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Numerical solution of time fractional nonlinear Schrödinger equation arising in quantum mechanics by cubic B-spline finite elements

Alaattin Esen[a,*] and Orkun Tasbozan [b]

[a]epartment of Mathematics, Faculty of Science and Art, Inönü University, Malatya, 44280, TURKEY.

[b]Department of Mathematics, Faculty of Science and Art, Mustafa Kemal University, Hatay, 31000, TURKEY..

Abstract

In the present article, we are going to investigate the numerical solutions of time fractional nonlinear Schrödinger equation which is frequently encountered in quantum mechanics by using cubic B-spline collocation method. To be able to control the efficiency of the proposed method, some sample problems have been studied in this article. The outstanding purpose of the paper is to indicate that the finite element method based on the cubic B-spline collocation method approach can also be suitable for the handling of the fractional differential equations. At the end, the results of numerical experiments are compared with those of analytical solution to ensure the accuracy and efficiency of the presented scheme.

Keywords: Finite element method, collocation method, time fractional nonlinear Schrödinger equation, cubic B-Spline, fractional quantum mechanics.

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1 Introduction

The study of fractional calculus has gained more importance for the formulation of natural phenomena. This results from the fact that fractional equations instead of integer order differential equations may be used for a better modeling of natural physics process and dynamic system processes. Moreover, since they have the memory effects, fractional differential equations can more suitably describe natural processes involving memory and hereditary characteristics. However, generally speaking, derivation and application of the analytical solutions of the fractional differential equations is not so easy in most cases. Therefore, obtaining some reliable and effective methods to solve fractional differential equations has gained more importance in recent years. Recently, it has increasingly become clear that most of the phenomena in various fields of science such as engineering, physics, chemistry and many others can be accurately described by mathematical tools from fractional calculus, that is, the theory of derivatives and integrals of fractional (non-integer) order [1]. The concept of differentiation and integration to non-integer order dates back very early in history. In fact, this subject was evident almost as early as the ideas of the classical calculus were known [2]. Many authors have pointed out that derivatives and integrals of non-integer order are more suitable for the description of the behavior of various materials. It has also become clear that new fractional-order models are more adequate than previously used integer-order ones. The increasing number of fractional derivative applications in many fields of science and engineering clearly shows that there is a tremendous demand for better mathematical models of real objects, and that the fractional calculus provides one possible approach on the way to more adequate mathematical modeling of real objects and processes. Even though there are a few analytical techniques [3] for dealing with the fractional equations, as also

*Corresponding author.

E-mail address: alaattin.esen@inonu.edu.tr (Alaattin Esen), otasbozan@mku.edu.tr(Orkun Tasbozan).

happens with ordinary (non-fractional) partial differential equations, in many cases the initial condition, and/or the external force are such that the only reasonable option is to resort to numerical methods. However, although there have been an increasing number of works on this topic during the last few years [4-12], this field of applied mathematics is by far much less developed and understood than its non fractional counterpart [13]. Although there have been many methods applied to solve fractional partial differential equations, there is still a long way to go in this field. There are several definitions of a fractional derivative of order $\alpha > 0$ [14]. The two most widely utilized are the Riemann-Liouville and Caputo. The main difference between the two is in the order of evaluation. We have just started with recalling the essentials of the fractional calculus. The fractional calculus is a name for the theory of integrals and derivatives of arbitrary order, which unifies and generalizes the notions of integer-order differentiation and n-fold integration. Now, we give some basic definitions and properties of the fractional calculus theory.

Definition 1 [8]. For $\mu \in \mathbb{R}$ and $x > 0$, a real function $f(x)$, is said to be in the space C_μ if there exists a real number $p > \mu$ such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C(0, \infty)$, and for $m \in \mathbb{N}$ it is said to be in the space C_μ^m if $f^m \in C_\mu$.

Definition 2 [8]. The Riemann-Liouville fractional integral operator of order $\alpha > 0$ for a function $f(x) \in C_\mu$, $\mu \geq -1$, is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\gamma-1} f(t)dt, \ \ \alpha > 0, \ \ x > 0, \ \ J^0 f(x) = f(x).$$

Also we have the following properties:

$$J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$$
$$J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$$
$$J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}.$$

Definition 3 [8]. If m be the smallest integer that exceeds α, the Caputo time fractional derivative operator of order $\alpha > 0$ is defined as

$${}_0^C D_t^\alpha U(x,t) = \frac{\partial^\alpha U(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{\partial^m U(x,s)}{\partial s^m} (t-s)^{m-\alpha-1} ds, & m-1 < \alpha < m, m \in \mathbb{N} \\ \frac{\partial^m U(x,t)}{\partial t^m}, & m = \alpha. \end{cases}$$

The finite element method, especially, has been an important method for solving both ordinary and partial differential equations. Besides, in this paper, the finite element method is applied to solve fractional differential equation, namely time fractional telegraph equation. The main idea behind the finite element method is to divide the whole region of the given problem into an equivalent system of finite elements with associated nodes and to choose the most appropriate element type to model most closely the actual physical behavior. Thus, by means of the finite element method, a huge problem is converted into many solvable small ones. For easy implementation, those elements must be made small enough to give usable results and yet large enough to reduce computational effort [15]. In this paper, we will use cubic B-spline finite element method to obtain the numerical solutions of the time fractional nonlinear Schrödinger equation by using the $L1$ discretizaton formulae of the fractional derivative as used by Ref.[5].

The Schrödinger equation is one of the fundamental equations arising in physics for describing quantum mechanical behavior [16, 17]. It is also often called the Schrödinger wave equation and is a partial differential equation that describes how the wave function of a physical system evolves over time. The fractional Schrödinger equation is a fundamental equation of fractional quantum mechanics [18]. It was discovered by Nick Laskin [19, 20] as a result of extending the Feynman path integral,from the Brownian-like to Levy-like quantum mechanical paths. After that, he considered the fractional Schrödinger equation for some particular cases like fractional Bohratom and one-dimensional fractional oscillator [20]. Some other cases of the fractional Schrödinger equation were discussed in [8, 21–24].

2 Governing Equation

In this study, we will consider the time fractional nonlinear Schrödinger equation as a model given as follows

$$i\frac{\partial^\gamma U(x,t)}{\partial t^\gamma} + \frac{\partial^2 U(x,t)}{\partial x^2} + |U(x,t)|^2 U(x,t) = f(x,t) \tag{2.1}$$

where

$$\frac{\partial^\gamma U(x,t)}{\partial t^\gamma} = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-\tau)^{-\gamma} \frac{\partial U(x,\tau)}{\partial \tau} d\tau \ ,0<\gamma<1$$

is the fractional derivatives given in the Caputo's sense [3, 14] and $i = \sqrt{-1}$. In this paper, for the time fractional nonlinear Schrödinger equation, we are going to take the boundary conditions of the model problem (2.1) given in the interval $a \le x \le b$ as

$$U(a,t) = h_1(t)\ , \quad U(b,t) = h_2(t), \quad t \ge 0 \tag{2.2}$$

and the initial condition as

$$U(x,0) = g(x)\ , \quad a \le x \le b. \tag{2.3}$$

In the numerical solution process, to be able to obtain a finite element scheme for solving time fractional nonlinear Schrödinger equation, we will also discretize the Caputo derivative by means of the so-called $L1$ formulae [5]:

$$\frac{\partial^\gamma f(t)}{\partial t^\gamma}|_{t_m} = \frac{(\Delta t)^{-\gamma}}{\Gamma(2-\gamma)} \sum_{k=0}^{m-1} b_k^\gamma \left[f(t_{m-k}) - f(t_{m-1-k}) \right]$$

where

$$b_k^\gamma = (k+1)^{1-\gamma} - k^{1-\gamma}.$$

Since $U(x,t)$ is complex valued function, we decompose $U(x,t)$ into its real and imaginary parts $R(x,t)$ and $S(x,t)$, respectively:

$$U(x,t) = R(x,t) + iS(x,t). \tag{2.4}$$

Substituting (2.4) into (2.1), the complex Eq. (2.1) can be rewritten as a system involving two time fractional partial differential equations:

$$\begin{array}{l} \frac{\partial^\gamma S(x,t)}{\partial t^\gamma} - \frac{\partial^2 R(x,t)}{\partial x^2} - \left(R(x,t)^2 + S(x,t)^2\right) R(x,t) = -f_r(x,t) \\ \frac{\partial^\gamma R(x,t)}{\partial t^\gamma} + \frac{\partial^2 S(x,t)}{\partial x^2} + \left(R(x,t)^2 + S(x,t)^2\right) S(x,t) = f_l(x,t). \end{array} \tag{2.5}$$

where $-f_r(x,t)$ and $f_l(x,t)$ are, respectively, the real and imaginary parts of the $f(x,t)$. Also, we have initial and boundary conditions of Eq. (2.1) as follows:

$$\begin{array}{l} R(a,t) = h_{1_r}(t)\ , \quad R(b,t) = h_{2_r}(t), \quad t \ge 0 \\ S(a,t) = h_{1_l}(t)\ , \quad S(b,t) = h_{2_l}(t), \quad t \ge 0 \end{array} \tag{2.6}$$

where $h_{1_r}(t)$ and $h_{1_l}(t)$ are, respectively, the real and imaginary parts of the $h_1(t)$ and $h_{2_r}(t)$ and $h_{2_l}(t)$ are, respectively, the real and imaginary parts of the $h_2(t)$. The initial conditions as

$$R(x,0) = g_r(x)\ , \quad S(x,0) = g_l(x)\ , \quad a \le x \le b. \tag{2.7}$$

where $g_r(x)$ and $g_l(x)$ are, respectively, the real and imaginary parts of the $g(x)$.

3 Cubic B-spline Finite Element Collocation Solutions

Before solving Eq. (2.5) with boundary conditions (2.6) and initial condition (2.7) by using collocation finite element method, first of all, we define cubic B-spline base functions. Let us assume that interval $[a,b]$ is partitioned into N finite elements of uniformly equal length by knots x_m, $m = 0,1,2,...,N$ such that $a = x_0 < x_1 \cdots < x_N = b$ and $h = x_{m+1} - x_m$. Cubic B-splines $\phi_m(x)$, $(m = -1(1)N+1)$, at knots x_m are defined over interval $[a,b]$ by [25]

$$\phi_m(x) = \frac{1}{h^3} \begin{cases} (x - x_{m-2})^3, & x \in [x_{m-2}, x_{m-1}], \\ h^3 + 3h^2(x - x_{m-1}) + 3h(x - x_{m-1})^2 - 3(x - x_{m-1})^3, & x \in [x_{m-1}, x_m], \\ h^3 + 3h^2(x_{m+1} - x) + 3h(x_{m+1} - x)^2 - 3(x_{m+1} - x)^3, & x \in [x_m, x_{m+1}], \\ (x_{m+2} - x)^3, & x \in [x_{m+1}, x_{m+2}], \\ 0 & \text{otherwise.} \end{cases}$$

The set of B-splines $\{\phi_{-1}(x), \phi_0(x), \ldots, \phi_{N+1}(x)\}$ forms a basis for the functions defined over $[a, b]$. Therefore, an approximation solutions $R_N(x,t)$ and $S_N(x,t)$ can be written in terms of the cubic B-splines trial functions as:

$$\begin{array}{l} R_N(x,t) = \sum_{m=-1}^{N+1} \delta_m(t)\phi_m(x) \\ S_N(x,t) = \sum_{m=-1}^{N+1} \sigma_m(t)\phi_m(x) \end{array} \tag{3.8}$$

where $\delta_m(t)$'s and $\sigma_m(t)$'s are unknown, time dependent quantities to be determined from the boundary and cubic B-spline collocation conditions. Each cubic B-spline covers four elements so that each element $[x_m, x_{m+1}]$ is covered by four cubic B-splines. For this problem, the finite elements are identified with the interval $[x_m, x_{m+1}]$ and the elements knots x_m, x_{m+1}. Using the nodal values R_m, $R_m^{'}$ and $R_m^{''}$ given in terms of the parameter $\delta_m(t)$

$$\begin{array}{l} R_m = R(x_m,t) = \delta_{m-1}(t) + 4\delta_m(t) + \delta_{m+1}(t), \\ R_m' = R'(x_m,t) = \frac{3}{h}(-\delta_{m-1}(t) + \delta_{m+1}(t)), \\ R_m'' = R''(x_m,t) = \frac{6}{h^2}(\delta_{m-1}(t) - 2\delta_m(t) + \delta_{m+1}(t)), \end{array} \tag{3.9}$$

the variation of $R_N(x,t)$ over the typical element $[x_m, x_{m+1}]$ is given by

$$R_N(x,t) = \sum_{j=m-1}^{m+2} \delta_j(t)\phi_j(x).$$

Using the nodal values S_m, $S_m^{'}$ and $S_m^{''}$ given in terms of the parameter $\sigma_m(t)$

$$\begin{array}{l} S_m = S(x_m,t) = \sigma_{m-1}(t) + 4\sigma_m(t) + \sigma_{m+1}(t), \\ S_m' = S'(x_m,t) = \frac{3}{h}(-\sigma_{m-1}(t) + \sigma_{m+1}(t)), \\ S_m'' = S''(x_m,t) = \frac{6}{h^2}(\sigma_{m-1}(t) - 2\sigma_m(t) + \sigma_{m+1}(t)), \end{array} \tag{3.10}$$

the variation of $S_N(x,t)$ over the typical element $[x_m, x_{m+1}]$ is given by

$$S_N(x,t) = \sum_{j=m-1}^{m+2} \sigma_j(t)\phi_j(x).$$

Firstly, if we substitute the global approximations in (3.8) and its required derivatives (3.9) and (3.10) into Eq.(2.1), we easily obtain the following set of γ-th order fractional ordinary differential equations:

$$\begin{array}{l} (\dot{\sigma}_{m-1}(t) + 4\dot{\sigma}_m(t) + \dot{\sigma}_{m+1}(t)) - \frac{6}{h^2}\left(\delta_{m-1}(t) - 2\delta_m(t) + \delta_{m+1}(t)\right) \\ -Z_m\left(\delta_{m-1}(t) + 4\delta_m(t) + \delta_{m+1}(t)\right) = -f_r(x,t) \\ \left(\dot{\delta}_{m-1}(t) + 4\dot{\delta}_m(t) + \dot{\delta}_{m+1}(t)\right) + \frac{6}{h^2}\left(\sigma_{m-1}(t) - 2\sigma_m(t) + \sigma_{m+1}(t)\right) \\ +Z_m\left(\sigma_{m-1}(t) + 4\sigma_m(t) + \sigma_{m+1}(t)\right) = f_l(x,t) \end{array} \tag{3.11}$$

where $\cdot$ denotes γ^{th} fractional derivative with respect to time and

$$Z_m = R^2 + S^2.$$

If time parameters $\delta_m(t)$'s and its fractional time derivatives $\dot{\delta}_m(t)$'s in Eq. (3.11) are discretized by the Crank-Nicolson formula, $L1$ formula, respectively:

$$\delta = \frac{1}{2}(\delta^n + \delta^{n+1}), \tag{3.12}$$

$$\dot{\delta} = \frac{d^{\gamma-1}\delta}{dt^{\gamma-1}} = \frac{(\Delta t)^{-\gamma}}{\Gamma(2-\gamma)}\sum_{k=0}^{n-1}\left[(k+1)^{1-\gamma} - k^{1-\gamma}\right]\left[\delta^{n-k} - \delta^{n-k-1}\right], \tag{3.13}$$

and if time parameters $\sigma_m(t)$'s and its fractional time derivatives $\dot{\sigma}_m(t)$'s in Eq. (3.11) are discretized by the Crank-Nicolson formula, $L1$ formula, respectively:

$$\sigma = \frac{1}{2}(\sigma^n + \sigma^{n+1}), \tag{3.14}$$

$$\dot{\sigma} = \frac{d^{\gamma-1}\delta}{dt^{\gamma-1}} = \frac{(\Delta t)^{-\gamma}}{\Gamma(2-\gamma)}\sum_{k=0}^{n-1}\left[(k+1)^{1-\gamma} - k^{1-\gamma}\right]\left[\sigma^{n-k} - \sigma^{n-k-1}\right], \tag{3.15}$$

we obtain a recurrence relationship between successive time levels relating unknown parameters $\delta_m^{n+1}(t)$ and $\sigma_m^{n+1}(t)$

$$\begin{aligned}
&\sigma_{m-1}^{n+1}+4\sigma_m^{n+1}+\sigma_{m+1}^{n+1}+\left(-6\alpha-Z_m h^2\alpha\right)\delta_{m-1}^{n+1}+\left(12\alpha-4Z_m h^2\alpha\right)\delta_m^{n+1}\\
&+\left(-6\alpha-Z_m h^2\alpha\right)\delta_{m+1}^{n+1}=\sigma_{m-1}^{n}+4\sigma_m^{n}+\sigma_{m+1}^{n}+\left(6\alpha+Z_m h^2\alpha\right)\delta_{m-1}^{n}\\
&+\left(-12\alpha+4Z_m h^2\alpha\right)\delta_m^{n}+\left(6\alpha+Z_m h^2\alpha\right)\delta_{m+1}^{n}-2h^2\alpha f_r(x,t)\\
&-\sum_{k=1}^{n}b_k^{\gamma}\left[(\sigma_{m-1}^{n-k+1}-\sigma_{m-1}^{n-k})+4(\sigma_m^{n-k+1}-\sigma_m^{n-k})\ +(\sigma_{m+1}^{n-k+1}-\sigma_{m+1}^{n-k})\right]\\
&\delta_{m-1}^{n+1}+4\delta_m^{n+1}+\delta_{m+1}^{n+1}+\left(6\alpha+Z_m h^2\alpha\right)\sigma_{m-1}^{n+1}+\left(-12\alpha+4Z_m h^2\alpha\right)\sigma_m^{n+1}\\
&+\left(6\alpha+Z_m h^2\alpha\right)\sigma_{m+1}^{n+1}=\delta_{m-1}^{n}+4\delta_m^{n}+\delta_{m+1}^{n}+\left(-6\alpha-Z_m h^2\alpha\right)\sigma_{m-1}^{n}\\
&+\left(12\alpha-4Z_m h^2\alpha\right)\sigma_m^{n}+\left(-6\alpha-Z_m h^2\alpha\right)\sigma_{m+1}^{n}+2h^2\alpha f_l(x,t)\\
&-\sum_{k=1}^{n}b_k^{\gamma}\left[(\delta_{m-1}^{n-k+1}-\delta_{m-1}^{n-k})+4(\delta_m^{n-k+1}-\delta_m^{n-k})\ +(\delta_{m+1}^{n-k+1}-\delta_{m+1}^{n-k})\right]
\end{aligned}\tag{3.16}$$

where

$$Z_m=(\delta_{m-1}(t)+4\delta_m(t)+\delta_{m+1}(t))^2+(\sigma_{m-1}(t)+4\sigma_m(t)+\sigma_{m+1}(t))^2$$

and

$$\alpha=\frac{(\Delta t)^{\gamma}\Gamma(2-\gamma)}{2h^2}.$$

The iterative system (3.16) consists of $2N+2$ linear equations involving $2N+6$ unknown parameters $(\delta_{-1},\ldots,\delta_{N+1},\sigma_{-1},\ldots,\sigma_{N+1})^T$. In order to be able to obtain a unique solution to these systems, four additional constraints are needed. Those are obtained from the boundary conditions and their second derivatives and after that they are used to eliminate δ_{-1}, δ_{N+1}, σ_{-1}, σ_{N+1} from system (3.16). Using the relations

$$R_N(x,0)=\sum_{m=-1}^{N+1}\delta_m(0)\phi_m(x)$$

and

$$S_N(x,0)=\sum_{m=-1}^{N+1}\sigma_m(0)\phi_m(x)$$

together with extra conditions, which can easily be obtained from $R''(x_0,0)=R''_N(x_0,0)$ and $S''(x_0,0)=S''_N(x_0,0)$, since the second derivatives of the approximate initial conditions shall agree with those of the exact initial conditions to discard δ_{-1}, δ_{N+1}, σ_{-1}, σ_{N+1}, we obtain initial vectors δ_m^0 and σ_m^0 can be respectively obtained from the following matrix equations:

$$\begin{bmatrix} 6 & 0 & & & & \\ 1 & 4 & 1 & & & \\ & 1 & 4 & 1 & & \\ & & & \ddots & & \\ & & & 1 & 4 & 1 \\ & & & & 0 & 6 \end{bmatrix}\begin{bmatrix}\delta_0\\ \delta_1\\ \delta_2\\ \vdots\\ \delta_{N-1}\\ \delta_N\end{bmatrix}=\begin{bmatrix}R_0-\frac{h^2}{6}R''_0\\ R_1\\ R_2\\ \vdots\\ R_{N-1}\\ R_N-\frac{h^2}{6}R''_N\end{bmatrix}$$

and

$$\begin{bmatrix} 6 & 0 & & & & \\ 1 & 4 & 1 & & & \\ & 1 & 4 & 1 & & \\ & & & \ddots & & \\ & & & 1 & 4 & 1 \\ & & & & 0 & 6 \end{bmatrix}\begin{bmatrix}\sigma_0\\ \sigma_1\\ \sigma_2\\ \vdots\\ \sigma_{N-1}\\ \sigma_N\end{bmatrix}=\begin{bmatrix}S_0-\frac{h^2}{6}S''_0\\ S_1\\ S_2\\ \vdots\\ S_{N-1}\\ S_N-\frac{h^2}{6}S''_N\end{bmatrix},$$

which can be solved using an appropriate algorithm. Therefore, the approximate solution functions for $R(x,t)$ and $S(x,t)$ can be obtained from δ^n and σ^n using Eq. (3.16).

4 Numerical examples and results

Now, we are going to present a numerical example which support numerical results for time fractional nonlinear Schrödinger equation are obtained by collocation method using cubic B-spline base functions. The accuracy of the present method is measured by the error norm L_2

$$L_2 = \left\| U^{exact} - U_N \right\|_2 \simeq \sqrt{h \sum_{j=0}^{N} \left| U_j^{exact} - (U_N)_j \right|^2}$$

and the error norm L_∞

$$L_\infty = \left\| U^{exact} - U_N \right\|_\infty \simeq \max_j \left| U_j^{exact} - (U_N)_j \right|.$$

We are going to consider the time fractional nonlinear Schrödinger equation (2.1) with boundary conditions

$$U(0,t) = it^2\ , \quad U(1,t) = it^2, \quad t \geq 0$$

and initial conditions as

$$U(x,0) = 0\ , \quad 0 \leq x \leq 1.$$

The corresponding forcing term $f(x,t)$ is of the form

$$\begin{aligned} f(x,t) \qquad &= -\frac{2t^{2-\gamma}}{\Gamma(3-\gamma)} \cos(2\pi x) + (t^6 - 4\pi^2 t^2) \sin(2\pi x) \\ &+ i \left(\frac{2t^{2-\gamma}}{\Gamma(3-\gamma)} \sin(2\pi x) + (t^6 - 4\pi^2 t^2) \cos(2\pi x) \right). \end{aligned}$$

The exact solution of the problem is given by [8]

$$U(x,t) = t^2(\sin(2\pi x) + i\cos(2\pi x)).$$

A comparison of the analytical solution and numerical solutions obtained for the values of different values of γ is given in Tables 1-2. As it is clearly seen from the table, the analytical and numerical solutions obtained by the present scheme are in good agreement with each other. As the value of γ increases, the values of error norms L_2 and L_∞ decrease for $S(x,t)$ imaginary part of $U(x,t)$ and increase for $R(x,t)$ real part of $U(x,t)$. In Tables 3-4, we demonstrate the numerical results for $\gamma = 0.5$, $\Delta t = 0.002$ and $t_f = 1$ and for different number of divisions of the region. Tables 3-4 clearly show that as the number of division increases, the obtained numerical results become more accurate. We see this from the decreasing values of the error norms L_2 and L_∞. In Tables 5-6, we demonstrate the numerical results for $\gamma = 0.5$, $N = 40$ and $t_f = 0.25$ and for different number of Δt. Tables 5-6 clearly show that as the number of Δt decreases, the obtained numerical results become more accurate. We see this from the decreasing values of the error norms L_2 and L_∞. In Table 7, the error norm L_∞ of the present study are better than those in Ref. [8] at $t_f = 1$. In Figure 1, we demonstrate the graphs of numerical solutions obtained for $\gamma = 0.50$ and $N = 40$ at different time levels.

Table 1: The comparison of the exact solutions with the numerical solutions of $R(x,t)$ real part of $U(x,t)$ with $N=40$, $\Delta t=0.002$ and $t_f=1$ for different values of γ and the error norms L_2 and L_∞.

x	$\gamma=0.1$	$\gamma=0.3$	$\gamma=0.7$	$\gamma=0.9$	Exact
0.0	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.587513	0.587497	0.587474	0.587460	0.587785
0.2	0.950666	0.950628	0.950542	0.950477	0.951057
0.3	0.950637	0.950580	0.950441	0.950328	0.951057
0.4	0.587431	0.587362	0.587185	0.587038	0.587785
0.5	-0.000224	-0.000299	-0.000491	-0.000651	0.000000
0.6	-0.587870	-0.587941	-0.588122	-0.588273	-0.587785
0.7	-0.951043	-0.951102	-0.951248	-0.951366	-0.951057
0.8	-0.951018	-0.951057	-0.951148	-0.951218	-0.951057
0.9	-0.587794	-0.587811	-0.587837	-0.587853	-0.587785
1.0	0.000000	0.000000	0.000000	0.000000	0.000000
$L_2\times10^3$	0.244241	0.282432	0.388333	0.481627	
$L_\infty\times10^3$	0.421466	0.476307	0.622788	0.754269	

Table 2: The comparison of the exact solutions with the numerical solutions of $S(x,t)$ imaginary part of $U(x,t)$ with $N=40$, $\Delta t=0.002$ and $t_f=1$ for different values of γ and the error norms L_2 and L_∞.

x	$\gamma=0.1$	$\gamma=0.3$	$\gamma=0.7$	$\gamma=0.9$	Exact
0.0	1.000000	1.000000	1.000000	1.000000	1.000000
0.1	0.809065	0.809048	0.808989	0.808932	0.809017
0.2	0.309182	0.309153	0.309045	0.308944	0.309017
0.3	-0.308704	-0.308741	-0.308876	-0.309001	-0.309017
0.4	-0.808581	-0.808619	-0.808762	-0.808889	-0.809017
0.5	-0.999514	-0.999548	-0.999678	-0.999789	-1.000000
0.6	-0.808572	-0.808599	-0.808703	-0.808785	-0.809017
0.7	-0.308687	-0.308706	-0.308777	-0.308828	-0.309017
0.8	0.309202	0.309191	0.309150	0.309123	0.309017
0.9	0.809078	0.809073	0.809055	0.809045	0.809017
1.0	1.000000	1.000000	1.000000	1.000000	1.000000
$L_2\times10^3$	0.299659	0.276193	0.191137	0.132380	
$L_\infty\times10^3$	0.485920	0.451507	0.327280	0.232999	

Table 3: The comparison of the exact solutions with the numerical solutions of $R(x,t)$ imaginary part of $U(x,t)$ with $\gamma=0.5$, $\Delta t=0.002$ and $t_f=1$ for different values of N and the error norms L_2 and L_∞.

x	$N=10$	$N=20$	$N=40$	Exact
0.0	0.000000	0.000000	0.000000	0.000000
0.1	0.563396	0.582546	0.587484	0.587785
0.2	0.910919	0.942434	0.950586	0.951057
0.3	0.907609	0.941667	0.950514	0.951057
0.4	0.554170	0.580413	0.587279	0.587785
0.5	-0.014876	-0.003451	-0.000388	0.000000
0.6	-0.582572	-0.587006	-0.588025	-0.587785
0.7	-0.932052	-0.947350	-0.951170	-0.951057
0.8	-0.929195	-0.946693	-0.951101	-0.951057
0.9	-0.574077	-0.585053	-0.587825	-0.587785
1.0	0.000000	0.000000	0.000000	0.000000
$L_2\times10^3$	25.493140	5.451631	0.330630	
$L_\infty\times10^3$	43.447435	9.389941	0.542810	

Table 4: The comparison of the exact solutions with the numerical solutions of $S(x,t)$ imaginary part of $U(x,t)$ with $\gamma = 0.5$, $\Delta t = 0.002$ and $t_f = 1$ for different values of N and the error norms L_2 and L_∞.

x	$N = 10$	$N = 20$	$N = 40$	Exact
0.0	1.000000	1.000000	1.000000	1.000000
0.1	0.816537	0.810582	0.809024	0.809017
0.2	0.333872	0.314208	0.309109	0.309017
0.3	-0.262747	-0.299328	-0.308796	-0.309017
0.4	-0.745014	-0.795588	-0.808678	-0.809017
0.5	-0.928686	-0.985022	-0.999602	-1.000000
0.6	-0.743595	-0.795271	-0.808643	-0.809017
0.7	-0.260341	-0.298783	-0.308736	-0.309017
0.8	0.336396	0.314784	0.309173	0.309017
0.9	0.818142	0.810950	0.809065	0.809017
1.0	1.000000	1.000000	1.000000	1.000000
$L_2 \times 10^3$	44.135685	9.266138	0.240051	
$L_\infty \times 10^3$	71.314360	14.977938	0.399538	

Table 5: The comparison of the exact solutions with the numerical solutions of $R(x,t)$ imaginary part of $U(x,t)$ with $\gamma = 0.5$, $N = 40$ and $t_f = 0.25$ for different values of Δt and the error norms L_2 and L_∞.

	$\Delta t = 0.01$	$\Delta t = 0.005$	$\Delta t = 0.0025$	$\Delta t = 0.002$	$\Delta t = 0.001$
L_2	1.7777×10^{-3}	0.8377×10^{-3}	0.3677×10^{-3}	0.2738×10^{-3}	0.0869×10^{-3}
L_∞	2.9810×10^{-3}	1.3993×10^{-3}	0.6087×10^{-3}	0.4506×10^{-3}	0.1343×10^{-3}

Table 6: The comparison of the exact solutions with the numerical solutions of $S(x,t)$ imaginary part of $U(x,t)$ with $\gamma = 0.5$, $N = 40$ and $t_f = 0.25$ for different values of Δt and the error norms L_2 and L_∞.

	$\Delta t = 0.01$	$\Delta t = 0.005$	$\Delta t = 0.0025$	$\Delta t = 0.002$	$\Delta t = 0.001$
L_2	2.7965×10^{-3}	1.3264×10^{-3}	0.5915×10^{-3}	0.4445×10^{-3}	0.1506×10^{-3}
L_∞	4.5839×10^{-3}	2.1733×10^{-3}	0.9682×10^{-3}	0.7273×10^{-3}	0.2453×10^{-3}

Table 7: The results obtained of numerical solutions of $R(x,t)$ real part and $S(x,t)$ imaginary part of $U(x,t)$ for $N = 40$ and $\Delta t = 0.002$ by proposed method in comparison with the in Ref. [8] and exact solution at $t_f = 1$.

	$\gamma = 0.1$		$\gamma = 0.3$	
L_∞	Real Part	Imaginary Part	Real Part	Imaginary Part
Present	4.2147×10^{-4}	4.8592×10^{-4}	4.7631×10^{-4}	4.5151×10^{-4}
[8]	2.8536×10^{-3}	2.1753×10^{-3}	2.8610×10^{-3}	2.1771×10^{-3}

Figure 1: The comparison of the exact(lines) and numerical solutions for $\gamma = 0.5$, $N = 40$ and $\Delta t = 0.002$ at $t = 0.5$ (stars), $t = 0.75$ (squares), and $t = 1$ (triangles).

5 Conclusion

For last words, in the present study, numerical solutions of the time fractional nonlinear Schrödinger equation encountered in quantum mechanics based on the cubic B-spline finite element method have been calculated and presented. The time fractional derivative is considered in the form of the Caputo sense. In this study, the fractional derivative appearing in the time fractional nonlinear Schrödinger equation arising in quantum mechanics is approximated by means of the so-called $L1$ formulae. A test problem is worked out to examine the performance of the present algorithm. The performance and efficiency of the method are shown by calculating error norms L_2 and L_∞. The obtained results show that the error norms are sufficiently small during all computer runs. The obtained results also indicate that the present method is a particularly successful numerical scheme to solve the time fractional nonlinear Schrödinger equation arising in quantum mechanics. As a conclusion, in future studies, the method can efficiently be applied to this type of non-linear time fractional problems arising in physics and mathematics with success. Moreover, the method can also be applied and tested on a more wide range of other physically important equations.

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