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A common fixed point theorem for weakly subsequentially continuous mappings satisfying implicit relation

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Abstract

In this paper, we prove a common fixed point theorem for two weakly subsequentially continuous and compatible of type (E) for two pairs of self mappings, which satisfying implicit relation in metric spaces, an example is given to illustrate our results, also we give an application to solve a partial differential equations, and the study of its generalized Hyers-Ulam stability, our results improve and extend some previous results.

Keywords: Common fixed point, weakly subsequentially continuous, compatible of type (E), implicit relation.

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1 Introduction

The generalization of Banach principle, for more than one mapping was been initiated by Jungck[17], where he introduced the concept of commuting mappings to establish a common fixed point theorem for two mappings in metric spaces, Sessa[34] defined the weakly commuting mappings which is a generalization to the commuting mappings, later Jungck[18] introduced the concept of compatibility mappings in metric space, it is weaker than the last notions. After that many authors introduced various type of compatibility, compatibility of type (A),of type (B),of type (C) and of type (P) for two self mappings f and g on metric space (X, d) respectively in [19], [28],[30] and [29] as follows: the pair$\{S, T\}$ is compatible of type (A) if

$$\lim_{n\to\infty} d(STx_n, T^2x_n) = 0 \text{ and } \lim_{n\to\infty} d(TSx_n, S^2x_n) = 0,$$

S and T are compatible of type (B) if

$$\lim_{n\to\infty} d(STx_n, T^2x_n) \leq \frac{1}{2}\Big[\lim_{n\to\infty} d(STx_n, Sz) + \lim_{n\to\infty} d(Sz, S^2x_n)\Big] \text{ and}$$

$$\lim_{n\to\infty} d(TSx_n, S^2x_n) \leq \frac{1}{2}\Big[\lim_{n\to\infty} d(TSx_n, gz) + \lim_{n\to\infty} d(Tz, T^2x_n)\Big],$$

they are compatible of type (C) if

$$\lim_{n\to\infty} d(STx_n, T^2x_n) \leq \frac{1}{3}[\lim_{n\to\infty} d(STx_n, Sz) + \lim_{n\to\infty} d(Sz, T^2x_n) + \lim_{n\to\infty} d(Sz, T^2x_n)]$$

and

$$\lim_{n\to\infty} d(TSx_n, S^2x_n) \leq \frac{1}{3}[\lim_{n\to\infty} d(TSx_n, Tz) + \lim_{n\to\infty} d(Tz, T^2x_n) + \lim_{n\to\infty} d(Tz, S^2x_n)],$$

and said to be compatible of type(P) if

$$\lim_{n\to\infty} d(S^2x_n, T^2x_n) = 0,$$

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whenever in the all above definitions, $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$, for some $z \in X$.
Aamri and Moutawakil [1] defined two self maps S and T on a metric space (X, d) are said to be satisfy property (E,A), if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z,$$

for some z in X.

2 Preliminaries

Pant[26] introduced the notion of reciprocal continuity as follows:

Definition 2.1. *Self maps S and T of a metric space (X, d) are said to be reciprocally continuous, if $\lim_{n\to\infty} STx_n = St$ and $\lim_{n\to\infty} TSx_n = Tt$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, for some $t \in X$.*

In 2009, Bouhadjera and Godet Thobie [9] introduced the concept of subcompatibility and subsequential continuity as follows:
Two self-mappings S and T on a metric space (X, d) are said to be subcompatible, if there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z \text{ and } \lim_{n\to\infty} d(STx_n, TSx_n) = 0,$$

for some $t \in X$

Definition 2.2. *The pair $\{S, T\}$ is called to be subsequentially continuous, if there exists a sequence $\{x_n\}$ in X, such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, for some $t \in X$ and $\lim_{n\to\infty} STx_n = St$, $\lim_{n\to\infty} TSx_n) = Tt$.*

Now, as a generalization to the Definition 2.2, define:

Definition 2.3. *Let S and T to be two self mappings of a metric space (X, d), the pair $\{S, T\}$ is said to be weakly subsequentially continuous (shortly wsc), if there exists a sequence $\{x_n\}$, such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in X$ and $\lim_{n\to\infty} STx_n = St$, $\lim_{n\to\infty} TSx_n) = Tt$.*

Notice that subsequentially continuous or, reciprocally continuous maps are weakly subsequentially continuous, but the converse may be not.

Definition 2.4. *The pair $\{S, T\}$ is said to be S-subsequentially continuous, if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, for some $t \in X$ and $\lim_{n\to\infty} STx_n = St$.*

Definition 2.5. *The pair $\{S, T\}$ is said to be S-subsequentially continuous, if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in X$ and $\lim_{n\to\infty} TSx_n = Tt$.*

Example 2.1. *Let $X = [0, 2]$ and d is the euclidian metric, we define S, T as follows:*

$$Sx = \begin{cases} 1 + x, & 0 \le x \le 1 \\ \frac{x+1}{2}, & 1 < x \le 2 \end{cases} , \quad Tx = \begin{cases} 1 - x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2 \end{cases}$$

Clearly that S and T are discontinuous at 1.
We consider a sequence $\{x_n\}$, which defined for each $n \ge 1$ by: $x_n = \frac{1}{n}$,
clearly that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Sx_n = 1$, also we have:

$$\lim_{n\to\infty} STx_n = \lim_{n\to\infty} S(2 - \frac{1}{n}) = 2 = S(1),$$

$$\lim_{n\to\infty} TSx_n = \lim_{n\to\infty} T(2 + \frac{1}{n}) = 2 \ne T(1),$$

then $\{S, T\}$ *is S-subsequentially continuous, so it is wsc .*

On other hand, let $\{y_n\}$ *be a sequence which defined or each* $n \geq 1$ *by:* $y_n = 1 + \frac{1}{n}$*, we have*

$$\lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Ty_n = 1,$$

but

$$\lim_{n\to\infty} STy_n = \lim_{n\to\infty} S(1 + \frac{2}{n}) = 1 \neq S(1),$$

$$\lim_{n\to\infty} TSy_n = \lim_{n\to\infty} T(4 + \frac{1}{n}) = 1 \neq T(1),$$

then S and T are never reciprocally continuous.

Singh and Mahendra Singh [36] introduced the notion of compatibility of type (E), and gave some properties about this type as follows:

Definition 2.6. *Self maps S and T on a metric space* (X, d)*, are said to be compatible of type (E), if* $\lim_{n\to\infty} T^2x_n = \lim_{n\to\infty} TSx_n = St$ *and* $\lim_{n\to\infty} S^2x_n = \lim_{n\to+\infty} STx_n = Tt$*, whenever* $\{x_n\}$ *is a sequence in X, such that* $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$*, for some* $t \in X$*.*

Remark 2.1. *If* $St = Tt$*, then compatible of type (E) implies compatible (compatible of type (A), compatible of type (B), compatible of type (C), compatible of type (P)), however the converse may be not true. Generally compatibility of type (E) implies compatibility of type (B).*

Definition 2.7. *Two self maps S and T of a metric space* (X, d) *are S-compatible of type (E), if* $\lim_{n\to\infty} S^2x_n = \lim_{n\to\infty} STx_n = Tt$*, for some* $t \in X$*.*

The pair $\{S, T\}$ *is said to be T-compatible of type (E), if* $\lim_{n\to\infty} T^2x_n = \lim_{n\to\infty} TSx_n = St$*, for some* $t \in X$*.*

Notice that if S and T are compatible of type (E), then they are S-compatible and T-compatible of type (E), but the converse is not true.

Example 2.2. *Let* $X = [0, \infty)$ *endowed with the euclidian metric, we define* S, T *as follows:*

$$Sx = \begin{cases} 2, & 0 \leq x \leq 2 \\ x+1, & x > 2 \end{cases} \qquad Tx = \begin{cases} \frac{x+2}{2}, & 0 \leq x \leq 2 \\ 0, & x > 2 \end{cases}$$

Consider the sequence $\{x_n\}$ *which defined by:* $x_n = 2 - \frac{1}{n}$*, for all* $n \geq 1$*.*

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = 2,$$

$$\lim_{n\to\infty} S^2x_n = \lim_{n\to\infty} STxn) = 2 = T(2)$$

$$\lim_{n\to\infty} T^2x_n = \lim_{n\to\infty} TSxn) = 2 = S(2)$$

then $\{S, T\}$ *is compatible of type (E).*

Let $\mathcal{F}$ be the set of all continuous functions $F : \mathbb{R}^6_+ \to \mathbb{R}$, which are satisfying:

(F_1) : :F is non decreasing in t_1 and non increasing in t_2, t_3, t_4, t_5, t_6.

(F_2) : :For all $u > 0$, $F(u, u, 0, 0, u, u) > 0$.

Example 2.3.

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \alpha \max(t_2, t_3, t_4, t_5) - \beta(t_5 + t_6),$$

where $\alpha, \beta \geq 0$*, and* $\alpha + 2\beta < 1$*.*

Example 2.4.

$$F(t_1, t_2, t_3, t_4, t_5) = t_1 - k \max(t_2, t_3, t_4, \frac{t_5 + t_6}{2}),$$

where $0 \leq k < 1$*.*

Example 2.5.

$$F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \psi(\max(t_2,t_3,t_4,t_5,t_6)),$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, is increasing function such $\psi(0) = 0$ and for all $t > 0, \psi(t) < t$.

The aim of this paper is to prove the existence and the uniqueness of a common fixed point, for two pairs of self-mappings in metric space, which satisfying implicit relation, by using the weak subsequential continuity with compatibility of type (E), due to Singh et al.[36], also to support our results we give an example and an application, concerning the existence and uniqueness of a solution and the generalized Hyers-Ulam stability of a Dirichlet problem of partial differential equation, our results generalize and improve some previous results.

3 Main results

Theorem 3.1. *Let (X,d) be a metric space, A,B,S are four self mappings a on X such for all $x,y \in X$ we have:*

$$F(d(Sx,Ty),d(Ax,By),d(Ax,Sx),d(By,Ty),d(Ax,Ty),d(By,Sx)) \le 0, \tag{3.1}$$

where $F \in \mathcal{F}$, if the two pairs $\{A,S\}$ and $\{B,T\}$ are weakly subsequentially continuous (wsc) and compatible of type (E), then A,B,S and T have a unique common fixed point in X.

Proof. Since $\{A,S\}$ is wsc, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$ and $\lim_{n\to\infty} ASx_n = Az, \lim_{n\to\infty} SAx_n = Sz$ again $\{,S\}$ is compatible of type (E) implies that

$$\lim_{n\to\infty} ASx_n = \lim_{n\to\infty} A^2x_n = Sz$$

and

$$\lim_{n\to\infty} SAx_n = \lim_{n\to\infty} S^2x_n = Az,$$

consequently we obtain $Az = Sz$ and z is a coincidence point for A and S. Similarly for B and T, since $\{B,T\}$ is wsc (suppose that it is B-subsequentially continuous) there exists a sequence $\{yn\}$ such

$$\lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t$$

for some $t \in X$ and

$$\lim_{n\to\infty} BTy_n = Bt,$$

again$\{B,T\}$ is compatible of type (E), we get

$$\lim_{n\to\infty} BTy_n = \lim_{n\to\infty} B^2y_n = Tt$$

$$\lim_{n\to\infty} TBy_n = \lim_{n\to\infty} T^2y_n = Bt,$$

so we have $Bt = Tt$.
We claim $Az = Bt$, if not by using (3.1) we get:

$$F(d(Sz,Tt),d(Az,Bt),d(Az,Sz),d(Bt,Tt),d(Az,Tt),d(Bt,Sz)) =$$

$$F(d(Az,Bt),d(Az,Bt),0,0,d(Az,Bt),d(Az,Bt)) \le 0,$$

which is a contradiction with (F_2).
Now we will prove $z = Az$, if not by using(3.1) we get:

$$F(d(Sx_n,Tt),d(Ax_n,Bt),d(Ax_n,Sx_n),d(Bt,Tt),d(Ax_n,Tt),d(Bt,Sx_n)) \le 0,$$

letting $n \to \infty$ we get:

$$F(d(z,Tt),d(z,Bt),0,0,d(z,Tt),d(Bt,z)) =$$

$$F(d(z,Az),d(z,Az),0,0,d(z,Az),d(z,Az)) \le 0,$$

which is a contradiction, then $z = Az = Sz$.
Nextly we shall prove $z = t$, if not by using (3.1) we get:

$$F(d(Sx_n, Tyn), d(Ax_n, Byn), d(Ax_n, Sx_n), d(Byn, Tyn), d(Ax_n, Tyn), d(Byn, Sx_n)) \leq 0,$$

letting $n \to \infty$ we get:

$$F(d(z,t), d(z,t), 0, 0, d(z,t), d(t,z)) \leq 0,$$

which is a contradiction, then z is a fixed point for A, B, S and T. For the uniqueness suppose that there is another fixed point w and using (3.1) we get:

$$F(d(Sz, Tw), d(Az, Bw), d(Az, Sw), d(Bw, Tw), d(Az, Tw), d(Bw, Sz)) =$$

$$F(d(z,w), d(z,w), 0, 0, d(z,w), d(z,w)) \leq 0,$$

which contradicts $(F2)$, then z is unique. □

If $A = B$ and $S = T$, we obtain the following corollary:

Corollary 3.1. *Let (X, d) be a metric space and let $S, A : X \to X$ two self mappings such for all $x, y \in X$ we have:*

$$F(d(Sx, Sy), d(Ax, Ay), d(Ax, Sx), d(Ay, Sy), d(Ax, Sy), d(Ay, Sx)) \leq 0,$$

where $F \in \mathcal{F}$, assume that the pair $\{A, S\}$ is wsc A-subsequentially continuous and A-compatible of type (E), then A and S have a unique common fixed point in X.

If we combine Theorem 3.1 with Example 2.3, we obtain:

Corollary 3.2. *For four self mappings A, B, S and T on metric space (X, d) such for all $x, y \in X$ we have:*

$$d(Sx, Ty)) \leq \alpha \max\{(d(Ax, By), d(Ax, Sx), d(By, Ty), d(Ax, Ty)) + \beta d(By, Sx)),$$

where α, β are nonnegative numbers such $\alpha + 2\beta < 1$, assume that the following conditions hold:

1. *$\{A, S\}$ is A-subsequentially continuous and A-compatible of type (E),*
2. *$\{B, T\}$ is B-subsequentially continuous and B-compatible of type (E),*

then A, B, S and T have a unique common fixed point.

If we combine Theorem 3.1 with Example 2.4, we obtain:

Corollary 3.3. *For four self mappings A, B, S and T on metric space (X, d) such for all $x, y \in X$ we have:*

$$d(Sx, Ty)) \leq k \max\{(d(Ax, By), d(Ax, Sx), d(By, Ty), \frac{d(Ax, Ty) + d(By, Sx)}{2}),$$

where $0 \leq k < 1$ if the four mappings satisfying:

1. *$\{A, S\}$ is A-subsequentially continuous and A-compatible of type (E),*
2. *$\{B, T\}$ is B-subsequentially continuous and B-compatible of type (E),*

then A, B, S and T have a unique common fixed point.

If we combine Example 2.5 with Theorem 3.1, we obtain the following corollary:

Corollary 3.4. *Let (X, d) be a space metric and let A, B, S and T self mappings on X such for all $x, y \in X$ we have:*

$$d(Sx, Ty)) \leq \varphi(\max\{(d(Ax, By), d(Ax, Sx), d(By, Ty), d(Ax, Ty), d(By, Sx)),$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, is increasing function such $\psi(0) = 0$ and for all $t > 0, \psi(t) < t$.
If the following conditions are satisfied:

1. *$\{A, S\}$ is S-subsequentially continuous and S-compatible of type (E),*

2. $\{B,T\}$ *is T-subsequentially continuous and T-compatible of type (E),*

then A,B,S and T have a unique common fixed point.

Now we can obtain the same result in Theorem **??**, by using the subsequential continuity with compatibility of type (E) as follows:

Theorem 3.2. *Let (X,d) be a space metric and let A,B,S and T be four self mappings on X such for all $x,y \in X$ we have:*

$$F(d(Sx,Ty)),d(Ax,By),d(Ax,Sx),d(By,Ty),d(Ax,Ty),d(By,Sx)) \leq 0,$$

where $F \in \mathcal{F}$ if the four mappings satisfying:

1. $\{A,S\}$ *is subsequentially continuous and S-compatible (or A-compatible) of type (E),*
2. $\{B,T\}$ *is subsequentially continuous and B-compatible (or T-compatible) of type (E),*

then A,B,S and T have a unique common fixed point.

Proof. It is similar as in proof of Theorem 3.1. □

Example 3.6. *Let $X=[0,1]$ and d is the euclidian metric, we define A,B,S and T by*

$$Ax = \begin{cases} x, & 0 \leq x \leq \frac{1}{2} \\ \frac{3}{4}, & 2 < x \leq 1 \end{cases} \qquad Bx = \begin{cases} 1-x, & 0 \leq x \leq \frac{1}{2} \\ 1, & 2 < x \leq 1 \end{cases}$$

$$Sx = \begin{cases} \frac{1}{2}, & 0 \leq x \leq \frac{1}{2} \\ 0, & \frac{1}{2} < x \leq 1 \end{cases} \qquad Tx = \begin{cases} \frac{x+1}{3}, & 0 \leq x \leq \frac{1}{2} \\ \frac{1}{4}, & \frac{1}{2} < x \leq 1 \end{cases}$$

We consider a sequence $\{x_n\}$ which defined for each $n \geq 1$ by:
$x_n = \frac{1}{2} - \frac{1}{n}$, clearly that $\lim_{n\to\infty} Ax_n = \frac{1}{2}$ and $\lim_{n\to\infty} Sx_n = \frac{1}{2}$, also we have:

$$\begin{aligned} \lim_{n\to\infty} ASx_n &= A(\frac{1}{2}) \\ &= S(\frac{1}{2}) = \frac{1}{2}, \end{aligned}$$

then $\{A,S\}$ is A-subsequentially continuous and A-compatible of type (E), on the other hand consider a sequence defined by: $y_n = \frac{1}{2} - e^{-n}$, for all $n > 1$.It is clear that $\lim_{n\to\infty} By_n = \lim_{n\to\infty} Tx_n = \frac{1}{2}$, and $\lim_{n\to\infty} BTy_n = B(\frac{1}{2}) = T(\frac{1}{2}) = \frac{1}{2}$, this yields that $\{B,T\}$ is B-subsequentially continuous and B-compatible of type (E).
For the contractive condition, we have the following cases:

1. *For $x,y \in [0,\frac{1}{2}]$, we have*

$$d(Sx,Ty) = \frac{1}{6}|2y-1| \leq \frac{1}{3}|2y-1| = \frac{2}{3}d(By,Ty)$$

2. *For $x \in [0,\frac{1}{2}]$ and $y \in (\frac{1}{2},1]$, we have*

$$d(Sx,Ty) = \frac{1}{4} \leq \frac{1}{2} = \frac{2}{3}d(By,Ty)$$

3. *For $x \in (\frac{1}{2},1]$ and $y \in [0,\frac{1}{2}]$, we have*

$$d(Sx,Ty) = \frac{1}{4}(y+1) \leq \frac{1}{2} = \frac{2}{3}d(Ax,Sx)$$

4. *For $x,y \in (\frac{1}{2},1]$, we have*

$$d(Sx,Ty) = \frac{1}{4} \leq \frac{1}{2} = \frac{2}{3}d(By,Ty)$$

Consequently all hypotheses of Corollary 3.3 with $k=\frac{2}{3}$ satisfy, therefore $\frac{1}{2}$ is the unique common fixed for A,B,S and T.

4 Application

In this section, we will use Corollary 3.4 to assert the existence of the solution for a Dirichlet problem of non linear partial differential equation, which has been studied by Lazer[23] in complete metric space where he applied theorem of Matkowski [25], also it has been studied on sobolev spaces in paper[7].
Let Ω be a bounded domain in $\mathbb{R}^n$ with sufficiently smooth border $\partial\Omega$, consider the following problem:

$$\begin{cases} -\Delta u = f(t, u(t)), \\ u_{\partial\Omega} = 0, \\ \end{cases} \tag{4.2}$$

where f is continuous function on $\overline{\Omega} \times \mathbb{R}$, suppose that $\mathcal{C}(\overline{\Omega}, \mathbb{R})$ is the set of continuous functions from $\overline{\Omega}$ to $\mathbb{R}$. It is clear that the space $\mathcal{C}(\overline{\Omega}, \mathbb{R})$ endowed with the metric

$$\forall u, v \in \mathcal{C}(\overline{\Omega}, \mathbb{R}), d(u, v) = \max |u - v|,$$

is a complete metric space.
It is clear and is well know in the partial differential equations theory, that under the above conditions the problem (4.2) equivalent to the following integral equation:

$$u(t) = \int_{\Omega} G(t, u(s)) f(s, u(s) ds, \tag{4.3}$$

where G is the Green function associated to the Laplace operator.
We recall for the following definitions:

Definition 4.8. *A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a comparison one if it is increasing and $\varphi^n \to 0$ as $n \to \infty$,*

as a consequence for the last definitions, we have for all $t > 0$, $\varphi(t) < t$, $\varphi(0) = 0$ and φ is continuous at 0. Many authors study the stability in the sense of Hyers-Ulam and generalized Hyers-Ulam for the functional equations (for example see[16]) also for the integral equations and differential equations(see[3]).

Definition 4.9. *The equation (4.3) is said to be generalized Hyers-Ulam (Hyers-Ulam-Rassias) stable if there exists a comparison function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such for each $\varepsilon > 0$ and each u satisfying the inequality*

$$|u(t) - \int_{\Omega} G(t,) f(s, u(s)) ds| \leq \varepsilon,$$

there exists a solution $u^* : \Omega \to X$ of (4.3), such that

$$\|u(t) - u^*(t)\| \leq \psi(\varepsilon), \forall t \in I,$$

If $\psi(t) = ct$, for each $t \geq 0$ with $c > 0$, then we said the equation (4.3) has a Hyers-Ulam stability.

Theorem 4.3. *Assume that:*

1. $f \in \mathcal{C}(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$
2. *there exists a continuous functions $\theta : \overline{\Omega} \to \mathbb{R}_+$ such*

$$\sup_{t \in \overline{\Omega}} \int_{\Omega} \theta(s) G(t, s) ds \leq 1,$$

3. *there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such for all $s \in \overline{\Omega}$ and $u, v \in \mathcal{C}(\overline{\Omega}, \mathbb{R})$, we have*

$$|f(s, u(s)) - f(s, v(s))| \leq \theta(s) \varphi(|u - v|),$$

then the problem (4.2) have a unique solution.

In additionally if the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such $\psi(t) = t - \varphi(t)$ is bijective, then the problem (4.2) is generalized Hyers-Ulam stable.

Proof. . Consider the mapping:

$$Tx(t) = \int_{\Omega} G(s, u(s)) f(s, u(s)) ds,$$

since f is continuous on into ,then $T : \mathcal{C}(\overline{\Omega}, \mathbb{R}) \to \mathcal{C}(\overline{\Omega}, \mathbb{R})$, i.e T is a self mapping on $\mathcal{C}(\overline{\Omega}, \mathbb{R})$, and so the problem (4.3) have a solution if and only if the self-mapping T have a fixed point in $\mathcal{C}(\overline{\Omega}, \mathbb{R})$.
T is continuous, then $\{id_{\mathcal{C}(\overline{\Omega},\mathbb{R})}, S\}$ (id the identity in the space $\mathcal{C}(\overline{\Omega}, \mathbb{R})$) is subsequentially continuous and compatible of type (E), further we have:

$$|Tu(t) - Tv(t)| = |\int_{\Omega} G(t,s)(f(s,u(s)) - f(s,v(s)))ds| \leq$$

$$\int_{\Omega} |G(t,s)(f(s,u(s)) - f(s,v(s)))|ds \leq \varphi(|u - v|) \int_{\Omega} |G(t,s)\theta(s)|ds,$$

$$\leq \varphi(|u - v|) \leq \varphi(\max(d(u,v), 0, d(v,Tv), d(u,Tv), d(Tv,u)),$$

consequently all the hypotheses of Corollary 3.4 (with $A = B = S = id_{\mathcal{C}(\overline{\Omega},\mathbb{R})}$) hold, then T have a unique fixed point and so the problem (4.2) have a unique solution.
For the stability, putting $\psi(t) = t - \varphi(t)$, we get

$$d(u, u^*) \leq d(u, Tu) + d(Tu, Tu^*) \leq$$

$$\leq d(u, Tu) + \varphi(|u - u^*|)$$

then

$$d(u, u^*| \leq \psi^{-1}(d(u, Tu)) \leq \psi^{-1}(\varepsilon),$$

consequently the problem (4.2) has a generalized Hyers-Ulam stability. □

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