

Letting $\varepsilon \to 0$, we have

$$F_{Au,z}(t_0) \geq F_{Au,z}\left(\frac{2}{k}t_0\right) \text{ and } L_{Au,z}(t_0) \leq L_{Au,z}\left(\frac{2}{k}t_0\right),$$

which contradicts (3.2) and so $Au = Su = z$. Therefore, u is a coincidence point of the pair (A,S).

Now, we assert that $Bv = Tv = z$. If $z \neq Bv$, putting $x = u$ and $y = v$ in the inequality (3.4), we get for some $t_0 > 0$

$$\begin{aligned}
(1 + \alpha F_{Su,Tv}(t_0))\, F_{Au,Bv}(t_0) &> \alpha F_{z,Bv}(t_0) + F_{z,Bv}\left(\frac{2}{k}t_0 - \varepsilon\right) \text{ and} \\
(1 + \beta L_{Su,Tv}(t_0))\, L_{Au,Bv}(t_0) &< L_{z,Bv}\left(\frac{2}{k}t_0 - \varepsilon\right)
\end{aligned}$$

for all $\varepsilon \in \left(0, \frac{2}{k}t_0\right)$. As $\varepsilon \to 0$ we have

$$F_{z,Bv}(t_0) \geq F_{z,Bv}\left(\frac{2}{k}t_0\right) \text{ and } L_{z,Bv}(t_0) \leq L_{z,Bv}\left(\frac{2}{k}t_0\right),$$

which contradicts (3.2) and so $Bv = Tv = z$. Therefore, v is a coincidence point of the pair (B,T).

Since the pair (A,S) is weakly compatible and $Au = Su$ we obtain $Az = Sz$. Now, we assert that z is a common fixed point of A and S. If $z \neq Az$, applying the inequality (3.4) with $x = z$ and $y = v$, we obtain for some $t_0 > 0$

$$(1 + \alpha F_{Sz,Tv}(t_0))\, F_{Az,Bv}(t_0) > \alpha (F_{Az,z}(t_0))^2 + \min\left\{F_{Az,z}(t_0), F_{Az,z}\left(\frac{2}{k}t_0\right)\right\}$$

and

$$(1 + \beta L_{Sz,Tv}(t_0))\, L_{Az,Bv}(t_0) < \beta (L_{Az,z}(t))^2 + \max\left\{L_{Az,z}(t_0), L_{Az,z}\left(\frac{2}{k}t_0\right)\right\}$$

Hence

$$\begin{aligned}
F_{Az,z}(t_0) &> \min\left\{F_{Az,z}(t_0), F_{Az,z}\left(\frac{2}{k}t_0\right)\right\} = F_{Az,z}(t_0) \text{ and} \\
L_{Az,z}(t_0) &< \max\left\{L_{Az,z}(t_0), L_{Az,z}\left(\frac{2}{k}t_0\right)\right\} = L_{Az,z}(t_0).
\end{aligned}$$

which is impossible and so $Az = z = Sz$, which shows that z is a common fixed point of A and S. Since the pair (B,T) is weakly compatible, we get $Bz = Tz$. Similarly, we can prove that z is a common fixed point of B and T. Therefore, z is a common fixed point of A, B, S and T. The uniqueness of z follows easily by the inequality (3.4). □

Remark 3.15. *Theorem 3.2 improves and generalizes theorem 3.2 of [8] and theorem 2.1 of [10].*

If $B = A$ and $T = S$ in theorems 3.1 and 3.2, we obtain a common fixed point for a pair of self-mappings.

Applying theorems 3.1, 3.2, 3.3 and 3.4, we deduce a common fixed point for four finite families of self-mappings given by the following corollary.

Corollary 3.1. *Let $\{A_i\}_{i=1}^{m}$, $\{B_r\}_{r=1}^{n}$, $\{S_k\}_{k=1}^{p}$ and $\{T_h\}_{h=1}^{q}$ be four finite families of self-mappings of an intuitionistic Menger space $(X, F, L, *, \diamondsuit)$, where $*$ is a continuous t-norm and $\diamondsuit$ is a continuous t-conorm with $A = A_1A_2....A_m$, $B = B_1B_2....B_n$, $S = S_1S_2....S_p$ and $T = T_1T_2....T_q$ satisfying the inequality (3.1) of lemma 3.4 or the the inequality (3.4) of lemma 3.5. Suppose that the pairs (A, S) and (B, T) verify the CLR_{ST} property. Then $\{A_i\}_{i=1}^{m}$, $\{B_r\}_{r=1}^{n}$, $\{S_k\}_{k=1}^{p}$ and $\{T_h\}_{h=1}^{q}$ a unique common fixed point in X provided that the pairs of families $\left(\{A_i\}_{i=1}^{m}, \{S_k\}_{k=1}^{p}\right)$ and $\left(\{B_r\}_{r=1}^{n}, \{T_h\}_{h=1}^{q}\right)$ commute pairwise.*

By setting $A_1 = A_2 = ... = A_m = A$, $B_1 = B_2 = ... = B_n = B$, $S_1 = S_2 = ... = S_p = S$ and $T_1 = T_2 = ... = T_q = T$ in corollary 3.1, we get thai A, B, S and T have a unique common fixed point in X provided that the pairs (A^m, S^p) and (B^n, T^q) commute pairwise.

In the proof of the following lemma, we don't need to prove the inequality (3.2).

Lemma 3.6. *Let A, B, S and T be self-mappings of an intuitionistic Menger space $(X, F, L, *, \diamondsuit)$ satisfying the conditions 1,2,3,4 of lemma 3.4 and*

$$\begin{aligned}
&(1+\alpha F_{Sx,Ty}(t))\,F_{Ax,By}(t) > \alpha \min\left\{F_{Ax,Sx}(t)\,F_{By,Ty}(t), F_{Sx,By}(t)\,F_{Ax,Ty}(t)\right\}\\
&\quad + \min\left\{\begin{array}{c} F_{Sx,Ty}(t), \sup_{t_1+t_2=\frac{2}{k}t} \max\left\{F_{Ax,Sx}(t_1), F_{By,Ty}(t_2)\right\}, \\ \sup_{t_3+t_4=2t} \max\left\{F_{Sx,By}(t_3), F_{Ax,Ty}(t_4)\right\} \end{array}\right\}\\
&(1+\beta L_{Sx,Ty}(t))\,L_{Ax,By}(t) < \beta \max\left\{L_{Ax,Sx}(t)\,L_{By,Ty}(t), L_{Sx,By}(t)\,L_{Ax,Ty}(t)\right\}\\
&\quad + \max\left\{\begin{array}{c} L_{Sx,Ty}(t), \inf_{t_1+t_2=\frac{2}{k}t} \min\left\{L_{Ax,Sx}(t_1), L_{By,Ty}(t_2)\right\}, \\ \inf_{t_3+t_4=2t} \min\left\{L_{Sx,By}(t_3), L_{Ax,Ty}(t_4)\right\} \end{array}\right\}
\end{aligned} \tag{3.5}$$

for all $x, y \in X$, $t > 0$, for some $\alpha, \beta \geq 0$ and $1 \leq k < 2$. Then the pairs (A, S) and (B, T) satisfy the CLR_{ST} property.

Proof. As in the proof of lemma 3.4, $Ax_n \to z$, $Sx_n \to z$ and $Ty_n \to z$. Now, we show that $By_n \to z$. We assert that $l = z$. Assume that $l \neq z$. Using the inequality (3.5) with $x = x_n$, $y = y_n$ and letting $n \to +\infty$ we have for all $\varepsilon \in \left(0, \frac{2}{k}t_0\right)$

$$F_{z,l}(t_0)\,1 \text{ and } L_{z,l}(t_0)\,0$$

and so we have $z = l$. Thus, the pairs (A, S) and $B, T)$ satisfy the CLR_{ST} property. □

Theorem 3.3. *Let A, B, S and T be self mappings of an intuitionistic Menger space $(X, F, L, *, \diamondsuit)$ satisfying the inequality (3.5) of lemma 3.6 . If the pairs (A, S) and (B, T) satisfy the CLR_{ST} property, then (A, S) and (B, T) have coincidence points. Moreover, if (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point in X.*

Proof. As in the proof of theorem 3.1, there exist $u, v \in X$ such that $Au = Su = Bv = Tv = z$. Therefore, u is a coincidence point of the pair (A, S) and v is a coincidence point of the pair (B, T).

Since the pair (A, S) is weakly compatible and $Au = Su$ we obtain $Az = Sz$. Now, we assert that z is a common fixed point of A and S. If $z \neq Az$, applying the inequality (3.5) with $x = z$ and $y = v$, we obtain for some $t_0 > 0$

$$(1+\alpha F_{Sz,Tv}(t_0))\,F_{Az,Bv}(t_0) > \alpha (F_{Az,z}(t_0))^2 + \min\left\{F_{Az,z}(t_0), F_{Az,z}(t_0)\right\}$$

and

$$(1+\beta L_{Sz,Tv}(t_0))\,L_{Az,Bv}(t_0)\,\beta (L_{Az,z}(t_0))^2 + \max\left\{L_{Az,z}(t_0), L_{Az,z}(t_0)\right\}.$$

Hence

$$F_{Az,z}(t_0) > F_{Az,z}(t_0) \text{ and } L_{Az,z}(t_0) < L_{Az,z}(t_0),$$

which is impossible and so $Az = z = Sz$, which shows that z is a common fixed point of A and S.

Since the pair (B, T) is weakly compatible, we get $Bz = Tz$. Similarly, we can prove that z is a common fixed point of B and T by putting $x = y = z$ in the inequality (3.5). Therefore, z is a common fixed point of A, B, S and T. The uniqueness of z follows easily by the inequality (3.5). □

Let Φ be the set of all non-decreasing and continuous functions $\varphi : (0,1] \to (0,1]$ such that $\varphi(t) > t$ for all $t \in (0,1]$ and Ψ be the set of all non-increasing and continuous functions $\psi : (0,1] \to (0,1]$ such that $\psi(t) < t$ for all $t \in (0,1]$.

Lemma 3.7. *Let A, B, S and T be self-mappings of an intuitionistic Menger space $(X, F, L, *, \diamondsuit)$ satisfying the conditions 1,2,3,4 of lemma 3.4 and*

$$\begin{array}{c} (1+\alpha F_{Sx,Ty}(t))\, F_{Ax,By}(t) \geq \alpha \min\left\{F_{Ax,Sx}(t)\, F_{By,Ty}(t), F_{Sx,By}(t)\, F_{Ax,Ty}(t)\right\} \\ +\varphi\left(\min\left\{\begin{array}{c} F_{Sx,Ty}(t), \sup_{t_1+t_2=\frac{2}{k}t} \min\left\{F_{Ax,Sx}(t_1), F_{By,Ty}(t_2)\right\}, \\ \sup_{t_3+t_4=\frac{2}{k}t} \min\left\{F_{Sx,By}(t_3), F_{Ax,Ty}(t_4)\right\} \end{array}\right\}\right) \\ (1+\beta L_{Sx,Ty}(t))\, L_{Ax,By}(t) \leq \beta \max\left\{L_{Ax,Sx}(t)\, L_{By,Ty}(t), L_{Sx,By}(t)\, L_{Ax,Ty}(t)\right\} \\ \psi\left(\max\left\{\begin{array}{c} L_{Sx,Ty}(t), \inf_{t_1+t_2=\frac{2}{k}t} \max\left\{L_{Ax,Sx}(t_1), L_{By,Ty}(t_2)\right\}, \\ \inf_{t_3+t_4=\frac{2}{k}t} \max\left\{L_{Sx,By}(t_3), L_{Ax,Ty}(t_4)\right\} \end{array}\right\}\right) \end{array} \tag{3.6}$$

for all $x, y \in X$, $t > 0$, for some α, $\beta \geq 0$ and $1 \leq k < 2$, where $\varphi \in \Phi$ and $\psi \in \Psi$. Then the pairs (A,S) and (B,T) satisfy the CLR_{ST} property.

Proof. It follows as in the proof of lemma 3.4. □

Remark 3.16. *Lemmas 3.4, 3.5, 3.6 and 3.7 remain true if we assume that the pair (B,T) satisfies the CLR_T property, $B(X) \subseteq S(X)$ and $S(X)$ is a closed subset of X.*

Theorem 3.4. *Let A, B, S and T be self mappings of an intuitionistic Menger space $(X, F, L, *, \diamondsuit)$ satisfying the inequality (3.6) of lemma 3.7. If the pairs (A,S) and (B,T) satisfy the CLR_{ST} property, then (A,S) and (B,T) have coincidence points. Moreover, if (A,S) and (B,T) are weakly compatible, then A, B, S and T have a unique common fixed point in X.*

Proof. As in the proof of theorem 3.1, $z = Au = Su = Bv = Tv$. Since the pair (A,S) is weakly compatible and $Au = Su$ we obtain $Az = Sz$. Now, we assert that z is a common fixed point of A and S. If $z \neq Az$, applying the inequality (3.6) with $x = z$ and $y = v$, we obtain for some $t_0 > 0$

$$\begin{aligned} (1+\alpha F_{Sz,Tv}(t_0))\, F_{Az,Bv}(t_0) &\geq \alpha (F_{Az,z}(t_0))^2 + \\ &\quad \varphi\left(\min\left\{F_{Az,z}(t_0), \min\left\{F_{Az,z}(\epsilon), F_{Az,z}\left(\frac{2}{k}t_0 - \epsilon\right)\right\}\right\}\right) \end{aligned}$$

and

$$\begin{aligned} (1+\beta L_{Sz,Tv}(t_0))\, L_{Az,Bv}(t_0) &\leq \beta (L_{Az,z}(t_0))^2 + \\ &\quad \psi\left(\max\left\{L_{Az,z}(t_0), \max\left\{L_{Az,z}(\epsilon), L_{Az,z}\left(\frac{2}{k}t_0 - \epsilon\right)\right\}\right\}\right). \end{aligned}$$

for all $\varepsilon \in \left(0, \frac{2}{k}t_0\right)$. Letting $\varepsilon \to 0$, we get

$$\begin{aligned} F_{Az,z}(t_0) &\geq \varphi\left(\min F_{Az,z}(t_0)\right) > F_{Az,z}(t_0) \\ L_{Az,z}(t_0) &\leq \psi\left(\max L_{Az,z}(t_0)\right) < L_{Az,z}(t_0) \end{aligned}$$

which is impossible and so $Az = z = Sz$, which shows that z is a common fixed point of A and S.

Since the pair (B,T) is weakly compatible, we get $Bz = Tz$. Similarly, we can prove that z is a common fixed point of B and T by putting $x = y = z$ in the inequality (3.6). Therefore, z is a common fixed point of A, B, S and T. The uniqueness of z follows easily by the inequality (3.6). □

Remark 3.17. *Theorem 3.4 improves and generalizes theorem 26 of [5], theorem 3.2 of [6], theorem 3.1 of [7], theorems 3.1, 3.2 of [20] and theorem 1 of [28].*

Remark 3.18. *In theorems 3.1, 3.2, 3.3 and 3.4, by a similar manner, we can prove that A, B, S and T have a unique common fixed point in X if we assume that the pairs (A,S) and (B,T) verify $JCLR_{ST}$ property or CLR_{AB} property instead of CLR_{ST} property.*

Remark 3.19. *It is easy to see that theorem 3.1 remains true if we replace*

$$\sup_{t_3+t_4=2t} \min\left\{F_{Sx,By}\left(t_3\right), F_{Ax,Ty}\left(t_4\right)\right\} \text{ and } \inf_{t_3+t_4=2t} \max\left\{L_{Sx,By}\left(t_3\right), L_{Ax,Ty}\left(t_4\right)\right\}$$

in the inequality (3.1) by

$$\sup_{t_3+t_4=2t} \max\left\{F_{Sx,By}\left(t_3\right), F_{Ax,Ty}\left(t_4\right)\right\} \text{ and } \inf_{t_3+t_4=2t} \min\left\{L_{Sx,By}\left(t_3\right), L_{Ax,Ty}\left(t_4\right)\right\}$$

respectively. Also, theorems 3.2 and 3.4 remain true if we replace

$$\sup_{t_3+t_4=\frac{2}{k}t} \min\left\{F_{By,Ty}\left(t_3\right), F_{Ax,Ty}\left(t_4\right)\right\} \text{ and } \inf_{t_3+t_4=\frac{2}{k}t} \max\left\{L_{By,Ty}\left(t_3\right), L_{Ax,Ty}\left(t_4\right)\right\}$$

in the inequalities (3.4) and (3.6) by

$$\sup_{t_3+t_4=\frac{2}{k}t} \max\left\{F_{By,Ty}\left(t_3\right), F_{Ax,Ty}\left(t_4\right)\right\} \text{ and } \inf_{t_3+t_4=\frac{2}{k}t} \min\left\{L_{By,Ty}\left(t_3\right), L_{Ax,Ty}\left(t_4\right)\right\}$$

respectively.

Theorem 3.5. *Let A, B, S and T be self-mappings of an intuitionistic Menger space $(X, F, L, *, \Diamond)$ satisfying the conditions of lemma 3.4 or lemma 3.5 or lemma 3.6 or lemma 3.7. If the pairs (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point in X.*

Proof. In view of lemma 3.4, lemma 3.5, lemma 3.6 and lemma 3.7, the pairs (A, S) and (B, T) verify the CLR$_{ST}$ property, therefore there exist two sequence $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to+\infty} Ax_n = \lim_{n\to+\infty} Sx_n = \lim_{n\to+\infty} By_n = \lim_{n\to+\infty} Ty_n = z,$$

where $z \in S(X) \cap T(X)$. The rest of the proof follows as in the proof of theorems 3.1, 3.2, 3.3 and 3.4. □

Remark 3.20. *Theorem 3.5 improves and generalizes theorem 28 of [5], theorem 3.3 of [7] and theorem 2.3 of [10].*

Example 3.4. *We retain A and B and replace S and T in the example 3.3 by the following mappings*

$$Sx = \begin{cases} 3 & \text{if} \quad x = 3 \\ 6 & \text{if} \quad x \in \left]3, 5\right] \\ \dfrac{x+1}{2} & \text{if} \quad x \in [5, 11) \end{cases}, \; Tx = \begin{cases} 3 & \text{if} \quad x = 3 \\ 9 & \text{if} \quad x \in \left]3, 5\right[\\ x-2 & \text{if} \quad x \in [5, 11) \end{cases}.$$

Therefore,

$$A(X) = \{3, 4\} \subset [3, 9] = T(X) \text{ and } B(X) = \{3, 5\} \subset [3, 6] = S(X).$$

Thus, all the conditions of theorem 3.3 are satisfied and 3 *is a unique common fixed point of the pairs (A, S) and (B, T). Also, it is noted that theorem 3.1 can not be used in the context of this example as $S(X)$ and $T(X)$ are closed subsets of X.*

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