Malaya J. Mat. 3(4)(2015) 423–490

Malaya Journal of Matematik
www.malayajournal.org

MJM
an international journal of mathematical sciences with computer applications...

ISSN : 2319-3786

Periodic boundary value problems for singular fractional differential equations with impulse effects

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Abstract

Firstly by using iterative method, we prove existence and uniqueness of solutions of Cauchy problems of differential equations involving Caputo fractional derivative, Riemann-Liouville and Hadamard fractional derivatives with order $q \in (0,1)$. Then we obtain exact expression of solutions of impulsive fractional differential equations, i.e., exact expression of piecewise continuous solutions. Finally, four classes of integral type periodic boundary value problems of singular fractional differential equations with impulse effects are proposed. Sufficient conditions are given for the existence of solutions of these problems. We allow the nonlinearity $p(t)f(t,x)$ in fractional differential equations to be singular at $t = 0,1$ and be involved a super-linear and sub-linear term. The analysis relies on Schaefer's fixed point theorem.

Keywords: singular fractional differential system, impulsive boundary value problem, Riemann-Liouville fractional derivative, Caputo fractional derivative, Hadanard fractional derivative, Caputo type Hadamard fractional derivative, fixed point theorem.

2010 MSC: 92D25, 34A37, 34K15.

Contents

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1 Introduction

One knows that the fractional derivatives (Riemann-Liouville fractional derivative, Caputo fractional derivative and Hadamard fractional derivative and other type see [40]) are actually nonlocal operators because integrals are nonlocal operators. Moreover, calculating time fractional derivatives of a function at some time requires all the past history and hence fractional derivatives can be used for modeling systems with memory.

Fractional order differential equations are generalizations of integer order differential equations. Using fractional order differential equations can help us to reduce the errors arising from the neglected parameters in modeling real life phenomena. Fractional differential equations have many applications see Chapter 10 in [63], books [41, 63, 66].

In recent years, there have been many results obtained on the existence and uniqueness of solutions of initial value problems or boundary value problems for nonlinear fractional differential equations, see [16, 18, 58, 61, 62, 64, 71, 85, 88].

Dynamics of many evolutionary processes from various fields such as population dynamics, control theory, physics, biology, and medicine. undergo abrupt changes at certain moments of time like earthquake, harvesting, shock, and so forth. These perturbations can be well approximated as instantaneous change of states or impulses.These processes are modeled by impulsive differential equations. In 1960, Milman and Myshkis introduced impulsive differential equations in their paper [56]. Based on their work, several monographs have been published by many authors like Samoilenko and Perestyuk [67], Lakshmikantham et al. [50], Bainov and Simeonov [21, 22], Bainov and Covachev [23], and Benchohra et al. [24].

Fractional differential equation was extended to impulsive fractional differential equations, since Agarwal and Benchohra published the first paper on the topic [20] in 2008. Since then many authors such as in [8, 27, 30, 39, 42, 43, 46–49, 60, 64, 70, 71, 84] studied the existence or uniqueness of solutions of impulsive initial or boundary value problems for fractional differential equations. For examples, impulsive anti-periodic boundary value problems see [10, 11, 20, 44, 72, 73], impulsive periodic boundary value problems see [69, 79], impulsive initial value problems see [25, 29, 59, 68], two-point, three-point or multi-point impulsive boundary value problems see [9, 72, 87], impulsive boundary value problems on infinite intervals see [86].

In [31], Feckan and Zhou pointed out that the formula of solutions for impulsive fractional differential equations in [2, 7, 13, 19] is incorrect and gave their correct formula. In [76, 78], the authors established a general framework to find the solutions for impulsive fractional boundary value problems and obtained some sufficient conditions for the existence of the solutions to a kind of impulsive fractional differential equations respectively. In [75], the authors illustrated their comprehensions for the counterexample in [31] and criticized the viewpoint in [31, 76, 78]. Next, in [32], Feckan et al. expounded for the counterexample in [31] and provided further five explanations in the paper.

Recently, in [33, 78, 89], the authors studied the existence and uniqueness of solutions of the following boundary value problem of impulsive fractional differential equation

$$\begin{cases} {}^CD_{0^+}^q x(t) = f(t, x(t)), t \in (t_i, t_{i+1}], i \in \mathbb{N}_0, \\ \Delta x|_{t=t_i} = I_i(x(t_i^-)), i \in \mathbb{N}, \\ ax(0) + bx(T) = x_0, \end{cases} \tag{1.1}$$

where $q \in (0, 1]$, ${}^CD_{0^+}^q$ is the standard Caputo fractional derivative of order q, $\mathbb{N}_0 = \{0, 1, \cdots, m\}$ and $\mathbb{N} = \{1, 2, \cdots, m\}$, $f : [0, T] \times \mathbb{R} \mapsto \mathbb{R}$ is is a jointly continuous function, $I_k : \mathbb{R} \mapsto \mathbb{R}(k = 1, 2, \cdots, m)$ are continuous functions, and $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $\Delta x|_{t=t_k} = \lim_{t \to t_k^+} x(t) - \lim_{t \to t_k^-} x(t) = x(t_k^+) - x(t_k^-)$ and $x(t_k^+), x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$ respectively, a, b, x_0 a constant with $a + b \neq 0$. One knows that the boundary condition $ax(0) + bx(T) = x_0$ in (1.1) becomes $x(0) - x(T) = \frac{x_0}{a}$ when $a + b = 0$, that is so called nonhomogeneous periodic type boundary condition.

Wang and Bai [69] studied the existence and uniqueness of solutions of the following periodic boundary value problems for nonlinear impulsive fractional differential equation

$$\begin{cases} {}^{RL}D_{0^+}^\alpha x(t) - \lambda x(t) = f(t, x(t)), t \in (0, 1] \setminus \{t_1\}, \\ \lim_{t \to 0} t^{1-\alpha} x(t) = x(1), \\ \lim_{t \to t_1^+} (t - t_1)^{1-\alpha}[x(t) - x(t_1)] = I(x(t_1)), \end{cases} \tag{1.2}$$

where $\alpha \in (0,1]$, ${}^{RL}D^{\alpha}$ is the standard Riemann-Liouville fractional derivative, $\lambda \in \mathbb{R}$, $0 < t_1 < 1$, $I \in C(\mathbb{R},\mathbb{R})$, f is continuous at every point $(t,u) \in [0,1] \times \mathbb{R}$. We note that the impulse effects in (1.2) change to $\lim_{t\to t_1^+}(t-t_1)^{1-\alpha}x(t) = I(x(t_1))$ when $\alpha \in (0,1)$. The assumptions imposed on f and I are as follows: (i) there exists a constant $M > 0$ such that $|f(t,u)| \le M$ and $|I(u)| \le M$ for all $t \in [0,1]$ and $u \in \mathbb{R}$; (ii) there exist positive constant k, and l such that $|f(t,u) - f(t,v)| \le k|u-v|$ and $|I(u) - I(v)| \le l|u-v|$ for all $t \in [0,1]$ and $u, v \in \mathbb{R}$.

One knows that $\lim_{t\to t_1^+}(t-t_1)^{1-\alpha}[x(t) - x(t_1)] = I(x(t_1))$ becomes $\lim_{t\to t_1^+}(t-t_1)^{1-\alpha}x(t) = I(x(t_1))$ if $\alpha \in (0,1)$. So it is easy to know that the results can not be applied to solve the following problem

$$\begin{cases} {}^{RL}D_{0^+}^{\alpha}x(t) - \lambda x(t) = f(t,x(t)), t \in (0,1] \setminus \{t_1\}, \\ \lim_{t\to 0} t^{1-\alpha}x(t) = x(1), \\ \lim_{t\to t_1^+}(t-t_1)^{1-\alpha}x(t) - x(t_1) = J(x(t_1)), \end{cases} \tag{1.3}$$

since $I(x) = x + J(x)$ in mentioned problem corresponding to (1.3) may be unbounded. Furthermore, it seems to be difficult to generalize the method in the proof of Lemma 2.1[69] to the following problem with multiple impulse point:

$$\begin{cases} {}^{RL}D_{0^+}^{\alpha}x(t) - \lambda x(t) = f(t,x(t)), t \in (t_i,t_{i+1}], i = 0,1,2,\cdots,m, \\ \lim_{t\to 0} t^{1-\alpha}x(t) = x(1), \\ \lim_{t\to t_i^+}(t-t_i)^{1-\alpha}x(t) - x(t_i) = I(t_i,x(t_i)), i = 1,2,\cdots,m. \end{cases} \tag{1.4}$$

In a fractional differential equation, there exist two cases: the firs case is $D^{\alpha} = D_{0^+}^{\alpha}$ in (1.1) or (1.3), i.e., the fractional derivative has a unique start point. Recently, Belmekki, Nieto and Rodriguez-Lopez [17] consider the second case in which D^{α} has multiple start points, i.e., $D^{\alpha} = D_{t_i^+}^{\alpha}$. They studied the existence and uniqueness of solutions of the following periodic boundary value problem of the impulsive fractional differential equation

$$\begin{cases} {}^{RL}D_{t_i^+}^{\alpha}u(t) - \lambda u(t) = f(t,u(t)), t \in (t_i,t_{i+1}], i = 0,1,2,\cdots,p, \\ \lim_{t\to 0^+} t^{1-\alpha}u(t) = u(1), \\ \lim_{t\to t_1^+}(t-t_i)^{1-\alpha}[u(t) - u(t_i)] = I_i(u(t_i)), i = 1,2,\cdots,p, \end{cases} \tag{1.5}$$

where $\alpha \in (0,1)$, $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = 1$, $\lambda \in \mathbb{R}$, ${}^{RL}D_{t_i^+}^{\alpha}$ represents the standard Riemann-Liouville fractional derivatives, $I_i \in C(\mathbb{R},\mathbb{R})(i = 1,2,\cdots,p)$, f is continuous at every point $(t,u) \in (t_i,t_{i+1}] \times \mathbb{R}$ for $i = 0,1,2,\cdots,p$. The assumptions imposed on f and I_i are similar to those used in [69].

We observed that in the above-mentioned work, the authors all require that the nonlinear term f is bounded and continuous, if the impulse functions I_k, J_k are bounded, it is easy to see that these conditions are very strongly restrictive and difficult to satisfy in applications. We observed that in the above-mentioned work, the authors all require that the nonlinear term f is bounded and continuous, if the impulse functions I_k, J_k are bounded, it is easy to see that these conditions are very strongly restrictive and difficult to satisfy in applications. Furthermore, there has been few papers discussed the existence of solutions of the periodic boundary value problems for impulsive fractional differential equations involving other fractional derivatives such as the impulsive Hadamard type fractional differential equation

$$\begin{cases} D_{0^+}^{\alpha}x(t) - \lambda x(t) = f(t,x(t)), t \in (t_i,t_{i+1}], i = 0,1,\cdots,m, \\ \Delta x(t_i) = I(t_i,x(t_i)), i = 1,2,\cdots,m, \end{cases} \tag{1.6}$$

where $D_{0^+}^{\alpha}$ is the so called Hadamard type fractional derivative of order $\alpha \in (0,1)$.

In this paper, we will study the existence of solutions of four classes of impulsive integral type boundary value problems of singular fractional differential systems. The first one is as follows:

$$\begin{cases} {}^{RL}D_{0^+}^{\alpha}x(t) - \lambda x(t) = p(t)f(t,x(t)), t \in (t_i,t_{i+1}], i = 0,1, \\ x(1) - \lim_{t\to 0} t^{1-\alpha}x(t) = \int_0^1 \phi(s)G(s,x(s))ds, \\ \lim_{t\to t_1^+}(t-t_1)^{1-\alpha}x(t) - x(t_1) = I(t_1,x(t_1)), \end{cases} \tag{1.7}$$

where

(a) $0 < \alpha < 1$, $\lambda \in R$, ${}^{RL}D_{0^+}^{\alpha}$ is the Riemann-Liouville fractional derivative of order α,

(b) $0 = t_0 < t_1 < t_2 = 1$,

(c) $\phi : (0,1) \mapsto \mathbb{R}$ satisfy $\phi|_{(0,t_1)}, \in L^1(0,t_1), \phi|_{(t_1,1)} \in L^1(t_1,1)$,

(d) $p : (0,1) \mapsto \mathbb{R}$ satisfy the growth conditions: there exist constants k,l with $k > -1$ and $\max\{-\alpha, -k-1\} < l \le 0$ such that $|p(t)| \le t^k(1-t)^l, t \in (0,1)$,

(e) f, G defined on $(0,1] \times \mathbb{R}$ are **impulsive II-Carathéodory functions**, $I : \{t_1\} \times \mathbb{R} \to \mathbb{R}$ is a **Discrete II-Carathéodory function**.

The second one is following

$$\begin{cases} {}^{C}D_{0^+}^{\alpha}x(t) - \lambda x(t) = p(t)f(t,x(t)), t \in (t_i, t_{i+1}], i = 0,1, \\ x(1) - \lim\limits_{t \to 0^+} x(t) = \int_0^1 \phi(s)G(s,x(s))ds, \\ \lim\limits_{t \to t_1^+} x(t) - x(t_1) = I(t_1, x(t_1)), \end{cases} \tag{1.8}$$

where

(f) $0 < \alpha < 1$, $\lambda \in \mathbb{R}$, ${}^{C}D_{0^+}^{\alpha}$ is the Caputo fractional derivative of order α, t_i satisfies (b), $\phi : (0,1) \mapsto \mathbb{R}$ satisfy (c), $p : (0,1) \mapsto \mathbb{R}$ satisfy that there exist constants k,l with $k > -1$, $l \le 0, l \le 0$ with $\alpha + l > 0$, $\alpha + k + l > 0$ such that $|p(t)| \le t^k(1-t)^l$, $t \in (0,1)$,

(g) f, G defined on $(0,1] \times \mathbb{R}$ are **impulsive I-Carathéodory functions**, $I : \{t_1\} \times \mathbb{R} \mapsto \mathbb{R}$ is a **Discrete I-Carathéodory function**.

We emphasize that much work on fractional boundary value problems involves either Riemann-Liouville or Caputo type fractional differential equations see [4–6, 11]. Another kind of fractional derivatives that appears side by side to Riemann-Liouville and Caputo derivatives in the literature is the fractional derivative due to Hadamard introduced in 1892 [35], which differs from the preceding ones in the sense that the kernel of the integral (in the definition of Hadamard derivative) contains logarithmic function of arbitrary exponent. Recent studies can be seen in [12, 14, 15].

Thirdly we study the following impulsive integral type boundary value problems of singular fractional differential systems

$$\begin{cases} {}^{RLH}D_{1^+}^{\alpha}x(t) - \lambda x(t) = p(t)f(t,x(t)), t \in (t_i, t_{i+1}], i = 0,1, \\ x(e) - \lim\limits_{t \to 1^+} (\log t)^{1-\alpha}x(t) = \int_1^e \phi(s)G(s,x(s))ds, \\ \lim\limits_{t \to t_1^+} \left(\log \frac{t}{t_1}\right)^{1-\alpha} x(t) - x(t_1) = I(t_1, x(t_1)), \end{cases} \tag{1.9}$$

where

(h) $0 < \alpha < 1$, $\lambda \in R$, ${}^{RLH}D_{1^+}^{\alpha}$ is the Hadamard fractional derivative of order α,

(i) $1 = t_0 < t_1 < t_2 = e$, $\phi \in L^1(1,e)$, $p : (1,e) \mapsto \mathbb{R}$ are continuous and satisfy the growth conditions: there exist constants k,l with $k > -1$ and $\max\{-\alpha, -k-1\} < l \le 0$ such that $|p(t)| \le (\log t)^k(1-\log t)^l$, $t \in (1,e)$,

(j) f, G defined on $(1,e] \times \mathbb{R}$ are **impulsive III-Carathéodory functions**, $I : \{t_1\} \times \mathbb{R} \mapsto \mathbb{R}$ is a **Discrete III-Carathéodory function**.

Finally we study the following impulsive integral type boundary value problems of singular fractional differential systems

$$\begin{cases} {}^{CH}D_{1^+}^{\alpha}x(t) - \lambda x(t) = p(t)f(t,x(t)), t \in (t_i, t_{i+1}], i = 0,1, \\ x(e) - \lim\limits_{t \to 1^+} x(t) = \int_1^e \phi(s)G(s,x(s))ds, \\ \lim\limits_{t \to t_1^+} x(t) - x(t_1) = I(t_1, x(t_1)), \end{cases} \tag{1.10}$$

where

(k) $0 < \alpha < 1$, $\lambda \in \mathbb{R}$, ${}^{CH}D_{1^+}^{\alpha}$ is the Caputo type Hadamard fractional derivative of order α'

(l) $1 = t_0 < t_1 < t_2 = e$, $\phi \in L^1(1,e)$ and $p : (1,e) \mapsto \mathbb{R}$ satisfies that there exist constants k,l with $k > -1$ and $\max\{-\alpha, -k-\alpha\} < l \le 0$ such that $|p(t)| \le (\log t)^k(1-\log t)^l$, $t \in (1,e)$,

(m) f, G defined on $(1, e] \times \mathbb{R}$ are **impulsive I-Carathéodory functions**, $I : \{t_1\} \times \mathbb{R} \mapsto \mathbb{R}$ is a **Discrete I-Carathéodory function**.

A function $x : (0, 1] \mapsto \mathbb{R}$ is called a solution of BVP(1.7) (or BVP(1.8)) if $x|_{(t_i,t_{i+1}]}(i = 0, 1)$ is continuous, the limits below exist $\lim_{t\to t_i^+}(t - t_i)^{1-\alpha}x(t), i = 0, 1$ (or $\lim_{t\to t_i^+} x(t)(i = 0, 1)$ and x satisfies all equations in (1.7) (or (1.8)).

A function $x : (1, e] \mapsto \mathbb{R}$ is called a solution of BVP(1.9) (or BVP(1.10)) if $x|_{(t_i,t_{i+1}]}(i = 0, 1)$ is continuous, the limits below exist $\lim_{t\to t_i^+}\left(\log \frac{t}{t_i}\right)^{1-\alpha} x(t), i = 0, 1$ (or $\lim_{t\to t_i^+} x(t)(i = 0, 1)$ and x satisfies all equations in (1.9) (or (1.10)).

To get solutions of a boundary value problem of fractional differential equations, we firstly define a Banach space X, then we transform the boundary value problem into a integral equation and define a nonlinear operator T on X by using the integral equation obtained, finally, we prove that T has fixed point in X. The fixed points are just solutions of the boundary value problem. Three difficulties occur in known papers: one is how to transform the boundary value problem into a integral equation; the other one is how to define and prove a Banach space and the completely continuous property of the nonlinear operator defined; the third one is to choose a suitable fixed point theorem and impose suitable growth conditions on functions to get the fixed points of the operator.

To the best of the authors knowledge, no one has studied the existence of solutions of BVP(1.i) $(i = 7, 8, 9, 10)$. This paper fills this gap. Another purpose of this paper is to illustrate the similarity and difference of these three kinds of fractional differential equations. We obtain results on the existence of at least one solution for BVP(1.i) $(i = 7, 8, 9, 10)$ respectively. Some examples are given to illustrate the efficiency of the main theorems. For simplicity we only consider the left-sided operators here. The right-sided operators can be treated similarly.

The remainder of this paper is as follows: in Section 2, we present related definitions; in Section 3 some preliminary results are given. In Sections 4, the main theorems and their proof are given. In Section 5, a mistake happened in cited paper is showed and a corrected expression of solutions is given.

2 Related definitions

For the convenience of the readers, we firstly present the necessary definitions from the fractional calculus theory. These definitions and results can be found in the literatures [41, 63, 66].

Let the Gamma function, Beta function and the classical Mittag-Leffler special function be defined by

$$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1}e^{-x}dx, \ \ \mathbf{B}(p, q) = \int_0^1 x^{p-1}(1 - x)^{q-1}dx, \ \ E_{\delta,\sigma}(x) = \sum_{k=0}^{+\infty} \frac{x^k}{\Gamma(\delta k+\sigma)}$$

respectively for $, \alpha > 0, p > 0, q > 0, \delta > 0, \sigma > 0$. We note that $E_{\delta,\delta}(x) > 0$ for all $x \in \mathbb{R}$ and $E_{\delta,\delta}(x)$ is strictly increasing in x. Then for $x > 0$ we have $E_{\delta,\delta}(-x) < E_{\delta,\delta}(0) = \frac{1}{\Gamma(\delta)} < E_{\delta,\delta}(x)$.

Definition 2.1. *[41]. Let $c \in \mathbb{R}$. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $g : (c, \infty) \mapsto \mathbb{R}$ is given by*

$$I^\alpha_{c^+}g(t) = \frac{1}{\Gamma(\alpha)} \int_c^t (t - s)^{\alpha-1}g(s)ds,$$

provided that the right-hand side exists.

Definition 2.2. *[41]. Let $c \in \mathbb{R}$. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $g : (c, +\infty) \mapsto \mathbb{R}$ is given by*

$$^{RL}D^\alpha_{c^+}g(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_c^t \frac{g(s)}{(t-s)^{\alpha-n+1}}ds,$$

where $\alpha < n < \alpha + 1$, i.e., $n = \lceil\alpha\rceil$, provided that the right-hand side exists.

Definition 2.3. *[41]. Let $c \in \mathbb{R}$. The Caputo fractional derivative of order $\alpha > 0$ of a function $g : (c, +\infty) \mapsto \mathbb{R}$ is given by*

$$^{C}D^\alpha_{c^+}g(t) = \frac{1}{\Gamma(n-\alpha)} \int_c^t \frac{g^{(n)}(s)}{(t-s)^{\alpha-n+1}}ds,$$

where $\alpha < n < \alpha + 1$, i.e., $n = \lceil\alpha\rceil$, provided that the right-hand side exists.

Definition 2.4. *[41]. Let $c > 0$. The Hadamard fractional integral of order $\alpha > 0$ of a function $g : [c, +\infty) \mapsto \mathbb{R}$ is given by*

$$ {}^{H}I_{c^+}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_c^t \left(\log \frac{t}{s}\right)^{\alpha-1} g(s) \frac{ds}{s}, $$

provided that the right-hand side exists.

Definition 2.5. *[41]. Let $c > 0$. The Hadamard fractional derivative of order $\alpha > 0$ of a function $g : [c, +\infty) \mapsto \mathbb{R}$ is given by*

$$ {}^{RLH}D_{c^+}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \left(t\frac{d}{dt}\right)^n \int_c^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} g(s) \frac{ds}{s}, $$

where $\alpha < n < \alpha + 1$, i.e., $n = \lceil \alpha \rceil$, provided that the right-hand side exists.

Definition 2.6. *[38]. Let $c > 0$. The Caputo type Hadamard fractional derivative of order $\alpha > 0$ of a function $g : [c, +\infty) \mapsto \mathbb{R}$ is given by*

$$ {}^{CH}D_{c^+}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_c^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \left(s\frac{d}{ds}\right)^n g(s) \frac{ds}{s}, $$

where $\alpha < n < \alpha + 1$, i.e., $n = \lceil \alpha \rceil$, provided that the right-hand side exists.

Definition 2.7. *We call $F : \bigcup_{i=0}^{1} (t_i, t_{i+1}) \times \mathbb{R} \mapsto \mathbb{R}$ an* **impulsive I-Carathéodory function** *if it satisfies*

(i) *$t \mapsto F(t, u)$ is measurable on $(t_i, t_{i+1})(i = 0, 1)$ for any $u \in \mathbb{R}$,*
(ii) *$u \mapsto F(t, u)$ are continuous on $\mathbb{R}$ for all $t \in (t_i, t_{i+1})(i = 0, 1)$,*
(iii) *for each $r > 0$ there exists $M_r > 0$ such that*

$$ |F(t, u)| \le M_r, t \in (t_i, t_{i+1}), |u| \le r, (i = 0, 1). $$

Definition 2.8. *We call $F : \bigcup_{i=0}^{1} (t_i, t_{i+1}) \times \mathbb{R} \mapsto \mathbb{R}$ an* **impulsive II-Carathéodory function** *if it satisfies*

(i) *$t \mapsto F\left(t, (t - t_i)^{\alpha-1} u\right)$ is measurable on $(t_i, t_{i+1})(i = 0, 1)$ for any $u \in \mathbb{R}$,*
(ii) *$u \mapsto F\left(t, (t - t_i)^{\alpha-1} u\right)$ are continuous on $\mathbb{R}$ for all $t \in (t_i, t_{i+1})(i = 0, 1)$,*
(iii) *for each $r > 0$ there exists $M_r > 0$ such that*

$$ \left|F\left(t, (t - t_i)^{\alpha-1} u\right)\right| \le M_r, t \in (t_i, t_{i+1}), |u| \le r, (i = 0, 1). $$

Definition 2.9. *We call $F : \bigcup_{i=0}^{1} (t_i, t_{i+1}) \times \mathbb{R} \mapsto \mathbb{R}$ an* **impulsive III-Carathéodory function** *if it satisfies*

(i) *$t \mapsto F\left(t, \left(\log \frac{t}{t_i}\right)^{\alpha-1} u\right)$ is measurable on $(t_i, t_{i+1})(i = 0, 1)$ for any $u \in \mathbb{R}$,*
(ii) *$u \mapsto F\left(t, \left(\log \frac{t}{t_i}\right)^{\alpha-1} u\right)$ are continuous on $\mathbb{R}$ for all $t \in (t_i, t_{i+1})(i = 0, 1)$,*
(iii) *for each $r > 0$ there exists $M_r > 0$ such that*

$$ \left|F\left(t, \left(\log \frac{t}{t_i}\right)^{\alpha-1} u\right)\right| \le M_r, t \in (t_i, t_{i+1}), |u| \le r, (i = 0, 1). $$

Definition 2.10. *We call $I : \{t_1\} \times \mathbb{R} \mapsto \mathbb{R}$ an* **discrete I-Carathéodory function** *if it satisfies*

(i) *$u \mapsto I(t_1, u)$ are continuous on $\mathbb{R}$,*
(ii) *for each $r > 0$ there exists $M_r > 0$ such that $|I(t_1, u)| \le M_r, |u| \le r$.*

Definition 2.11. *We call $I : \{t_1\} \times \mathbb{R} \mapsto \mathbb{R}$ an* **discrete II-Carathéodory function** *if it satisfies*

(i) *$u \mapsto I\left(t_1, t_1^{\alpha-1} u\right)$ are continuous on $\mathbb{R}$,*
(ii) *for each $r > 0$ there exists $M_r > 0$ such that $\left|I\left(t_1, t_1^{\alpha-1} u\right)\right| \le M_r, |u| \le r$.*

Definition 2.12. *We call $I : \{t_1\} \times \mathbb{R} \mapsto \mathbb{R}$ an* **discrete III-Carathéodory function** *if it satisfies*

(i) *$u \mapsto I\left(t_1, (\log t_1)^{\alpha-1} u\right)$ are continuous on $\mathbb{R}$,*
(ii) *for each $r > 0$ there exists $M_r > 0$ such that $\left|I\left(t_1, (\log t_1)^{\alpha-1} u\right)\right| \le M_r, |u| \le r$.*

Definition 2.13. *[57].Let E and F be Banach spaces. A operator $T : E \mapsto F$ is called a completely continuous operator if T is continuous and maps any bounded set into relatively compact set.*

The following Banach spaces are used:

(i) *Let $a < b$ be constants. $C(a,b]$ denote the set of all continuous functions on $(a,b]$ with the limit $\lim\limits_{t\to a^+} x(t)$ existing, and the norm $||x|| = \sup\limits_{t\in(a,b]} |x(t)|$;*

(ii) *Let $a < b$ be constants. $C_{1-\alpha}(a,b]$ the set of all continuous functions on $(a,b]$ with the limit $\lim\limits_{t\to a^+}(t-a)^{1-\alpha}x(t)$ existing, the norm $||x||_{1-\alpha} = \sup\limits_{t\in(a,b]} (t-a)^{1-\alpha}|x(t)|$;*

(iii) *Let $0 < a < b$. $LC_{1-\alpha}(a,b]$ denote the set of all continuous functions on $(a,b]$ with the limit $\lim\limits_{t\to a^+} \left(\log \frac{t}{a}\right)^{1-\alpha} x(t)$ existing, and the norm $||x|| = \sup\limits_{t\in(a,b]} \left(\log \frac{t}{a}\right)^{1-\alpha} |x(t)|$.*

Let m be a positive integer and $\mathbb{N}_0 = \{0,1,2,\cdots,m\}$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$. The following Banach spaces are also used in this paper:

$$P_mC_{1-\alpha}(0,1] = \left\{x : (0,1] \mapsto \mathbb{R} : \ x|_{(t_i,t_{i+1}]} \in C_{1-\alpha}(t_i,t_{i+1}] : i \in \mathbb{N}_0 \ \right\}$$

with the norm

$$||x|| = ||x||_{P_mC_{1-\alpha}} = \max\left\{\sup_{t\in(t_i,t_{i+1}]} (t-t_i)^{1-\alpha}|x(t)| : i \in \mathbb{N}_0\right\},$$

$$P_mC(0,1] = \left\{x : (0,1] \mapsto \mathbb{R} : \ x|_{(t_i,t_{i+1}]} \in C(t_i,t_{i+1}] : i \in \mathbb{N}_0 \ \right\}$$

with the norm

$$||x|| = ||x||_{P_mC(0,1]} = \max\left\{\sup_{t\in(t_i,t_{i+1}]} |x(t)| : i \in \mathbb{N}_0\right\}.$$

Let $1 = t_0 < t_1 < \cdots < t_m < t_{m+1} = e$. We also use the Banach spaces

$$LP_mC_{1-\alpha}(1,e] = \left\{ x : (1,e] \mapsto \mathbb{R} : \begin{array}{c} x|_{(t_i,t_{i+1}]} \in C(t_i,t_{i+1}], i \in \mathbb{N}_0, \\ \text{there exist the limits} \\ \lim\limits_{t\to t_i^+} \left(\log \frac{t}{t_i}\right)^{1-\alpha} x(t), i \in \mathbb{N}_0 \end{array} \right\}$$

with the norm

$$||x|| = ||x||_{LP_mC_{1-\alpha}} = \max\left\{\sup_{t\in(t_i,t_{i+1}]} \left(\log \frac{t}{t_i}\right)^{1-\alpha} |x(t)|, i \in \mathbb{N}_0\right\},$$

$$P_mC(1,e] = \left\{x : (1,e] \mapsto \mathbb{R} : \ x|_{(t_i,t_{i+1}]} \in C(t_i,t_{i+1}], i \in \mathbb{N}_0 \ \right\}$$

with the norm

$$||x|| = ||x||_{P_mC} = \max\left\{\sup_{t\in(t_i,t_{i+1}]} |x(t)|, i \in \mathbb{N}_0\right\}.$$

3 Some preliminary results

In this section, we present some preliminary results that can be used in next sections for get solutions of BVP(1.i)(i=7,8,9,10) respectively.

3.1 Basic theory for linear fractional differential equation

Lakshmikantham et al. [51–54] investigated the basic theory of initial value problems for fractional differential equations involving Riemann-Liouville differential operators of order $q \in (0,1)$. The existence and uniqueness of solutions of the following initial value problems of fractional differential equations were discussed under the assumption that $f \in C_r[0,1]$. We will establish existence and uniqueness results for these problems under more weaker assumptions see (A1)-(A4) in the sequel.

Let $\eta \in R$, $F, A : (0,1) \mapsto \mathbb{R}$ and $B, G : (1,e) \mapsto \mathbb{R}$ are continuous functions. We will consider the following four classes of initial value problems of non-homogeneous linear fractional differential equations:

$$\begin{cases} {}^{C}D_{0^+}^{\alpha}x(t) = A(t)x(t) + F(t),\ t \in (0,1), \\ \lim\limits_{t\to 0^+} x(t) = \eta, \end{cases} \tag{3.1.1}$$

$$\begin{cases} {}^{RL}D_{0^+}^{\alpha}x(t) = A(t)x(t) + F(t),\ t \in (0,1), \\ \lim\limits_{t\to 0^+} t^{1-\alpha}x(t) = \eta, \end{cases} \tag{3.1.2}$$

$$\begin{cases} {}^{RLH}D_{0^+}^{\alpha}x(t) = B(t)x(t) + G(t),\ t \in (1,e), \\ \lim\limits_{t\to 1^+} (\log t)^{1-\alpha}\, x(t) = \eta, \end{cases} \tag{3.1.3}$$

$$\begin{cases} {}^{CH}D_{0^+}^{\alpha}x(t) = B(t)x(t) + G(t),\ t \in (1,e), \\ \lim\limits_{t\to 1^+} x(t) = \eta, \end{cases} \tag{3.1.4}$$

To get solutions of (3.1.1), we need the following assumptions:

(A1) there exists constants $k_i > -1$, $l_i \le 0$ with $l_i > \{-\alpha, -\alpha - k_i\}$ $(i = 1,2)$, $M_A \ge 0$ and $M_F \ge 0$ such that $|A(t)| \le M_A t^{k_1}(1-t)^{l_1}$ and $|F(t)| \le M_F t^{k_2}(1-t)^{l_2}$ for all $t \in (0,1)$.

Choose Picard function sequence as

$$\phi_0(t) = \eta,\ t \in (0,1],$$

$$\phi_n(t) = \eta + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)\phi_{n-1}(s) + F(s)]ds, t \in (0,1], n = 1,2,\cdots.$$

Claim 1. $\phi_n \in C(0,1]$. One sees $\phi_0 \in C(0,1]$. Then ϕ_1 is continuous on $(0,1]$, together with

$$\left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)\phi_0(s) + F(s)]ds\right| \le ||x|| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|A(s)|ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|F(s)|ds$$

$$\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[M_A|\eta|s^{k_1}(1-s)^{l_1} + M_F s^{k_2}(1-s)^{l_2}]ds$$

$$\le M_A|\eta| \int_0^t \frac{(t-s)^{\alpha+l_1-1}}{\Gamma(\alpha)}s^{k_1}ds + M_F \int_0^t \frac{(t-s)^{\alpha+l_2-1}}{\Gamma(\alpha)}s^{k_2}ds$$

$$= M_A|\eta|t^{\alpha+k_1+l_1} \int_0^1 \frac{(1-w)^{\alpha+l_1-1}}{\Gamma(\alpha)}w^{k_1}dw + M_F t^{\alpha+k_2+l_2} \int_0^1 \frac{(1-w)^{\alpha+l_2-1}}{\Gamma(\alpha)}w^{k_2}dw$$

$$= M_A|\eta|t^{\alpha+k_1+l_1}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)} + M_F t^{\alpha+k_2+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)} \to 0 \text{ as } t \to 0^+,$$

we see that $\lim\limits_{t\to 0^+} \phi_1(t)$ exists. So $\phi_1 \in C(0,1]$. By mathematical induction method, we can prove that $\phi_n \in C(0,1]$.

Claim 2. $\{\phi_n\}$ is convergent uniformly on $(0,1]$. In fact we have for $t \in (0,1]$ that

$$|\phi_1(t) - \phi_0(t)| = \left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)\phi_0(s) + F(s)]ds\right|$$

$$\le M_A|\eta| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^{k_1}(1-s)^{l_1}ds + M_F \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^{k_2}(1-s)^{l_2}ds$$

$$\le M_A|\eta| \int_0^t \frac{(t-s)^{\alpha+l_1-1}}{\Gamma(\alpha)}s^{k_1}ds + M_F \int_0^t \frac{(t-s)^{\alpha+l_2-1}}{\Gamma(\alpha)}s^{k_2}ds$$

$$= M_A|\eta|t^{\alpha+k_1+l_1}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)} + M_F t^{\alpha+k_2+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}.$$

So

$$|\phi_2(t)-\phi_1(t)| = \left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}A(s)[\phi_1(s)-\phi_0(s)]ds\right|$$

$$\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}M_A s^{k_1}(1-s)^{l_1}\left(M_A|\eta|s^{\alpha+k_1+l_1}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}+M_F s^{\alpha+k_2+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\right)ds$$

$$\leq |\eta|M_A^2\int_0^t \frac{(t-s)^{\alpha+l_1-1}}{\Gamma(\alpha)}s^{\alpha+2k_1+l_1}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}ds$$

$$+M_AM_F\int_0^t \frac{(t-s)^{\alpha+l_1-1}}{\Gamma(\alpha)}s^{\alpha+k_1+k_2+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}ds$$

$$= |\eta|M_A^2 t^{2\alpha+2k_1+2l_1}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,\alpha+2k_1+l_1+1)}{\Gamma(\alpha)}$$

$$+M_AM_F t^{2\alpha+k_1+k_2+l_1+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,\alpha+k_1+k_2+l_2+1)}{\Gamma(\alpha)}.$$

Now suppose that

$$|\phi_j(t)-\phi_{j-1}(t)| \leq |\eta|M_A^j t^{j\alpha+jk_1+jl_1}\prod_{i=0}^{j-1}\frac{\mathbf{B}(\alpha+l_1,i\alpha+(i+1)k_1+il_1+1)}{\Gamma(\alpha)}$$

$$+M_A^{j-1}M_F t^{j\alpha+(j-1)k_1+k_2+(j-1)l_1+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\prod_{i=1}^{j-1}\frac{\mathbf{B}(\alpha+l_1,i\alpha+ik_1+k_2+(i-1)l_1+l_2+1)}{\Gamma(\alpha)}.$$

We get that

$$|\phi_{j+1}(t)-\phi_j(t)| = \left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}A(s)[\phi_j(s)-\phi_{j-1}(s)]ds\right|$$

$$\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}M_A\left(|\eta|M_A^j s^{j\alpha+jk_1+jl_1}\prod_{i=0}^{j-1}\frac{\mathbf{B}(\alpha+l_1,i\alpha+(i+1)k_1+il_1+1)}{\Gamma(\alpha)}\right.$$

$$\left.+M_A^{j-1}M_F s^{j\alpha+(j-1)k_1+k_2+(j-1)l_1+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\prod_{i=1}^{j-1}\frac{\mathbf{B}(\alpha+l_1,i\alpha+ik_1+k_2+(i-1)l_1+l_2+1)}{\Gamma(\alpha)}\right)s^{k_1}(1-s)^{l_1}ds$$

$$\leq \int_0^t \frac{(t-s)^{\alpha+l_1-1}}{\Gamma(\alpha)}M_A\left(|\eta|M_A^j s^{j\alpha+jk_1+jl_1}\prod_{i=0}^{j-1}\frac{\mathbf{B}(\alpha+l_1,i\alpha+(i+1)k_1+il_1+1)}{\Gamma(\alpha)}\right.$$

$$\left.+M_A^{j-1}M_F s^{j\alpha+(j-1)k_1+k_2+(j-1)l_1+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\prod_{i=1}^{j-1}\frac{\mathbf{B}(\alpha+l_1,i\alpha+ik_1+k_2+(i-1)l_1+l_2+1)}{\Gamma(\alpha)}\right)s^{k_1}ds$$

$$\leq |\eta|M_A^{j+1}t^{(j+1)\alpha+(j+1)k_1+(j+1)l_1}\prod_{i=0}^{j}\frac{\mathbf{B}(\alpha+l_1,i\alpha+(i+1)k_1+il_1+1)}{\Gamma(\alpha)}$$

$$+M_A^jM_F t^{(j+1)\alpha+jk_1+k_2+jl_1+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\prod_{i=1}^{j}\frac{\mathbf{B}(\alpha+l_1,i\alpha+ik_1+k_2+(i-1)l_1+l_2+1)}{\Gamma(\alpha)}.$$

From the mathematical induction method, we get for every $n=1,2,\cdots$ that

$$|\phi_{n+1}(t)-\phi_n(t)| \leq |\eta|M_A^{n+1}t^{(n+1)\alpha+(n+1)k_1+(n+1)l_1}\prod_{i=0}^{n}\frac{\mathbf{B}(\alpha+l_1,i\alpha+(i+1)k_1+il_1+1)}{\Gamma(\alpha)}$$

$$+M_A^nM_F t^{(n+1)\alpha+nk_1+k_2+nl_1+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\prod_{i=1}^{n}\frac{\mathbf{B}(\alpha+l_1,i\alpha+ik_1+k_2+(i-1)l_1+l_2+1)}{\Gamma(\alpha)}$$

$$\leq |\eta|M_A^{n+1}\prod_{i=0}^{n}\frac{\mathbf{B}(\alpha+l_1,i\alpha+(i+1)k_1+il_1+1)}{\Gamma(\alpha)}$$

$$+M_A^nM_F\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\prod_{i=1}^{n}\frac{\mathbf{B}(\alpha+l_1,i\alpha+ik_1+k_2+(i-1)l_1+l_2+1)}{\Gamma(\alpha)}, t\in[0,1].$$

Consider

$$\sum_{n=1}^{+\infty} u_n = \sum_{n=1}^{+\infty} |\eta| M_A^{n+1} \prod_{i=0}^{n} \frac{\mathbf{B}(\alpha+l_1, i\alpha+(i+1)k_1+il_1+1)}{\Gamma(\alpha)},$$

$$\sum_{n=1}^{+\infty} v_n = \sum_{n=1}^{+\infty} M_A^n M_F \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)} \prod_{i=1}^{n} \frac{\mathbf{B}(\alpha+l_1, i\alpha+ik_1+k_2+(i-1)l_1+l_2+1)}{\Gamma(\alpha)}.$$

One sees for sufficiently large n that

$$\frac{u_{n+1}}{u_n} = M_A \frac{\mathbf{B}(\alpha+l_1,(n+1)\alpha+(n+1)k_1+(n+1)l_1)}{\Gamma(\alpha)} = M_A \int_0^1 (1-x)^{\alpha+l_1-1} x^{(n+1)\alpha+(n+1)k_1+(n+1)l_1} dx$$

$$\leq M_A \int_0^\delta (1-x)^{\alpha+l_1-1} x^{(n+1)\alpha+(n+1)k_1+(n+1)l_1} dx + M_A \int_\delta^1 (1-x)^{\alpha+l_1-1} dx \text{ with } \delta \in (0,1)$$

$$\leq M_A \int_0^\delta (1-x)^{\alpha+l_1-1} dx \delta^{(n+1)\alpha+(n+1)k_1+(n+1)l_1} + \frac{M_A}{\alpha+l_1}\delta^{\alpha+l_1}$$

$$\leq \frac{M_A}{\alpha+l_1}\delta^{(n+1)\alpha+(n+1)k_1+(n+1)l_1} + \frac{M_A}{\alpha+l_1}\delta^{\alpha+l_1}.$$

For any $\epsilon > 0$, it is easy to see that there exists $\delta \in (0,1)$ such that $\frac{M_a}{\alpha+l_1}\delta^{\alpha+l_1} < \frac{\epsilon}{2}$. For this δ, there exists an integer $N > 0$ sufficiently large such that $\frac{M_a}{\alpha+l_1}\delta^{(n+1)\alpha+(n+1)k_1+(n+1)l_1} < \frac{\epsilon}{2}$ for all $n > N$. So $0 < \frac{u_{n+1}}{u_n} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ for all $n > N$. It follows that $\lim_{n\to+\infty} \frac{u_{n+1}}{u_n} = 0$. Then $\sum_{n=1}^{+\infty} u_n$ is convergent. Similarly we get $\sum_{n=1}^{+\infty} v_n$ is convergent. Hence

$$\phi_0(t) + [\phi_1(t) - \phi_0(t)] + [\phi_2(t) - \phi_1(t)] + \cdots + [\phi_n(t) - \phi_{n-1}(t)] + \cdots, t \in [0,1]$$

is uniformly convergent. Then $\{\phi_n(t)\}$ is convergent uniformly on $(0,1]$.

Claim 3. $\phi(t) = \lim_{n\to+\infty} \phi_n(t)$ defined on $(0,1]$ is a unique continuous solution of the integral equation

$$x(t) = \eta + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}[A(s)x(s) + F(s)]ds, t \in (0,1]. \ l \tag{3.1.5}$$

Proof. By $\phi(t) = \lim_{n\to+\infty} \phi_n(t)$ and the uniformly convergence, we see $\phi(t)$ is continuous on $[0,1]$ by definging $x(t)|_{t=0} = \lim_{t\to 0^+} x(t)$. From

$$\left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)\phi_{n-1}(s) + F(s)]ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)\phi_{m-1}(s) + F(s)]ds \right|$$

$$\leq M_A ||\phi_{n-1} - \phi_{m-1}|| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{k_1}(1-s)^{l_1} ds$$

$$\leq M_A ||\phi_{n-1} - \phi_{m-1}|| t^{\alpha+k_1+l_1} \frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}$$

$$\leq M_A ||\phi_{n-1} - \phi_{m-1}|| \frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)} \to 0 \text{ as } m,n \to +\infty,$$

it follows that

$$\phi(t) = \lim_{n\to+\infty} \phi_n(t) = \lim_{n\to+\infty} \left[\eta + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)\phi_{n-1}(s) + F(s)]ds\right]$$

$$= \eta + \lim_{n\to+\infty} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)\phi_{n-1}(s) + F(s)]ds$$

$$= \eta + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left[A(s) \lim_{n\to+\infty} \phi_{n-1}(s) + F(s)\right] ds$$

$$= \eta + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} [A(s)\phi(s) + F(s)] ds.$$

Then ϕ is a continuous solution of (3.1.5) defined on $(0,1]$.

Suppose that ψ defined on $(0,1]$ is also a solution of (3.1.5). Then

$$\psi(t) = \eta + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)\psi(s) + F(s)]ds, t \in (0,1].$$

We need to prove that $\phi(t) \equiv \psi(t)$ on $[0,1]$. Then

$$|\psi(t)-\phi_0(t)| = \left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|A(s)\psi(s)+F(s)|ds\right|$$

$$\leq |\eta| M_A t^{\alpha+k_1+l_1}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)} + M_F t^{\alpha+k_2+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}.$$

Furthermore, we have

$$|\psi(t)-\phi_1(t)| = \left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}A(s)[\psi(s)-\phi_0(s)]ds\right|$$

$$\leq |\eta| M_A^2 t^{2\alpha+2k_1+2l_1}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,\alpha+2k_1+l_1+1)}{\Gamma(\alpha)}$$

$$+M_A M_F t^{2\alpha+k_1+k_2+l_1+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,\alpha+k_1+k_2+l_2+1)}{\Gamma(\alpha)}.$$

Now suppose that

$$|\psi(t)-\phi_{j-1}(t)| \leq |\eta| M_A^j t^{j\alpha+jk_1+jl_1}\prod_{i=0}^{j-1}\frac{\mathbf{B}(\alpha+l_1,i\alpha+(i+1)k_1+il_1+1)}{\Gamma(\alpha)}$$

$$+M_A^{j-1}M_F t^{j\alpha+(j-1)k_1+k_2+(j-1)l_1+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\prod_{i=1}^{j-1}\frac{\mathbf{B}(\alpha+l_1,i\alpha+ik_1+k_2+(i-1)l_1+l_2+1)}{\Gamma(\alpha)}.$$

Then

$$|\psi(t)-\phi_j(t)| = \left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}A(s)[\psi(s)-\phi_{j-1}(s)]ds\right|$$

$$\leq |\eta| M_A^{j+1} t^{(j+1)\alpha+(j+1)k_1+(j+1)l_1}\prod_{i=0}^{j}\frac{\mathbf{B}(\alpha+l_1,i\alpha+(i+1)k_1+il_1+1)}{\Gamma(\alpha)}$$

$$+M_A^{j}M_F t^{(j+1)\alpha+jk_1+k_2+jl_1+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\prod_{i=1}^{j}\frac{\mathbf{B}(\alpha+l_1,i\alpha+ik_1+k_2+(i-1)l_1+l_2+1)}{\Gamma(\alpha)}.$$

Hence

$$|\psi(t)-\phi_n(t)| \leq |\eta| M_A^{n+1} t^{(n+1)\alpha+(n+1)k_1+(n+1)l_1}\prod_{i=0}^{n}\frac{\mathbf{B}(\alpha+l_1,i\alpha+(i+1)k_1+il_1+1)}{\Gamma(\alpha)}$$

$$+M_A^{n}M_F t^{(n+1)\alpha+nk_1+k_2+nl_1+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\prod_{i=1}^{n}\frac{\mathbf{B}(\alpha+l_1,i\alpha+ik_1+k_2+(i-1)l_1+l_2+1)}{\Gamma(\alpha)}$$

$$\leq |\eta| M_A^{n+1}\prod_{i=0}^{n}\frac{\mathbf{B}(\alpha+l_1,i\alpha+(i+1)k_1+il_1+1)}{\Gamma(\alpha)}$$

$$+M_A^{n}M_F\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\prod_{i=1}^{n}\frac{\mathbf{B}(\alpha+l_1,i\alpha+ik_1+k_2+(i-1)l_1+l_2+1)}{\Gamma(\alpha)} \text{ for all } n=1,2,\cdots.$$

Similarly we have

$$\lim_{n\to+\infty}|\eta| M_A^{n+1}\prod_{i=0}^{n}\frac{\mathbf{B}(\alpha+l_1,i\alpha+(i+1)k_1+il_1+1)}{\Gamma(\alpha)} = 0,$$

$$\lim_{n\to+\infty}M_A^{n}M_F\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\prod_{i=1}^{n}\frac{\mathbf{B}(\alpha+l_1,i\alpha+ik_1+k_2+(i-1)l_1+l_2+1)}{\Gamma(\alpha)} = 0.$$

Then $\lim_{n\to+\infty}\phi_n(t) = \psi(t)$ uniformly on $(0,1]$. Then $\phi(t) \equiv \psi(t)$. Then (3.1.5) has a unique solution ϕ. The proof is complete.

Theorem 3.1. *Suppose that (A1) holds. Then x is a solution of IVP(3.1.1) if and only if x is a solution of the integral equation (3.1.5).*

Proof. Suppose that $x \in C(0,1]$ is a solution of IVP(3.1.1). Then $\lim_{t\to 0^+} x(t) = \eta$ and $||x|| = r < +\infty$, From (A1), we have

$$\left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds\right| \le ||x|| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|A(s)|ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|F(s)|ds$$

$$\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[M_A r s^{k_1}(1-s)^{l_1} + M_F s^{k_2}(1-s)^{l_2}]ds$$

$$\le M_A r \int_0^t \frac{(t-s)^{\alpha+l_1-1}}{\Gamma(\alpha)} s^{k_1} ds + M_F \int_0^t \frac{(t-s)^{\alpha+l_2-1}}{\Gamma(\alpha)} s^{k_2} ds$$

$$= M_A r t^{\alpha+k_1+l_1} \int_0^1 \frac{(1-w)^{\alpha+l_1-1}}{\Gamma(\alpha)} w^{k_1} dw + M_F t^{\alpha+k_2+l_2} \int_0^1 \frac{(1-w)^{\alpha+l_2-1}}{\Gamma(\alpha)} w^{k_2} dw$$

$$= M_A r t^{\alpha+k_1+l_1} \frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)} + M_F t^{\alpha+k_2+l_2} \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}.$$

So $t \to \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds$ is defined on $(0,1]$ and

$$\lim_{t\to 0^+} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} A(s)x(s)ds = \lim_{t\to 0^+} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F(s)ds = 0. \tag{3.1.6}$$

Furthermore, we have for $t_1, t_2 \in (0,1]$ with $t_1 < t_2$ that

$$\left|\int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds - \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds\right|$$

$$\le \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}|A(s)x(s)+F(s)|ds + \int_0^{t_1} \frac{|(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}|}{\Gamma(\alpha)}|A(s)x(s)+F(s)|ds$$

$$\le M_A r \left[\int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} s^{k_1}(1-s)^{l_1} ds + \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} s^{k_1}(1-s)^{l_1} ds\right]$$

$$+M_F \left[\int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} s^{k_2}(1-s)^{l_2} ds + \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} s^{k_2}(1-s)^{l_2} ds\right]$$

$$\le M_A r \left[\int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha+l_1-1}}{\Gamma(\alpha)} s^{k_1} ds + \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} s^{k_1}(t_2-s)^{l_1} ds\right]$$

$$+M_F \left[\int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha+l_2-1}}{\Gamma(\alpha)} s^{k_2} ds + \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} s^{k_2}(t_2-s)^{l_2} ds\right]$$

$$= M_A r \left[t_2^{\alpha+k_1+l_1} \int_{\frac{t_1}{t_2}}^1 \frac{(1-w)^{\alpha+l_1-1}}{\Gamma(\alpha)} w^{k_1} dw + \int_0^{t_1} \frac{(t_1-s)^{\alpha+l_1-1}}{\Gamma(\alpha)} s^{k_1} ds - \int_0^{t_1} \frac{(t_2-s)^{\alpha+l_1-1}}{\Gamma(\alpha)} s^{k_1} ds\right]$$

$$+M_F \left[t_2^{\alpha+k_2+l_2} \int_{\frac{t_1}{t_2}}^1 \frac{(1-w)^{\alpha+l_2-1}}{\Gamma(\alpha)} w^{k_2} dw + \int_0^{t_1} \frac{(t_1-s)^{\alpha+l_2-1}}{\Gamma(\alpha)} s^{k_2} ds - \int_0^{t_1} \frac{(t_2-s)^{\alpha+l_2-1}}{\Gamma(\alpha)} s^{k_2} ds\right]$$

$$= M_A r \left[t_2^{\alpha+k_1+l_1} \int_{\frac{t_1}{t_2}}^1 \frac{(1-w)^{\alpha+l_1-1}}{\Gamma(\alpha)} w^{k_1} dw\right.$$

$$\left.+t_1^{\alpha+k_1+l_1} \int_0^1 \frac{(1-w)^{\alpha+l_1-1}}{\Gamma(\alpha)} w^{k_1} dw - t_2^{\alpha+k_1+l_1} \int_0^{\frac{t_1}{t_2}} \frac{(1-w)^{\alpha+l_1-1}}{\Gamma(\alpha)} w^{k_1} dw\right]$$

$$+M_F \left[t_2^{\alpha+k_2+l_2} \int_{\frac{t_1}{t_2}}^1 \frac{(1-w)^{\alpha+l_2-1}}{\Gamma(\alpha)} w^{k_2} dw\right.$$

$$+t_1^{\alpha+k_2+l_2}\int_0^1\frac{(1-w)^{\alpha+l_2-1}}{\Gamma(\alpha)}w^{k_2}dw-t_2^{\alpha+k_2+l_2}\int_0^{\frac{t_1}{t_2}}\frac{(1-w)^{\alpha+l_2-1}}{\Gamma(\alpha)}w^{k_2}dw\Bigg]$$

$$=M_Ar\left[t_2^{\alpha+k_1+l_1}\int_{\frac{t_1}{t_2}}^1\frac{(1-w)^{\alpha+l_1-1}}{\Gamma(\alpha)}w^{k_1}dw\right.$$

$$\left.+|t_1^{\alpha+k_1+l_1}-t_2^{\alpha+k_1+l_1}|\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}-t_2^{\alpha+k_1+l_1}\int_{\frac{t_1}{t_2}}^1\frac{(1-w)^{\alpha+l_1-1}}{\Gamma(\alpha)}w^{k_1}dw\right]$$

$$+M_F\left[t_2^{\alpha+k_2+l_2}\int_{\frac{t_1}{t_2}}^1\frac{(1-w)^{\alpha+l_2-1}}{\Gamma(\alpha)}w^{k_2}dw\right.$$

$$\left.+|t_1^{\alpha+k_2+l_2}-t_2^{\alpha+k_2+l_2}|\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}-t_2^{\alpha+k_2+l_2}\int_{\frac{t_1}{t_2}}^1\frac{(1-w)^{\alpha+l_2-1}}{\Gamma(\alpha)}w^{k_2}dw\right]\to 0 \text{ as } t_1\to t_2.$$

So $t\mapsto\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds$ is continuous on $(0,1]$ by defining

$$\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds\Big|_{t=0}=\lim_{t\to0^+}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds.$$

We have $I_{0^+}^{\alpha}\,{}^CD_{0^+}^{\alpha}x(t)=I_{0^+}^{\alpha}[A(t)x(t)+F(t)]$. So

$$\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds=I_{0^+}^{\alpha}[A(t)x(t)+F(t)]=I_{0^+}^{\alpha}\,{}^CD_{0^+}^{\alpha}x(t)$$

$$=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left(\frac{1}{\Gamma(1-\alpha)}\int_0^s(s-w)^{-\alpha}x'(w)dw\right)ds \text{ interchange the order of ingrals}$$

$$=\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^t\int_w^t(t-s)^{\alpha-1}(s-w)^{-\alpha}dsx'(w)dw \text{ use } \frac{s-w}{t-w}=u$$

$$=\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^t\int_0^1(1-u)^{\alpha-1}u^{-\alpha}dux'(w)dw \text{ by } \mathbf{B}(\alpha,1-\alpha)=\frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\Gamma(1)}$$

$$=\int_0^t x'(w)dw=x(t)-\lim_{t\to0^+}x(t)=x(t)-\eta.$$

Then $x\in C(0,1]$ is a solution of (3.1.5).

On the other hand, if x is a solution of (3.1.5), together Cases 1,2 and 3, we have $x\in C(0,1]$ and $\lim_{t\to0^+}x(t)=\eta$. So $x\in C(0,1]$. Furthermore, we have

$${}^CD_{0^+}^{\alpha}x(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-s)^{-\alpha}x'(s)ds$$

$$=\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-s)^{-\alpha}\left(\eta+\int_0^s\frac{(s-w)^{\alpha-1}}{\Gamma(\alpha)}[A(w)x(w)+F(w)]dw\right)'ds$$

$$=\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-s)^{-\alpha}\left(\int_0^s\frac{(s-w)^{\alpha-1}}{\Gamma(\alpha)}[A(w)x(w)+F(w)]dw\right)'ds$$

$$=\left[\frac{1}{\Gamma(2-\alpha)}\int_0^t(t-s)^{1-\alpha}\frac{1}{\Gamma(1-(1-\alpha))}\left(\int_0^s(s-w)^{-(1-\alpha)}[A(w)x(w)+F(w)]dw\right)'ds\right]'$$

$$=\frac{1}{\Gamma(2-\alpha)}\left[(t-s)^{1-\alpha}\frac{1}{\Gamma(1-(1-\alpha))}\int_0^s(s-w)^{-(1-\alpha)}[A(w)x(w)+F(w)]dw\Big|_0^t\right.$$

$$\left.+(1-\alpha)\int_0^t(t-s)^{-\alpha}\frac{1}{\Gamma(1-(1-\alpha))}\int_0^s(s-w)^{-(1-\alpha)}[A(w)x(w)+F(w)]dwds\right]'$$

$$= \frac{1}{\Gamma(1-\alpha)}\left[\int_0^t (t-s)^{-\alpha}\frac{1}{\Gamma(1-(1-\alpha))}\int_0^s (s-w)^{-(1-\alpha)}[A(w)x(w)+F(w)]dwds\right]' \text{ by (3.1.6)}$$

$$= \frac{1}{\Gamma(1-\alpha)}\frac{1}{\Gamma(\alpha)}\left[\int_0^t\int_w^t (t-s)^{-\alpha}(s-w)^{-(1-\alpha)}ds[A(w)x(w)+F(w)]dw\right]' \text{ by chenging the order of integrals}$$

$$= \frac{1}{\Gamma(1-\alpha)}\frac{1}{\Gamma(\alpha)}\left[\int_0^t\int_0^1 (1-u)^{-\alpha}u^{\alpha-1}du[A(w)x(w)+F(w)]dw\right]' \text{ by } \frac{s-w}{t-w}=u$$

$$= \left[\int_0^t [A(w)x(w)+F(w)]dw\right]' \text{ by } \mathbf{B}(1-\alpha,\alpha)=\Gamma(1-\alpha)\Gamma(\alpha) = A(t)x(t)+F(t).$$

So $x \in C(0,1]$ is a solution of IVP(3.1.1). The proof is completed. □

Theorem 3.2. *Suppose that (A1) holds. Then (3.1.1) has a unique solution. If there exists constants $k_2 > -1$, $l_2 \le 0$ with $l_2 > \{-\alpha, -\alpha-k_2\}$, $M_F \ge 0$ such that $|F(t)| \le M_F t^{k_2}(1-t)^{l_2}$ for all $t \in (0,1)$, then the following special problem*

$$\begin{cases} {}^C D_{0^+}^{\mathbf{ff}} x(t) = \lambda x(t) + F(t),\ t \in (0,1], \\ \lim\limits_{t\to 0^+} x(t) = \eta \end{cases} \tag{3.1.7}$$

has a unique solution

$$x(t) = \eta E_{\alpha,1}(\lambda t^\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha)F(s)ds,\ t \in (0,1]. \tag{3.1.8}$$

Proof. From Claims 1, 2 and 3, Theorem 3.1 implies that (3.1.1) has a unique solution. From the assumption and $A(t) \equiv \lambda$, it is easy to see that (A1) holds with $k_1 = l_1 = 0$ and k_2, l_2 mentioned. Thus (3.1.7) has a unique solution. We get from the Picard function sequence that

$$\phi_n(t) = \eta + \lambda\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_{n-1}(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}F(s)ds$$

$$= \eta + \eta\lambda\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds + \lambda^2\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\int_0^s \frac{(s-w)^{\alpha-1}}{\Gamma(\alpha)}\phi_{n-2}(w)dwds$$

$$+\lambda\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{(s-w)^{\alpha-1}}{\Gamma(\alpha)}F(w)dwds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}F(s)ds$$

$$= \eta + \frac{\eta\lambda}{\Gamma(\alpha+1)}t^\alpha + \lambda^2\int_0^t\int_w^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{(s-w)^{\alpha-1}}{\Gamma(\alpha)}ds\phi_{n-2}(w)dw$$

$$+\lambda\int_0^t\int_w^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{(s-w)^{\alpha-1}}{\Gamma(\alpha)}dsF(w)dw + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}F(s)ds$$

$$= \eta + \frac{\eta\lambda}{\Gamma(\alpha+1)}t^\alpha + \lambda^2\int_0^t (t-w)^{2\alpha-1}\int_0^1 \frac{(1-u)^{\alpha-1}}{\Gamma(\alpha)}\frac{u^{\alpha-1}}{\Gamma(\alpha)}du\phi_{n-2}(w)dw$$

$$+\lambda\int_0^t (t-w)^{2\alpha-1}\int_0^1 \frac{(1-u)^{\alpha-1}}{\Gamma(\alpha)}\frac{u^{\alpha-1}}{\Gamma(\alpha)}duF(w)dw + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}F(s)ds$$

$$= \eta\left(1+\frac{\lambda t^\alpha}{\Gamma(\alpha+1)}\right) + \lambda^2\int_0^t \frac{(t-w)^{2\alpha-1}}{\Gamma(2\alpha)}\phi_{n-2}(w)dw + \int_0^t (t-s)^{\alpha-1}\left(\frac{\lambda(t-s)^\alpha}{\Gamma(2\alpha)} + \frac{1}{\Gamma(\alpha)}\right)F(s)ds$$

$$= \cdots\cdots\cdots$$

$$= \eta\sum_{j=0}^{n-1}\frac{\lambda^j t^{j\alpha}}{\Gamma(j\alpha+1)} + \eta\lambda^n\int_0^t \frac{(t-w)^{n\alpha-1}}{\Gamma(n\alpha)}dw + \int_0^t (t-s)^{\alpha-1}\left(\sum_{j=0}^{n-1}\frac{\lambda^j(t-s)^{j\alpha}}{\Gamma((j+1)\alpha)}\right)F(s)ds$$

$$= \eta\sum_{j=0}^{n}\frac{\lambda^j t^{j\alpha}}{\Gamma(j\alpha+1)} + \int_0^t (t-s)^{\alpha-1}\left(\sum_{j=0}^{n}\frac{\lambda^j(t-s)^{j\alpha}}{\Gamma((j+1)\alpha)}.\right)F(s)ds$$

$$\to \eta E_{\alpha,1}(\lambda t^\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha)F(s)ds.$$

Then we get (3.1.8). The proof is complete. □

To get solutions of (3.1.2), we need the following assumptions:

(A2) there exists constants $k_i > -\alpha$, $l_i \le 0$ with $l_1 > \{-\alpha, -\alpha - k_1\}$, $l_2 > \max\{-\alpha, -1 - k_2\}$, $M_A \ge 0$ and $M_F \ge 0$ such that $|A(t)| \le M_A t^{k_1}(1-t)^{l_1}$ and $|F(t)| \le M_F t^{k_2}(1-t)^{l_2}$ for all $t \in (0,1)$.

Choose Picard function sequence as

$$\phi_0(t) = \eta t^{\alpha-1},\ t \in (0,1],$$

$$\phi_n(t) = \eta t^{\alpha-1} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)\phi_{n-1}(s) + F(s)]ds, t \in (0,1], n = 1,2,\cdots.$$

Claim 1. $\phi_n \in C_{1-\alpha}(0,1]$. Since $\phi_0 \in C_{1-\alpha}(0,1]$, then ϕ_1 is continuous on $(0,1]$, together with

$$t^{1-\alpha}\left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)\phi_0(s) + F(s)]ds\right|$$

$$= t^{1-\alpha}\left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)s^{\alpha-1}s^{1-\alpha}\phi_0(s) + F(s)]ds\right|$$

$$\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[M_A|\eta|s^{\alpha-1}s^{k_1}(1-s)^{l_1} + M_F s^{k_2}(1-s)^{l_2}]ds$$

$$\le t^{1-\alpha}M_A|\eta|\int_0^t \frac{(t-s)^{\alpha+l_1-1}}{\Gamma(\alpha)}s^{\alpha+k_1-1}ds + t^{1-\alpha}M_F\int_0^t \frac{(t-s)^{\alpha+l_2-1}}{\Gamma(\alpha)}s^{k_2}ds$$

$$= M_A|\eta|t^{\alpha+k_1+l_1}\frac{\mathbf{B}(\alpha+l_1,\alpha+k_1)}{\Gamma(\alpha)} + M_F t^{1+k_2+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)} \to 0 \text{ as } t \to 0^+,$$

we see $\phi_1 \in C_{1-\alpha}(0,1]$. By mathematical induction method, we can prove that $\phi_n \in C_{1-\alpha}(0,1]$.

Claim 2. $\{t \to t^{1-\alpha}\phi_n(t)\}$ is convergent uniformly on $(0,1]$. In fact we have for $t \in (0,1]$ that

$$t^{1-\alpha}|\phi_1(t) - \phi_0(t)| = \left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)\phi_0(s) + F(s)]ds\right|$$

$$\le t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[M_A|\eta|s^{k_1}(1-s)^{l_1} + M_F s^{k_2}(1-s)^{l_2}]ds$$

$$\le |\eta|M_A t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha+l_1-1}}{\Gamma(\alpha)}s^{k_1}ds + M_F t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha+l_2-1}}{\Gamma(\alpha)}s^{k_2}ds$$

$$= |\eta|M_A t^{k_1+l_1+1}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)} + M_F t^{k_2+l_2+1}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}.$$

So

$$t^{1-\alpha}|\phi_2(t) - \phi_1(t)| = \left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}A(s)[\phi_1(s) - \phi_0(s)]ds\right|$$

$$\le t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}M_A s^{k_1}(1-s)^{l_1}\left(|\eta|M_A s^{k_1+l_1+1}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)} + M_F s^{k_2+l_2+1}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\right)ds$$

$$\le |\eta|M_A^2 t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha+l_1-1}}{\Gamma(\alpha)}s^{2k_1+l_1+1}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}ds$$

$$+M_A M_F t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha+l_1-1}}{\Gamma(\alpha)}s^{k_1+k_2+l_2+1}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}ds$$

$$= |\eta|M_A^2 t^{2k_1+2l_1+2}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2k_1+l_1+2)}{\Gamma(\alpha)}$$

$$+M_A M_F t^{k_1+k_2+l_1+l_2+2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,k_1+k_2+l_2+2)}{\Gamma(\alpha)}.$$

$$t^{1-\alpha}|\phi_3(t)-\phi_2(t)|=\left|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}A(s)[\phi_2(s)-\phi_1(s)]ds\right|$$

$$\le t^{1-\alpha}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}M_As^{k_1}(1-s)^{l_1}\left(|\eta|M_A^2s^{2k_1+2l_1+2}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2k_1+l_1+2)}{\Gamma(\alpha)}\right.$$

$$\left.+M_AM_Fs^{k_1+k_2+l_1+l_2+2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2+k_1+k_2+l_2)}{\Gamma(\alpha)}\right)ds$$

$$\le|\eta|M_A^3t^{1-\alpha}\int_0^t\frac{(t-s)^{\alpha+l_1-1}}{\Gamma(\alpha)}s^{3k_1+2l_1+2}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2+2k_1+l_1)}{\Gamma(\alpha)}ds$$

$$+M_A^2M_Ft^{1-\alpha}\int_0^t\frac{(t-s)^{\alpha+l_1-1}}{\Gamma(\alpha)}s^{2k_1+k_2+l_1+l_2+2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,k_1+k_2+l_2+2)}{\Gamma(\alpha)}ds$$

$$=|\eta|M_A^3t^{3k_1+3l_1+3}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2k_1+l_1+2)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,3k_1+2l_1+3)}{\Gamma(\alpha)}$$

$$+M_A^2M_Ft^{2k_1+k_2+2l_1+l_2+3}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,k_1+k_2+l_2+2)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2k_1+k_2+l_1+l_2+3)}{\Gamma(\alpha)}.$$

$$t^{1-\alpha}|\phi_4(t)-\phi_3(t)|=\left|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}A(s)[\phi_3(s)-\phi_2(s)]ds\right|$$

$$\le t^{1-\alpha}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}M_As^{k_1}(1-s)^{l_1}\left(|\eta|M_A^3s^{3k_1+3l_1+3}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2k_1+l_1+2)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,3k_1+2l_1+3)}{\Gamma(\alpha)}\right.$$

$$\left.+M_A^2M_Fs^{2k_1+k_2+2l_1+l_2+3}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,k_1+k_2+l_2+2)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2k_1+k_2+l_1+l_2+3)}{\Gamma(\alpha)}\right)ds$$

$$\le|\eta|M_A^4t^{1-\alpha}\int_0^t\frac{(t-s)^{\alpha+l_1-1}}{\Gamma(\alpha)}s^{4k_1+3l_1+3}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2k_1+l_1+2)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,3k_1+2l_1+3)}{\Gamma(\alpha)}ds$$

$$+M_A^3M_Ft^{1-\alpha}\int_0^t\frac{(t-s)^{\alpha+l_1-1}}{\Gamma(\alpha)}s^{3k_1+k_2+2l_1+l_2+3}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,k_1+k_2+l_2+2)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2k_1+k_2+l_1+l_2+3)}{\Gamma(\alpha)}ds$$

$$=|\eta|M_A^4t^{4k_1+4l_1+4}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2k_1+l_1+2)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,3k_1+2l_1+3)}{\Gamma(\alpha)}ds\frac{\mathbf{B}(\alpha+l_1,4k_1+3l_1+4)}{\Gamma(\alpha)}$$

$$+M_A^3M_Ft^{3k_1+k_2+3l_1+l_2+4}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,k_1+k_2+l_2+2)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2k_1+k_2+l_1+l_2+3)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,3k_1+k_2+2l_1+l_2+4)}{\Gamma(\alpha)}.$$

Similarly by the mathematical induction method, we get for every $n=1,2,\cdots$ that

$$t^{1-\alpha}|\phi_n(t)-\phi_{n-1}(t)|\le|\eta|M_A^nt^{nk_1+nl_1+n}\prod_{i=0}^{n-1}\frac{\mathbf{B}(\alpha+l_1,(i+1)k_1+il_1+(i+1))}{\Gamma(\alpha)}$$

$$+M_A^{n-1}M_Ft^{(n-1)k_1+k_2+(n-1)l_1+l_2+n}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\prod_{i=1}^{n-1}\frac{\mathbf{B}(\alpha+l,ik_1+k_2+(i-1)l_1+l_2+i+1)}{\Gamma(\alpha)}$$

$$\le|\eta|M_A^n\prod_{i=0}^{n-1}\frac{\mathbf{B}(\alpha+l_1,(i+1)k_1+il_1+(i+1))}{\Gamma(\alpha)}$$

$$+M_A^{n-1}M_F\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\prod_{i=1}^{n-1}\frac{\mathbf{B}(\alpha+l,ik_1+k_2+(i-1)l_1+l_2+i+1)}{\Gamma(\alpha)},t\in[0,1].$$

Similarly we can prove that both

$$\sum_{n=1}^{+\infty}u_n=\sum_{n=1}^{+\infty}|\eta|M_A^n\prod_{i=0}^{n-1}\frac{\mathbf{B}(\alpha+l_1,(i+1)k_1+il_1+(i+1))}{\Gamma(\alpha)},$$

$$\sum_{n=1}^{+\infty}v_n=\sum_{n=1}^{+\infty}M_A^{n-1}M_F\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\prod_{i=1}^{n-1}\frac{\mathbf{B}(\alpha+l,ik_1+k_2+(i-1)l_1+l_2+i+1)}{\Gamma(\alpha)}$$

are convergent. Hence

$$t^{1-\alpha}\phi_0(t)+t^{1-\alpha}[\phi_1(t)-\phi_0(t)]+t^{1-\alpha}[\phi_2(t)-\phi_1(t)]+\cdots+t^{1-\alpha}[\phi_n(t)-\phi_{n-1}(t)]+\cdots,t\in[0,1]$$

is uniformly convergent. Then $\{t \to t^{1-\alpha}\phi_n(t)\}$ is convergent uniformly on $(0,1]$.

Claim 3. $\phi(t) = t^{\alpha-1} \lim_{n\to+\infty} t^{1-\alpha}\phi_n(t)$ defined on $(0,1]$ is a unique continuous solution of the integral equation

$$x(t) = \eta t^{\alpha-1} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s) + F(s)]ds, t \in (0,1]. \tag{3.1.9}$$

Proof. By $\lim_{n\to+\infty} t^{1-\alpha}\phi_n(t) = t^{1-\alpha}\phi(t)$ and the uniformly convergence, we see $\phi(t)$ is continuous on $(0,1]$. From

$$t^{1-\alpha}\left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)\phi_{n-1}(s) + F(s)]ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)\phi_{m-1}(s) + F(s)]ds\right|$$

$$\leq M_A||\phi_{n-1} - \phi_{m-1}||t^{1-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{k_1}(1-s)^{l_1}s^{\alpha-1}ds$$

$$\leq M_A||\phi_{n-1} - \phi_{m-1}||t^{1-\alpha} \int_0^t \frac{(t-s)^{\alpha+l_1-1}}{\Gamma(\alpha)} s^{\alpha+k_1-1}ds$$

$$\leq M_A||\phi_{n-1} - \phi_{m-1}||t^{\alpha+k_1+l_1} \frac{\mathbf{B}(\alpha+l_1,\alpha+k_1)}{\Gamma(\alpha)}$$

$$\leq M_A||\phi_{n-1} - \phi_{m-1}|| \frac{\mathbf{B}(\alpha+l_1,\alpha+k_1)}{\Gamma(\alpha)} \to 0 \text{ uniformly as } m,n \to +\infty,$$

we know that

$$\phi(t) = t^{\alpha-1} \lim_{n\to\infty} t^{1-\alpha}\phi_n(t) = \lim_{n\to+\infty}\left[\eta + t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)\phi_{n-1}(s) + F(s)]ds\right]$$

$$= \eta t^{\alpha-1} + t^{\alpha-1} \lim_{n\to+\infty} t^{1-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)\phi_{n-1}(s) + F(s)]ds$$

$$= \eta + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)\phi(s) + F(s)]\, ds.$$

Then ϕ is a continuous solution of (3.1.9) defined on $(0,1]$.

Suppose that ψ defined on $(0,1]$ is also a solution of (3.1.9). Then

$$\psi(t) = \eta t^{\alpha-1} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)\psi(s) + F(s)]ds, t \in [0,1].$$

We need to prove that $\phi(t) \equiv \psi(t)$ on $(0,1]$. Then

$$t^{1-\alpha}|\psi(t) - \phi_0(t)| = t^{1-\alpha}\left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|A(s)\psi(s) + F(s)|ds\right|$$

$$\leq |\eta|M_A t^{k_1+l_1+1}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)} + M_F t^{k_2+l_2+1}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}.$$

Furthermore, we have

$$t^{1-\alpha}|\psi(t) - \phi_1(t)| = t^{1-\alpha}\left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}A(s)[\psi(s) - \phi_0(s)]ds\right|$$

$$\leq |\eta|M_A^2 t^{2k_1+2l_1+2}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2k_1+l_1+2)}{\Gamma(\alpha)}$$

$$+M_A M_F t^{k_1+k_2+l_1+l_2+2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,k_1+k_2+l_2+2)}{\Gamma(\alpha)}.$$

By mathematical induction method, we can get that

$$t^{1-\alpha}|\psi(t) - \phi_n(t)| = t^{1-\alpha}\left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}A(s)[\psi(s) - \phi_{n-1}(s)]ds\right|$$

$$\leq |\eta|M_A^n t^{nk_1+nl_1+n}\prod_{i=0}^{n-1}\frac{\mathbf{B}(\alpha+l_1,(i+1)k_1+il_1+(i+1))}{\Gamma(\alpha)}$$

$$+M_A^{n-1}M_F t^{(n-1)k_1+k_2+(n-1)l_1+l_2+n}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\prod_{i=1}^{n-1}\frac{\mathbf{B}(\alpha+l,ik_1+k_2+(i-1)l_1+l_2+i+1)}{\Gamma(\alpha)}.$$

Hence

$$t^{1-\alpha}|\psi(t)-\phi_n(t)| \le |\eta| M_A^n \prod_{i=0}^{n-1} \frac{\mathbf{B}(\alpha+l_1,(i+1)k_1+il_1+(i+1))}{\Gamma(\alpha)}$$

$$+M_A^{n-1}M_F \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)} \prod_{i=1}^{n-1} \frac{\mathbf{B}(\alpha+l,ik_1+k_2+(i-1)l_1+l_2+i+1)}{\Gamma(\alpha)}, \text{ for all } n=1,2,\cdots.$$

Similarly we have $\lim_{n\to+\infty} t^{1-\alpha}\phi_n(t) = t^{1-\alpha}\psi(t)$ uniformly on $(0,1]$. Then $\phi(t) \equiv \psi(t)$ on $(0,1]$. Then (3.1.9) has a unique solution ϕ. The proof is complete.

Theorem 3.3. *Suppose that (A2) holds. Then $x \in C_{1-\alpha}(0,1]$ is a solution of IVP(3.1.2) if and only if $x \in C_{1-\alpha}(0,1]$ is a solution of the integral equation (3.1.9).*

Proof. Suppose that $x \in C_{1-\alpha}(0,1]$ is a solution of IVP(3.1.2). Then $t \to t^{1-\alpha}x(t)$is continuous on $(0,1]$ by defining $t^{1-\alpha}x(t)|_{t=0} = \lim_{t\to0^+} t^{1-\alpha}x(t)$ and $||x|| = r < +\infty$. So by $\frac{w}{s}=u$, we get

$$\lim_{s\to0^+} \int_0^s (s-w)^{-\alpha}x(w)dw = \lim_{s\to0^+} \int_0^s (s-w)^{-\alpha}w^{\alpha-1}w^{1-\alpha}x(w)dw$$

$$= \lim_{s\to0^+} \xi^{1-\alpha}x(\xi) \int_0^s (s-w)^{-\alpha}w^{\alpha-1}dw \text{ by mean value theorem of integral, } \xi \in (0,s)$$

$$= \lim_{s\to0^+} \xi^{1-\alpha}x(\xi) \int_0^1 (1-u)^{-\alpha}u^{\alpha-1}du = \eta\mathbf{B}(1-\alpha,\alpha).$$

From (A2), we have

$$t^{1-\alpha}\left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds\right|$$

$$= t^{1-\alpha}\left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)s^{\alpha-1}s^{1-\alpha}x(s)+F(s)]ds\right|$$

$$\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[M_A r s^{\alpha-1}s^{k_1}(1-s)^{l_1}+M_F s^{k_2}(1-s)^{l_2}]ds$$

$$\le t^{1-\alpha}M_A r \int_0^t \frac{(t-s)^{\alpha+l_1-1}}{\Gamma(\alpha)}s^{\alpha+k_1-1}ds + t^{1-\alpha}M_F \int_0^t \frac{(t-s)^{\alpha+l_2-1}}{\Gamma(\alpha)}s^{k_2}ds$$

$$= M_A r t^{\alpha+k_1+l_1}\frac{\mathbf{B}(\alpha+l_1,\alpha+k_1)}{\Gamma(\alpha)} + M_F t^{1+k_2+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}.$$

So $t \to t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds$ is defined on $(0,1]$ and

$$\lim_{t\to0^+} t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds = 0. \tag{3.1.10}$$

Furthermore, we have similarly to Theorem 3.1 that $t \to \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds$ is continuous on $(0,1]$. So $t \to t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds$ is continuous on $[0,1]$ by defining

$$t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds\Big|_{t=0} = \lim_{t\to0^+} t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds. \tag{3.1.11}$$

We have $I_{0^+}^{\alpha}\,{}^{RL}D_{0^+}^{\alpha}x(t) = I_{0^+}^{\alpha}[A(t)x(t)+F(t)]$. So

$$\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[A(s)x(s)+F(s)]ds = I_{0^+}^{\alpha}[A(t)x(t)+F(t)] = I_{0^+}^{\alpha}\,{}^{RL}D_{0^+}^{\alpha}x(t)$$

$$= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left[\frac{1}{\Gamma(1-\alpha)}\left(\int_0^s (s-w)^{-\alpha}x(w)dw\right)'\right]ds$$

$$= \left[\int_0^t \frac{(t-s)^\alpha}{\Gamma(\alpha+1)} \frac{1}{\Gamma(1-\alpha)} \left(\int_0^s (s-w)^{-\alpha} x(w)dw\right)' ds\right]'$$

$$= \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha+1)} \left[(t-s)^\alpha \int_0^s (s-w)^{-\alpha} x(w)dw\Big|_0^t + \alpha \int_0^t (t-s)^{\alpha-1} \int_0^s (s-w)^{-\alpha} x(w)dwds\right]'$$

$$= \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha+1)} \left[-t^\alpha \lim_{s\to 0^+} \int_0^s (s-w)^{-\alpha} x(w)dw + \alpha \int_0^t (t-s)^{\alpha-1} \int_0^s (s-w)^{-\alpha} x(w)dwds\right]'$$

$$= \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha+1)} \left[\alpha \int_0^t \int_w^t (t-s)^{\alpha-1} (s-w)^{-\alpha} dsx(w)dw\right]' - \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} t^{\alpha-1} \lim_{s\to 0^+} \int_0^s (s-w)^{-\alpha} x(w)dw$$

$$= \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha+1)} \left[\alpha \int_0^t \int_0^1 (1-u)^{\alpha-1} u^{-\alpha} dux(w)dw\right]' - \frac{t^{\alpha-1}}{\Gamma(1-\alpha)\Gamma(\alpha)} \lim_{s\to 0^+} \int_0^s (s-w)^{-\alpha} x(w)dw \text{ by } \frac{s-w}{t-w} = u$$

$$= \left[\int_0^t x(w)dw\right]' = x(t) - \frac{t^{\alpha-1}}{\Gamma(1-\alpha)\Gamma(\alpha)} \eta \mathbf{B}(1-\alpha,\alpha) = x(t) - \eta t^{\alpha-1}.$$

Then $x \in C_{1-\alpha}(0,1]$ is a solution of (3.1.9).

On the other hand, if $x \in C_{1-\alpha}(0,1]$ is a solution of (3.1.9), together with (3.1.10)-(3.1.11) implies $\lim_{t\to 0^+} t^{1-\alpha} x(t) = \eta$. Furthermore, we have

$${}^{RL}D_{0^+}^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \left(\int_0^t (t-s)^{-\alpha} x(s)ds\right)$$

$$= \frac{1}{\Gamma(1-\alpha)} \left(\int_0^t (t-s)^{-\alpha} \left(\eta s^{\alpha-1} + \int_0^s \frac{(s-w)^{\alpha-1}}{\Gamma(\alpha)} [A(w)x(w) + F(w)]dw\right) ds\right)'$$

$$= \frac{1}{\Gamma(1-\alpha)} \left(\eta \int_0^t (t-s)^{-\alpha} s^{\alpha-1} ds\right)' + \frac{1}{\Gamma(1-\alpha)} \left(\int_0^t (t-s)^{-\alpha} \int_0^s \frac{(s-w)^{\alpha-1}}{\Gamma(\alpha)} [A(w)x(w) + F(w)]dwds\right)'$$

$$= \frac{1}{\Gamma(1-\alpha)} \left(\eta \int_0^1 (1-u)^{-\alpha} u^{\alpha-1} du\right)' + \frac{1}{\Gamma(1-\alpha)} \left(\int_0^t \int_w^t (t-s)^{-\alpha} \frac{(s-w)^{\alpha-1}}{\Gamma(\alpha)} ds[A(w)x(w) + F(w)]dw\right)'$$

$$= \frac{1}{\Gamma(1-\alpha)} \left(\int_0^t \int_0^1 (1-u)^{-\alpha} \frac{u^{\alpha-1}}{\Gamma(\alpha)} du[A(w)x(w) + F(w)]dw\right)' = A(t)x(t) + F(t).$$

So $x \in C_{1-\alpha}(0,1]$ is a solution of IVP(3.1.2). The proof is completed. □

Theorem 3.4. *Suppose that (A2) holds. Then (3.1.2) has a unique solution. If $A(t) \equiv \lambda$ and there exists constants $k_2 > -1$, $l_2 \le 0$ with $l_2 > \{-\alpha, -1-k_1\}$ and $M_F \ge 0$ such that $|F(t)| \le M_F t^{k_2}(1-t)^{l_2}$ for all $t \in (0,1)$, then following special problem*

$$\begin{cases} {}^{RL}D_{0^+}^{\text{ff}} x(t) = \lambda x(t) + F(t), \ t \in (0,1], \\ \lim_{t\to 0^+} t^{1-\alpha} x(t) = \eta \end{cases} \tag{3.1.12}$$

has a unique solution

$$x(t) = \eta \Gamma(\alpha) t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^\alpha) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha) F(s)ds, \ t \in (0,1]. \tag{3.1.13}$$

Proof. From Claims 1, 2 and 3, (3.1.2) and Theorem 3.3 has a unique solution. From the assumption and $A(t) \equiv \lambda$, one sees that (A2) holds with $k_1 = l_1 = 0$ and k_2, l_2 mentioned. Thus (3.1.12) has a unique solution. We get from the Picard function sequence that

$$\phi_n(t) = \eta t^{\alpha-1} + \lambda \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_{n-1}(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F(s)ds$$

$$= \eta t^{\alpha-1} + \eta\lambda \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{\alpha-1} ds + \lambda^2 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^s \frac{(s-w)^{\alpha-1}}{\Gamma(\alpha)} \phi_{n-2}(w)dwds$$

$$+\lambda \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \frac{(s-w)^{\alpha-1}}{\Gamma(\alpha)} F(w)dwds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F(s)ds$$

$$= \eta t^{\alpha-1} + \frac{\eta\lambda\Gamma(\alpha)t^{2\alpha-1}}{\Gamma(2\alpha)} + \lambda^2 \int_0^t \int_w^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \frac{(s-w)^{\alpha-1}}{\Gamma(\alpha)} ds \phi_{n-2}(w)dw$$

$$+\lambda \int_0^t \int_w^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \frac{(s-w)^{\alpha-1}}{\Gamma(\alpha)} dsF(w)dw + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F(s)ds$$

$$= \cdots\cdots\cdots$$

$$= \eta\Gamma(\alpha)t^{\alpha-1} \sum_{j=0}^{n-1} \frac{\lambda^j t^{j\alpha}}{\Gamma((j+1)\alpha)} + \eta\lambda^n \int_0^t \frac{(t-w)^{n\alpha-1}}{\Gamma(n\alpha)} dw + \int_0^t (t-s)^{\alpha-1} \left(\sum_{j=0}^{n-1} \frac{\lambda^j (t-s)^{j\alpha}}{\Gamma((j+1)\alpha)} \right) F(s)ds$$

$$= \eta\Gamma(\alpha)t^{\alpha-1} \sum_{j=0}^{n} \frac{\lambda^j t^{j\alpha}}{\Gamma((j+1)\alpha)} + \int_0^t (t-s)^{\alpha-1} \left(\sum_{j=0}^{n} \frac{\lambda^j (t-s)^{j\alpha}}{\Gamma((j+1)\alpha)}. \right) F(s)ds$$

$$\to \eta\Gamma(\alpha)t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^\alpha) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha) F(s)ds.$$

Then we get (3.1.13). The proof is complete. □

To get solutions of (3.1.3), we need the following assumptions:

(A3) there exists constants $k_i > -\alpha$, $l_i \le 0$ with $l_1 > \{-\alpha, -\alpha - k_1\}$, $l_2 > \max\{-\alpha, -1 + k_2\}$, $M_B \ge 0$ and $M_G \ge 0$ such that $|B(t)| \le M_B (\log t)^{k_1} (1 - \log t)^{l_1}$ and $|G(t)| \le M_G (\log t)^{k_2} (1 - \log t)^{l_2}$ for all $t \in (1, e)$.

Choose Picard function sequence as

$$\phi_0(t) = \eta(\log t)^{\alpha-1},\ t \in (1, e],$$

$$\phi_n(t) = \eta(\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} [B(s)\phi_{n-1}(s) + G(s)] \frac{ds}{s}, t \in (1, e], n = 1, 2, \cdots.$$

Claim 1. $\phi_n \in LC_{1-\alpha}(1, e]$. In fact, $\phi_0 \in LC_{1-\alpha}(1, e]$ and

$$(\log t)^{1-\alpha} \left| \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} [B(s)\phi_0(s) + G(s)] \frac{ds}{s} \right|$$

$$\le (\log t)^{1-\alpha} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} [M_B |\eta| (\log s)^{\alpha-1} (\log s)^{k_1} (1 - \log s)^{l_1} + M_G (\log s)^{k_2} (1 - \log s)^{l_2}] \frac{ds}{s}$$

$$\le (\log t)^{1-\alpha} M_B |\eta| \int_1^t \left(\log \frac{t}{s}\right)^{\alpha+l_1-1} (\log s)^{\alpha+k_1-1} \frac{ds}{s} + (\log t)^{1-\alpha} M_G \int_1^t \left(\log \frac{t}{s}\right)^{\alpha+l_2-1} (\log s)^{k_2} \frac{ds}{s}$$

$$= M_B |\eta| (\log t)^{\alpha+k_1+l_1} \mathbf{B}(\alpha + l_1, k_1 + \alpha) + M_G (\log t)^{1+k_1+l_1} \mathbf{B}(\alpha + l_2, k_2 + 1) \to 0 \text{ as } t \to 0^+,$$

we know that $t \to \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} [B(s)\phi_0(s) + G(s)] \frac{ds}{s}$ is continuous on $(1, e]$ and $\lim_{t \to 0^+} (\log t)^{1-\alpha} \phi_1(t)$ exists. Then $\phi_1 \in LC_{1-\alpha}(1, e]$. By mathematical induction method, we can show $\phi_n \in LC_{1-\alpha}(1, e]$.

Claim 2. $\{t \to (\log t)^{1-\alpha} \phi_n(t)\}$ is convergent uniformly on $(1, e]$. In fact we have for $t \in (1, e]$ that

$$(\log t)^{1-\alpha} |\phi_1(t) - \phi_0(t)| = \frac{1}{\Gamma(\alpha)} (\log t)^{1-\alpha} \left| \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} [B(s)\phi_0(s) + G(s)] \frac{ds}{s} \right|$$

$$\le \frac{1}{\Gamma(\alpha)} (\log t)^{1-\alpha} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} [|\eta| M_B (\log s)^{k_1} (1 - \log s)^{l_1} + M_G (\log s)^{k_2} (1 - \log s)^{l_2}] \frac{ds}{s}$$

$$\le |\eta| M_B \frac{1}{\Gamma(\alpha)} (\log t)^{1-\alpha} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha+l_1-1} (\log s)^{k_1} \frac{ds}{s}$$

$$+ M_G \frac{1}{\Gamma(\alpha)} (\log t)^{1-\alpha} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} (\log s)^{k_2} (1 - \log s)^{l_2} \frac{ds}{s}$$

$$= |\eta| M_B (\log t)^{k_1+l_1+1} \frac{\mathbf{B}(\alpha+l_1, k_1+1)}{\Gamma(\alpha)} + M_G (\log t)^{k_2+l_2+1} \frac{\mathbf{B}(\alpha+l_2, k_2+1)}{\Gamma(\alpha)}.$$

So

$$(\log t)^{1-\alpha} |\phi_2(t) - \phi_1(t)| = \frac{1}{\Gamma(\alpha)} (\log t)^{1-\alpha} \left| \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} B(s)[\phi_1(s) - \phi_0(s)] \frac{ds}{s} \right|$$

$$\le (\log t)^{1-\alpha} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} M_B (\log s)^{k_1} (1 - \log s)^{l_1} \left(|\eta| M_B (\log s)^{k_1+l_1+1} \frac{\mathbf{B}(\alpha+l_1, k_1+1)}{\Gamma(\alpha)} \right.$$

$$+M_G(\log s)^{k_2+l_2+1}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\Big)\frac{ds}{s}$$

$$\leq |\eta|M_B^2(\log t)^{2k_1+2l_1+2}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2k_1+l_1+2)}{\Gamma(\alpha)}$$

$$+M_BM_G(\log t)^{k_1+k_2+l_1+l_2+2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,k_1+k_2+l_2+2)}{\Gamma(\alpha)}.$$

$$(\log t)^{1-\alpha}|\phi_3(t)-\phi_2(t)| = \frac{1}{\Gamma(\alpha)}(\log t)^{1-\alpha}\left|\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}B(s)[\phi_2(s)-\phi_1(s)]ds\right|$$

$$\leq \frac{1}{\Gamma(\alpha)}(\log t)^{1-\alpha}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha+l_1-1}M_B(\log s)^{k_1}\Big(|\eta|M_B^2(\log s)^{2k_1+2l_1+2}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2k_1+l_1+2)}{\Gamma(\alpha)}$$

$$+M_BM_G(\log s)^{k_1+k_2+l_1+l_2+2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,k_1+k_2+l_2+2)}{\Gamma(\alpha)}\Big)\frac{ds}{s}$$

$$\leq |\eta|M_B^3(\log t)^{3k_1+3l_1+3}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2k_1+l_1+2)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,3k_1+2l_1+3)}{\Gamma(\alpha)}$$

$$+M_B^2M_G(\log t)^{2k_1+k_2+2l_1+l_2+3}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,k_1+k_2+l_2+2)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2k_1+k_2+l_1+l_2+3)}{\Gamma(\alpha)}.$$

$$(\log t)^{1-\alpha}|\phi_4(t)-\phi_3(t)| = \frac{1}{\Gamma(\alpha)}(\log t)^{1-\alpha}\left|\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}B(s)[\phi_3(s)-\phi_2(s)]ds\right|$$

$$\leq \frac{1}{\Gamma(\alpha)}(\log t)^{1-\alpha}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha+l_1-1}M_B(\log s)^{k_1}\Big(|\eta|M_B^3(\log s)^{3k_1+3l_1+3}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2k_1+l_1+2)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,3k_1+2l_1+3)}{\Gamma(\alpha)}$$

$$+M_B^2M_G(\log s)^{2k_1+k_2+2l_1+l_2+3}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,k_1+k_2+l_2+2)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2k_1+k_2+l_1+l_2+3)}{\Gamma(\alpha)}\Big)\frac{ds}{s}$$

$$\leq |\eta|M_B^4(\log t)^{4k_1+4l_1+4}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2k_1+l_1+2)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,3k_1+2l_1+3)}{\Gamma(\alpha)}ds\frac{\mathbf{B}(\alpha+l_1,4k_1+3l_1+4)}{\Gamma(\alpha)}$$

$$+M_B^3M_G(\log t)^{3k_1+k_2+3l_1+l_2+4}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,k_1+k_2+l_2+2)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,2k_1+k_2+l_1+l_2+3)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,3k_1+k_2+2l_1+l_2+4)}{\Gamma(\alpha)}.$$

Similarly by the mathematical induction method, we get for every $n=1,2,\cdots$ that

$$(\log t)^{1-\alpha}|\phi_n(t)-\phi_{n-1}(t)| \leq |\eta|M_B^n(\log t)^{nk_1+nl_1+n}\prod_{i=0}^{n-1}\frac{\mathbf{B}(\alpha+l_1,(i+1)k_1+il_1+(i+1))}{\Gamma(\alpha)}$$

$$+M_B^{n-1}M_G(\log t)^{(n-1)k_1+k_2+(n-1)l_1+l_2+n}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\prod_{i=1}^{n-1}\frac{\mathbf{B}(\alpha+l_1,ik_1+k_2+(i-1)l_1+l_2+i+1)}{\Gamma(\alpha)}$$

$$\leq |\eta|M_B^n\prod_{i=0}^{n-1}\frac{\mathbf{B}(\alpha+l_1,(i+1)k_1+il_1+(i+1))}{\Gamma(\alpha)}$$

$$+M_B^{n-1}M_G\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\prod_{i=1}^{n-1}\frac{\mathbf{B}(\alpha+l,ik_1+k_2+(i-1)l_1+l_2+i+1)}{\Gamma(\alpha)}, t\in(1,e].$$

Similarly we can prove that both

$$\sum_{n=1}^{+\infty}u_n = \sum_{n=1}^{+\infty}|\eta|M_B^n\prod_{i=0}^{n-1}\frac{\mathbf{B}(\alpha+l_1,(i+1)k_1+il_1+(i+1))}{\Gamma(\alpha)},$$

$$\sum_{n=1}^{+\infty}v_n = \sum_{n=1}^{+\infty}M_B^{n-1}M_G\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\prod_{i=1}^{n-1}\frac{\mathbf{B}(\alpha+l,ik_1+k_2+(i-1)l_1+l_2+i+1)}{\Gamma(\alpha)}$$

are convergent. Hence

$$(\log t)^{1-\alpha}\phi_0(t)+(\log t)^{1-\alpha}[\phi_1(t)-\phi_0(t)]+(\log t)^{1-\alpha}[\phi_2(t)-\phi_1(t)]+\cdots+(\log t)^{1-\alpha}[\phi_n(t)-\phi_{n-1}(t)]+\cdots,$$

$t\in(1,e]$ is uniformly convergent. Then $\{t\to(\log t)^{1-\alpha}\phi_n(t)\}$ is convergent uniformly on $(1,e]$.

Claim 3. $\phi(t) = (\log t)^{\alpha-1} \lim_{n\to+\infty} (\log t)^{1-\alpha}\phi_n(t)$ defined on $(1,e]$ is a unique continuous solution of the integral equation

$$x(t) = \eta(\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} [B(s)x(s) + G(s)] \frac{ds}{s}, t \in (1,e]. \tag{3.1.14}$$

Proof. By $\lim_{n\to+\infty} (\log t)^{1-\alpha}\phi_n(t) = (\log t)^{1-\alpha}\phi(t)$ and the uniformly convergence, we see $\phi(t)$ is continuous on $(1,e]$. From

$$(\log t)^{1-\alpha} \left| \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} [A(s)\phi_{n-1}(s) + F(s)] \frac{ds}{s} - \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} [B(s)\phi_{m-1}(s) + G(s)] \frac{ds}{s} \right|$$

$$\le M_B||\phi_{n-1} - \phi_{m-1}||(\log t)^{1-\alpha} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha+l_1-1} (\log s)^{k_1} (\log s)^{\alpha-1} \frac{ds}{s}$$

$$\le M_B||\phi_{n-1} - \phi_{m-1}||(\log t)^{1-\alpha} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha+l_1-1} (\log s)^{\alpha+k_1-1} \frac{ds}{s}$$

$$\le M_B||\phi_{n-1} - \phi_{m-1}||(\log t)^{\alpha+k_1+l_1} \mathbf{B}(\alpha + l_1, \alpha + k_1)$$

$$\le M_B||\phi_{n-1} - \phi_{m-1}||\mathbf{B}(\alpha + l_1, \alpha + k_1) \to 0 \text{ uniformly as } m, n \to +\infty,$$

we know that

$$\phi(t) = (\log t)^{\alpha-1} \lim_{n\to+\infty} (\log t)^{1-\alpha}\phi_n(t) = \lim_{n\to+\infty} \left[\eta + (\log t)^{1-\alpha} \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} [B(s)\phi_{n-1}(s) + G(s)] \frac{ds}{s}\right]$$

$$= \eta(\log t)^{\alpha-1} + (\log t)^{\alpha-1} \lim_{n\to+\infty} (\log t)^{1-\alpha} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} [B(s)\phi_{n-1}(s) + G(s)] \frac{ds}{s}$$

$$= \eta(\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} [B(s)\phi(s) + G(s)] \frac{ds}{s}.$$

Then ϕ is a continuous solution of (3.1.14) defined on $(1,e]$.

Suppose that ψ defined on $(1,e]$ is also a solution of (3.1.14). Then

$$\psi(t) = \eta(\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} [B(s)\psi(s) + G(s)] \frac{ds}{s}, t \in (1,e].$$

We need to prove that $\phi(t) \equiv \psi(t)$ on $(1,e]$. Then

$$(\log t)^{1-\alpha}|\psi(t) - \phi_0(t)| = (\log t)^{1-\alpha} \left| \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |B(s)\psi(s) + G(s)| \frac{ds}{s} \right|$$

$$\le |\eta|M_B(\log t)^{k_1+l_1+1} \frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)} + M_G(\log t)^{k_2+l_2+1} \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}.$$

Furthermore, we have

$$(\log t)^{1-\alpha}|\psi(t) - \phi_1(t)| = (\log t)^{1-\alpha} \frac{1}{\Gamma(\alpha)} \left| \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} B(s)[\psi(s) - \phi_0(s)] \frac{ds}{s} \right|$$

$$\le |\eta|M_B^2(\log t)^{2k_1+2l_1+2} \frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l_1,2k_1+l_1+2)}{\Gamma(\alpha)}$$

$$+M_B M_G(\log t)^{k_1+k_2+l_1+l_2+2} \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l_1,k_1+k_2+l_2+2)}{\Gamma(\alpha)}.$$

By mathematical induction method, we can get that

$$(\log t)^{1-\alpha}|\psi(t) - \phi_n(t)| = (\log t)^{1-\alpha} \frac{1}{\Gamma(\alpha)} \left| \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} B(s)[\psi(s) - \phi_{n-1}(s)]ds \right|$$

$$\leq |\eta| M_B^n (\log t)^{nk_1+nl_1+n} \prod_{i=0}^{n-1} \frac{\mathbf{B}(\alpha+l_1,(i+1)k_1+il_1+(i+1))}{\Gamma(\alpha)}$$

$$+M_B^{n-1} M_G (\log t)^{(n-1)k_1+k_2+(n-1)l_1+l_2+n} \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)} \prod_{i=1}^{n-1} \frac{\mathbf{B}(\alpha+l,ik_1+k_2+(i-1)l_1+l_2+i+1)}{\Gamma(\alpha)}$$

$$\leq |\eta| M_B^n \prod_{i=0}^{n-1} \frac{\mathbf{B}(\alpha+l_1,(i+1)k_1+il_1+(i+1))}{\Gamma(\alpha)}$$

$$+M_B^{n-1} M_G \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)} \prod_{i=1}^{n-1} \frac{\mathbf{B}(\alpha+l,ik_1+k_2+(i-1)l_1+l_2+i+1)}{\Gamma(\alpha)}, \text{ for all } n = 1, 2, \cdots.$$

Similarly we have $\lim_{n\to+\infty} (\log t)^{1-\alpha} \phi_n(t) = (\log t)^{1-\alpha} \psi(t)$ uniformly on $(1, e]$. Then $\phi(t) \equiv \psi(t)$ on $(1, e]$. Then (3.1.14) has a unique solution ϕ. The proof is complete.

Theorem 3.5. *Suppose that (A3) holds. Then $x \in LC_{1-\alpha}(1, e]$ is a solution of IVP(3.1.3) if and only if $x \in LC_{1-\alpha}(1, e]$ is a solution of the integral equation (3.1.14).*

Proof. Suppose that $x \in C_{1-\alpha}(0, 1]$ is a solution of IVP(3.1.3). Then $t \to (\log t)^{1-\alpha} x(t)$is continuous on $(1, e]$ by defining $(\log t)^{1-\alpha} x(t)|_{t=1} = \lim_{t\to 1^+} (\log t)^{1-\alpha} x(t)$ and $||x|| = r < +\infty$. So

$$\lim_{s\to 1^+} \int_1^s \left(\log \frac{s}{w}\right)^{-\alpha} x(w) \frac{dw}{w} = \lim_{s\to 1^+} \int_1^s \left(\log \frac{s}{w}\right)^{-\alpha} (\log w)^{\alpha-1} (\log w)^{1-\alpha} x(w) \frac{dw}{w}$$

$$= \lim_{s\to 1^+} (\log \xi)^{1-\alpha} x(\xi) \int_1^s \left(\log \frac{s}{w}\right)^{-\alpha} (\log w)^{\alpha-1} \frac{dw}{w} \text{ by mean value theorem of integral, } \xi \in (1, s)$$

$$= \lim_{s\to 1^+} (\log \xi)^{1-\alpha} x(\xi) \int_0^1 (1-u)^{-\alpha} u^{\alpha-1} du \text{ by } \frac{\log w}{\log s} = u$$

$$= \eta \mathbf{B}(1-\alpha, \alpha).$$

From (A3), we have

$$(\log t)^{1-\alpha} \left| \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} [B(s)x(s) + G(s)] \frac{ds}{s} \right|$$

$$\leq (\log t)^{1-\alpha} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} [M_B r (\log s)^{\alpha-1} (\log s)^{k_1} (1-\log s)^{l_1} + M_G (\log s)^{k_2} (1-\log s)^{l_2}] \frac{ds}{s}$$

$$\leq (\log t)^{1-\alpha} M_B r \int_1^t \left(\log \frac{t}{s}\right)^{\alpha+l_1-1} (\log s)^{\alpha+k_1-1} \frac{ds}{s} + (\log t)^{1-\alpha} M_G \int_1^t \left(\log \frac{t}{s}\right)^{\alpha+l_2-1} (\log s)^{k_2} \frac{ds}{s}$$

$$= M_B r (\log t)^{\alpha+k_1+l_1} \mathbf{B}(\alpha+l_1, k_1+\alpha) + M_G (\log t)^{1+k_1+l_1} \mathbf{B}(\alpha+l_2, k_2+1).$$

So $t \to (\log t)^{1-\alpha} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} [B(s)x(s) + G(s)] \frac{ds}{s}$ is defined on $(1, e]$ and

$$\lim_{t\to 1^+} (\log t)^{1-\alpha} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} [B(s)x(s) + G(s)] \frac{ds}{s} = 0. \tag{3.1.15}$$

Furthermore, we have similarly to Theorem 3.1.1 that $t \to \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} [B(s)x(s) + G(s)] \frac{ds}{s}$ is continuous on $(1, e]$. So $t \to (\log t)^{1-\alpha} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} [B(s)x(s) + G(s)] \frac{ds}{s}$ is continuous on $[1, e]$ by defining

$$(\log t)^{1-\alpha} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} [B(s)x(s) + G(s)] \frac{ds}{s}\bigg|_{t=1} = \lim_{t\to 1^+} (\log t)^{1-\alpha} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} [B(s)x(s) + G(s)] \frac{ds}{s}. \tag{3.1.16}$$

We have ${}^{H}I_{1^+}^{\alpha}\,{}^{RLH}D_{1^+}^{\alpha}x(t)={}^{H}I_{1^+}^{\alpha}[B(t)x(t)+G(t)]$. So

$$\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}[B(s)x(s)+G(s)]\frac{ds}{s}={}^{H}I_{1^+}^{\alpha}[B(t)x(t)+G(t)]={}^{H}I_{1^+}^{\alpha}\,{}^{RLH}D_{1^+}^{\alpha}x(t)$$

$$=\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{1}{\Gamma(1-\alpha)}s\left(\int_1^s\left(\log\frac{s}{w}\right)^{-\alpha}x(w)\frac{dw}{w}\right)'\frac{ds}{s}$$

$$=\frac{1}{\Gamma(\alpha+1)}\frac{1}{\Gamma(1-\alpha)}t\left[\int_1^t\left(\log\frac{t}{s}\right)^{\alpha}\left(\int_1^s\left(\log\frac{s}{w}\right)^{-\alpha}x(w)\frac{dw}{w}\right)'ds\right]'$$

$$=\frac{1}{\Gamma(1-\alpha)\Gamma(\alpha+1)}t\left[\left(\log\frac{t}{s}\right)^{\alpha}\int_1^s\left(\log\frac{s}{w}\right)^{-\alpha}x(w)\frac{dw}{w}\Big|_1^t+\alpha\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\int_1^s\left(\log\frac{s}{w}\right)^{-\alpha}x(w)\frac{dw}{w}\frac{ds}{s}\right]'$$

$$=\frac{1}{\Gamma(1-\alpha)\Gamma(\alpha+1)}t\left[(\log t)^{\alpha}\lim_{s\to1^+}\int_1^s\left(\log\frac{s}{w}\right)^{-\alpha}x(w)\frac{dw}{w}+\alpha\int_1^t\int_w^t\left(\log\frac{t}{s}\right)^{\alpha-1}\left(\log\frac{s}{w}\right)^{-\alpha}\frac{ds}{s}x(w)\frac{dw}{w}\right]'$$

$$=\frac{1}{\Gamma(1-\alpha)\Gamma(\alpha+1)}t\left[(\log t)^{\alpha}\eta\mathbf{B}(1-\alpha,\alpha)+\alpha\int_1^t\mathbf{B}(\alpha,1-\alpha)x(w)\frac{dw}{w}\right]'$$

$$=x(t)-\eta(\log t)^{\alpha-1}.$$

Then $x\in LC_{1-\alpha}(1,e]$ is a solution of (3.1.14).

On the other hand, if x is a solution of (3.1.14), together with Cases 1, 2, 3 and (3.1.15) implies $\lim_{t\to1^+}(\log t)^{1-\alpha}x(t)=\eta$. Then $x\in LC_{1-\alpha}(1,e]$. Furthermore, we have by Definition 2.5 that

$${}^{RLH}D_{1^+}^{\alpha}x(t)=\frac{1}{\Gamma(1-\alpha)}t\left(\int_1^t\left(\log\frac{t}{s}\right)^{-\alpha}x(s)\frac{ds}{s}\right)'$$

$$=\frac{1}{\Gamma(1-\alpha)}t\left[\int_1^t\left(\log\frac{t}{s}\right)^{-\alpha}\left(\eta(\log s)^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_1^s\left(\log\frac{s}{w}\right)^{\alpha-1}[A(w)x(w)+F(w)]\frac{dw}{w}\right)\frac{ds}{s}\right]'$$

$$=\frac{1}{\Gamma(1-\alpha)}t\left[\eta\int_1^t\left(\log\frac{t}{s}\right)^{-\alpha}(\log s)^{\alpha-1}\frac{ds}{s}\right]'$$

$$+\frac{1}{\Gamma(1-\alpha)}t\left[\frac{1}{\Gamma(\alpha)}\int_1^s\left(\log\frac{s}{w}\right)^{\alpha-1}[A(w)x(w)+F(w)]\frac{dw}{w}\frac{ds}{s}\right]'$$

$$=\frac{1}{\Gamma(1-\alpha)}t\left[\eta\mathbf{B}(1-\alpha,\alpha)\right]'+\frac{1}{\Gamma(1-\alpha)}t\left[\frac{1}{\Gamma(\alpha)}\int_1^t\int_w^t\left(\log\frac{t}{s}\right)^{-\alpha}\left(\log\frac{s}{w}\right)^{\alpha-1}\frac{ds}{s}[A(w)x(w)+F(w)]\frac{dw}{w}\right]'$$

$$=\frac{1}{\Gamma(1-\alpha)}t\left[\frac{1}{\Gamma(\alpha)}\int_1^t\mathbf{B}(1-\alpha,\alpha)[B(w)x(w)+G(w)]\frac{dw}{w}\right]'$$

$$=B(t)x(t)+G(t).$$

So $x\in LC_{1-\alpha}(1,e]$ is a solution of IVP(3.1.3). The proof is completed. □

Theorem 3.6. *Suppose that (A3) holds. Then (3.1.14) has a unique solution. If $B(t)\equiv\lambda$ and there exists constants $k_2>-1$, $l_2\le0$ with $l_2>\{-\alpha,-1-k_2\}$ and $M_G\ge0$ such that $|G(t)|\le M_G(\log t)^{k_2}(1-\log t)^{l_2}$ for all $t\in(1,e)$, then following special problem*

$$\begin{cases}{}^{RLH}D_{1^+}^{\mathbf{ff}}x(t)=\lambda x(t)+G(t),\ t\in(1,e],\\ \lim_{t\to0^+}(\log t)^{1-\alpha}x(t)=\eta\end{cases}\qquad(3.1.17)$$

has a unique solution

$$x(t)=\eta\Gamma(\alpha)(\log t)^{\alpha-1}E_{\alpha,\alpha}(\lambda(\log t)^{\alpha})+\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{s}\right)^{\alpha}\right)G(s)\frac{ds}{s},\ t\in(1,e].\qquad(3.1.18)$$

Proof. From Claims 1, 2 and 3, (3.1.14) has a unique solution. From the assumption and $B(t)\equiv\lambda$, one sees that (A3) holds with $k_1=l_1=0$ and k_2,l_2 mentioned in assumption. Thus (3.1.17) has a unique solution. We get from the Picard function sequence that

$$\phi_n(t)=\eta(\log t)^{\alpha-1}+\lambda\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\phi_{n-1}(s)ds+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}G(s)\frac{ds}{s}$$

$$=\eta(\log t)^{\alpha-1}+\frac{\eta\lambda}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}(\log s)^{\alpha-1}\frac{ds}{s}+\frac{\lambda^2}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{1}{\Gamma(\alpha)}\int_1^s\left(\log\frac{s}{w}\right)^{\alpha-1}\phi_{n-2}(w)\frac{dw}{w}\frac{ds}{s}$$

$$+\frac{\lambda}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{1}{\Gamma(\alpha)}\left(\log\frac{s}{w}\right)^{\alpha-1}G(w)\frac{dw}{w}\frac{ds}{s}+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}G(s)\frac{ds}{s}$$

$$=\eta(\log t)^{\alpha-1}+\frac{\eta\lambda(\log t)^{2\alpha-1}}{\Gamma(2\alpha)}+\frac{\lambda^2}{\Gamma(\alpha)}\int_1^t\int_w^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{1}{\Gamma(\alpha)}\left(\log\frac{s}{w}\right)^{\alpha-1}\frac{ds}{s}\phi_{n-2}(w)\frac{dw}{w}$$

$$+\frac{\lambda}{\Gamma(\alpha)}\int_1^t\int_w^t\left(\log\frac{t}{s}\right)^{\alpha-1}\left(\log\frac{s}{w}\right)^{\alpha-1}\frac{1}{\Gamma(\alpha)}\frac{ds}{s}G(w)\frac{dw}{w}+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}G(s)\frac{ds}{s}$$

$$=\cdots\cdots\cdots$$

$$=\eta\Gamma(\alpha)(\log t)^{\alpha-1}\sum_{j=0}^{n-1}\frac{\lambda^j(\log t)^{j\alpha}}{\Gamma((j+1)\alpha)}+\frac{\eta\lambda^n}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{dw}{w}+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\left(\sum_{j=0}^{n-1}\frac{\lambda^j(t-s)^{j\alpha}}{\Gamma((j+1)\alpha)}\right)F(s)\frac{ds}{s}$$

$$=\eta\Gamma(\alpha)(\log t)^{\alpha-1}\sum_{j=0}^{n}\frac{\lambda^j(\log t)^{j\alpha}}{\Gamma((j+1)\alpha)}+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\left(\sum_{j=0}^{n}\frac{\lambda^j\left(\frac{t}{s}\right)^{j\alpha}}{\Gamma((j+1)\alpha)}\right)G(s)\frac{ds}{s}$$

$$\to\eta\Gamma(\alpha)(\log t)^{\alpha-1}E_{\alpha,\alpha}(\lambda(\log t)^{\alpha})+\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{s}\right)^{\alpha}\right)G(s)\frac{ds}{s}.$$

Then we get (3.1.18). The proof is complete. □

To get solutions of (3.1.4), we need the following assumptions:

(A4) there exists constants $k_i>-1$, $l_i\le 0$ with $l_i>\{-\alpha,-\alpha-k_i\}$, $M_B\ge 0$ and $M_G\ge 0$ such that $|B(t)|\le M_B(\log t)^{k_1}(1-\log t)^{l_1}$ and $|G(t)|\le M_G(\log t)^{k_2}(1-\log t)^{l_2}$ for all $t\in(1,e)$.

Choose Picard function sequence as

$$\phi_0(t)=\eta,\ t\in(1,e],$$

$$\phi_n(t)=\eta+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}[B(s)\phi_{n-1}(s)+G(s)]\frac{ds}{s},t\in(1,e],n=1,2,\cdots.$$

Claim 1. $\phi_n\in C(1,e]$. Since $\phi_0\in C(1,e]$, then ϕ_1 is continuous on $(1,e]$ and

$$\left|\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}[B(s)\phi_0(s)+G(s)]\frac{ds}{s}\right|$$

$$\le\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}[M_B|\eta|(\log s)^{k_1}(1-\log s)^{l_1}+M_G(\log s)^{k_2}(1-\log s)^{l_2}]\frac{ds}{s}$$

$$\le M_B|\eta|\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}(\log s)^{k_1}(1-\log s)^{l_1}\frac{ds}{s}$$

$$+M_G\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}(\log s)^{k_2}(1-\log s)^{l_2}\frac{ds}{s}$$

$$=M_B|\eta|(\log t)^{\alpha+k_1+l_1}\mathbf{B}(\alpha+l_1,k_1+1)+M_G(\log t)^{\alpha+k_2+l_2}\mathbf{B}(\alpha+l_2,k_2+1)\to 0\text{ as }t\to 0^+,$$

we get that $\lim_{t\to 1^+}\phi_1(s)$ exists. Then $\phi_1\in C(1,e]$. By mathematical induction method, we see that $\phi_n\in C(1,e]$.

Claim 2. ϕ_n is convergent uniformly on $(1,e]$. In fact we have for $t\in(1,e]$ that

$$|\phi_1(t)-\phi_0(t)|=\left|\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}[B(s)\phi_0(s)+G(s)]\frac{ds}{s}\right|$$

$$\le\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}[M_B|\eta|(\log s)^{k_1}(1-\log s)^{l_1}+M_G(\log s)^{k_2}(1-\log s)^{l_2}]\frac{ds}{s}$$

$$\leq |\eta| M_B \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha+l_1-1} (\log s)^{k_1} \frac{ds}{s} + M_G \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha+l_2-1} (\log s)^{k_2} \frac{ds}{s}$$

$$= |\eta| M_B (\log t)^{\alpha+k_1+l_1} \frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)} + M_G (\log t)^{\alpha+k_2+l_2} \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}.$$

So

$$|\phi_2(t)-\phi_1(t)| = \left|\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} B(s)[\phi_1(s)-\phi_0(s)] \frac{ds}{s}\right|$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} [M_B(\log s)^{k_1}(1-\log s)^{l_1} \left(|\eta| M_B (\log s)^{\alpha+k_1+l_1} \frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\right.$$

$$\left.+ M_G (\log s)^{\alpha+k_2+l_2} \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\right) \frac{ds}{s}$$

$$\leq \frac{1}{\Gamma(\alpha)} |\eta| M_B^2 \int_1^t \left(\log\frac{t}{s}\right)^{\alpha+l_1-1} (\log s)^{\alpha+2k_1+l_1} \frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)} \frac{ds}{s}$$

$$+ M_B M_G \leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha+l_1-1} (\log s)^{\alpha+k_1+k_2+l_2} \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)} \frac{ds}{s}$$

$$= |\eta| M_B^2 (\log t)^{2k_1+2l_1+2} \frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l_1,\alpha+2k_1+l_1+1)}{\Gamma(\alpha)}$$

$$+ M_B M_G (\log t)^{2\alpha+k_1+k_2+l_1+l_2} \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l_1,\alpha+k_1+k_2+l_2+1)}{\Gamma(\alpha)},$$

and

$$|\phi_3(t)-\phi_2(t)| = \left|\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} B(s)[\phi_2(s)-\phi_1(s)] \frac{ds}{s}\right|$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} M_B(\log s)^{k_1}(1-\log s)^{l_1} \left(|\eta| M_B^2 (\log s)^{2k_1+2l_1+2} \frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l_1,\alpha+2k_1+l_1+1)}{\Gamma(\alpha)}\right.$$

$$\left.+ M_B M_G (\log s)^{2\alpha+k_1+k_2+l_1+l_2} \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l_1,\alpha+k_1+k_2+l_2+1)}{\Gamma(\alpha)}\right) \frac{ds}{s}$$

$$\leq |\eta| M_B^3 \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha+l_1-1} (\log) s^{2\alpha+3k_1+2l_1} \frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l_1,\alpha+2k_1+l_1+1)}{\Gamma(\alpha)} \frac{ds}{s}$$

$$+ M_B^2 M_G \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha+l_1-1} (\log s)^{2\alpha+2k_1+k_2+l_1+l_2} \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l_1,\alpha+k_1+k_2+l_2+1)}{\Gamma(\alpha)} \frac{ds}{s}$$

$$= |\eta| M_B^3 (\log t)^{3\alpha+3k_1+3l_1} \frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l_1,\alpha+2k_1+l_1+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l_1,2\alpha+3k_1+2l_1+1)}{\Gamma(\alpha)}$$

$$+ M_B^2 M_G (\log t)^{3\alpha+2k_1+k_2+2l_1+l_2} \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l_1,\alpha+k_1+k_2+l_2+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l_1,2\alpha+2k_1+k_2+l_1+l_2+1)}{\Gamma(\alpha)}.$$

$$|\phi_4(t)-\phi_3(t)| = \left|\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} B(s)[\phi_3(s)-\phi_2(s)] \frac{ds}{s}\right|$$

$$\frac{|\eta| M_B^4}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha+l_1-1} (\log s)^{3\alpha+4k_1+3l_1} \frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l_1,\alpha+2k_1+l_1+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l_1,2\alpha+3k_1+2l_1+1)}{\Gamma(\alpha)} \frac{ds}{s}$$

$$+ \frac{M_B^3 M_G}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha+l_1-1} (\log s)^{3\alpha+3k_1+k_2+2l_1+l_2} \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l_1,\alpha+k_1+k_2+l_2+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l_1,2\alpha+2k_1+k_2+l_1+l_2+1)}{\Gamma(\alpha)} \frac{ds}{s}$$

$$\leq |\eta| M_B^4 (\log t)^{4\alpha+4k_1+4l_1} \frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l_1,\alpha+2k_1+l_1+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l_1,2\alpha+3k_1+2l_1+1)}{\Gamma(\alpha)} \frac{ds}{s} \frac{\mathbf{B}(\alpha+l_1,3\alpha+4k_1+3l_1+1)}{\Gamma(\alpha)}$$

$$+ M_B^3 M_G (\log t)^{4\alpha+3k_1+k_2+3l_1+l_2} \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l_1,\alpha+k_1+k_2+l_2+1)}{\Gamma(\alpha)} \frac{\mathbf{B}(\alpha+l_1,2\alpha+2k_1+k_2+l_1+l_2+1)}{\Gamma(\alpha)} \times$$

$$\frac{\mathbf{B}(\alpha+l_1,3\alpha+3k_1+k_2+2l_1+l_2+1)}{\Gamma(\alpha)}.$$

Similarly by the mathematical induction method, we get for every $n = 1, 2, \cdots$ that

$$|\phi_n(t) - \phi_{n-1}(t)| \le |\eta| M_B^n (\log t)^{n\alpha + nk_1 + nl_1} \prod_{i=0}^{n-1} \frac{\mathbf{B}(\alpha+l_1, i\alpha+(i+1)k_1+il_1+1}{\Gamma(\alpha)}$$

$$+M_B^{n-1} M_G (\log t)^{n\alpha+(n-1)k_1+k_2+(n-1)l_1+l_2} \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)} \prod_{i=1}^{n-1} \frac{\mathbf{B}(\alpha+l, i\alpha+ik_1+k_2+(i-1)l_1+l_2+1)}{\Gamma(\alpha)}$$

$$\le |\eta| M_B^n \prod_{i=0}^{n-1} \frac{\mathbf{B}(\alpha+l_1, i\alpha+(i+1)k_1+il_1+1}{\Gamma(\alpha)}$$

$$+M_B^{n-1} M_G \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)} \prod_{i=1}^{n-1} \frac{\mathbf{B}(\alpha+l, i\alpha+ik_1+k_2+(i-1)l_1+l_2+1)}{\Gamma(\alpha)}, t \in (1, e].$$

Similarly we can prove that both

$$\sum_{n=1}^{+\infty} u_n = \sum_{n=1}^{+\infty} |\eta| M_B^n \prod_{i=0}^{n-1} \frac{\mathbf{B}(\alpha+l_1, i\alpha+(i+1)k_1+il_1+1}{\Gamma(\alpha)},$$

and

$$\sum_{n=1}^{+\infty} v_n = \sum_{n=1}^{+\infty} M_B^{n-1} M_G \frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)} \prod_{i=1}^{n-1} \frac{\mathbf{B}(\alpha+l, i\alpha+ik_1+k_2+(i-1)l_1+l_2+1)}{\Gamma(\alpha)}$$

are convergent. Hence

$$t^{1-\alpha}\phi_0(t) + t^{1-\alpha}[\phi_1(t) - \phi_0(t)] + t^{1-\alpha}[\phi_2(t) - \phi_1(t)] + \cdots + t^{1-\alpha}[\phi_n(t) - \phi_{n-1}(t)] + \cdots, t \in (1, e]$$

is uniformly convergent. Then $\{\phi_n(t)\}$ is convergent uniformly on $(1, e]$.

Claim 3. $\phi(t) = \lim_{n\to+\infty} \phi_n(t)$ defined on $(1, e]$ is a unique continuous solution of the integral equation

$$x(t) = \eta + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} [B(s)x(s) + G(s)] \frac{ds}{s}. \tag{3.1.19}$$

Proof. By $\lim_{n\to+\infty} \phi_n(t) = \phi(t)$ and the uniformly convergence, we see $\phi(t)$ is continuous on $(1, e]$. From

$$\left| \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} [B(s)\phi_{n-1}(s) + G(s)] \frac{ds}{s} - \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} [B(s)\phi_{m-1}(s) + G(s)] \frac{ds}{s} \right|$$

$$\le M_B ||\phi_{n-1} - \phi_{m-1}|| \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} (\log s)^{k_1} (1 - \log s)^{l_1} \frac{ds}{s}$$

$$\le M_B ||\phi_{n-1} - \phi_{m-1}|| \int_1^t \left(\log \frac{t}{s}\right)^{\alpha+l_1-1} (\log s)^{k_1} \frac{ds}{s}$$

$$\le M_B ||\phi_{n-1} - \phi_{m-1}|| (\log t)^{\alpha+k_1+l_1} \frac{\mathbf{B}(\alpha+l_1, k_1+1)}{\Gamma(\alpha)}$$

$$\le M_B ||\phi_{n-1} - \phi_{m-1}|| \frac{\mathbf{B}(\alpha+l_1, k_1+1)}{\Gamma(\alpha)} \to 0 \text{ uniformly as } m, n \to +\infty,$$

we know that

$$\phi(t) = \lim_{n\to\infty} \phi_n(t) = \lim_{n\to+\infty} \left[\eta + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} [B(s)\phi_{n-1}(s) + G(s)] \frac{ds}{s}\right]$$

$$= \eta + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} [B(s)\phi(s) + G(s)] \frac{ds}{s}.$$

Then ϕ is a continuous solution of (3.1.19) defined on $(1, e]$.

Suppose that ψ defined on $(1, e]$ is also a solution of (3.19). Then

$$\psi(t) = \eta + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} [B(s)\phi(s) + G(s)] \frac{ds}{s}, t \in (1, e].$$

We need to prove that $\phi(t) \equiv \psi(t)$ on $(0,1]$. Now we have

$$|\psi(t)-\phi_0(t)| = \frac{1}{\Gamma(\alpha)}\left|\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}[B(s)\phi(s)+G(s)]\frac{ds}{s}\right|$$

$$\leq |\eta| M_B (\log t)^{\alpha+k_1+l_1}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)} + M_G(\log t)^{\alpha+k_2+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}.$$

Furthermore, we have

$$|\psi(t)-\phi_1(t)| = \frac{1}{\Gamma(\alpha)}\left|\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}B(s)[\psi(s)-\phi_0(s)]\frac{ds}{s}\right|$$

$$\leq |\eta| M_B^2(\log t)^{2k_1+2l_1+2}\frac{\mathbf{B}(\alpha+l_1,k_1+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,\alpha+2k_1+l_1+1)}{\Gamma(\alpha)}$$

$$+M_BM_G(\log t)^{2\alpha+k_1+k_2+l_1+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\frac{\mathbf{B}(\alpha+l_1,\alpha+k_1+k_2+l_2+1)}{\Gamma(\alpha)}.$$

By mathematical induction method, we can get that

$$|\psi(t)-\phi_n(t)| = \frac{1}{\Gamma(\alpha)}\left|\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}B(s)[\psi(s)-\phi_{n-1}(s)]\frac{ds}{s}\right|$$

$$\leq |\eta| M_B^n(\log t)^{n\alpha+nk_1+nl_1}\prod_{i=0}^{n-1}\frac{\mathbf{B}(\alpha+l_1,i\alpha+(i+1)k_1+il_1+1}{\Gamma(\alpha)}$$

$$+M_B^{n-1}M_G(\log t)^{n\alpha+(n-1)k_1+k_2+(n-1)l_1+l_2}\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\prod_{i=1}^{n-1}\frac{\mathbf{B}(\alpha+l,i\alpha+ik_1+k_2+(i-1)l_1+l_2+1)}{\Gamma(\alpha)}$$

$$\leq |\eta| M_B^n\prod_{i=0}^{n-1}\frac{\mathbf{B}(\alpha+l_1,i\alpha+(i+1)k_1+il_1+1}{\Gamma(\alpha)}$$

$$+M_B^{n-1}M_G\frac{\mathbf{B}(\alpha+l_2,k_2+1)}{\Gamma(\alpha)}\prod_{i=1}^{n-1}\frac{\mathbf{B}(\alpha+l,i\alpha+ik_1+k_2+(i-1)l_1+l_2+1)}{\Gamma(\alpha)}, t\in(1,e].$$

Similarly we have $\lim_{n\to+\infty}\phi_n(t)=\psi(t)$ uniformly on $(1,e]$. Then $\phi(t)\equiv\psi(t)$ on $(1,e]$. Then (3.1.19) has a unique solution ϕ. The proof is complete.

Theorem 3.7. *Suppose that (A4) holds. Then $x\in C(1,e]$ is a solution of IVP(3.1.4) if and only if $x\in C(1,e]$ is a solution of the integral equation (3.1.19).*

Proof. Suppose that $x\in C(1,e]$ is a solution of IVP(3.1.4). Then $t\to x(t)$is continuous on $[0,1]$ by defining $x(t)|_{t=0}=\lim_{t\to0^+}x(t)$ and $||x||=r<+\infty$. One can see that

$$\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}(\log s)^{k_1}(1-\log s)^{l_1}\frac{ds}{s}\leq\int_1^t\left(\log\frac{t}{s}\right)^{\alpha+l_1-1}(\log s)^{k_1}\frac{ds}{s}\text{ by }\frac{\log s}{\log t}=u$$

$$=(\log t)^{\alpha+k_1+l_1}\int_0^1(1-u)^{\alpha+l_1-1}u^{k_1}du\leq(\log t)^{\alpha+k_1+l_1}\int_0^1(1-u)^{\alpha+l_1-1}u^{k_1}du$$

$$=(\log t)^{\alpha+k_1+l_1}\mathbf{B}(\alpha+l_1,k_1+1).$$

From (A4), we have for $t\in(1,e]$ that

$$\left|\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}[B(s)x(s)+G(s)]\frac{ds}{s}\right|$$

$$\leq\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}[M_Br(\log s)^{k_1}(1-\log s)^{l_1}+M_G(\log s)^{k_2}(1-\log s)^{l_2}]\frac{ds}{s}$$

$$\leq M_Br\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}(\log s)^{k_1}(1-\log s)^{l_1}\frac{ds}{s}$$

$$+M_G\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}(\log s)^{k_2}(1-\log s)^{l_2}\frac{ds}{s}$$

$$= M_B r(\log t)^{\alpha+k_1+l_1}\mathbf{B}(\alpha+l_1,k_1+1)$$

$$+M_G(\log t)^{\alpha+k_2+l_2}\mathbf{B}(\alpha+l_2,k_2+1).$$

So $t \to \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1}[B(s)x(s)+G(s)]\frac{ds}{s}$ is defined on $(1,e]$ and

$$\lim_{t\to 1^+}\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1}[B(s)x(s)+G(s)]\frac{ds}{s}=0. \tag{3.1.20}$$

Furthermore, we have similarly to Theorem 3.1 that $t \to \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1}[B(s)x(s)+G(s)]\frac{ds}{s}$ is continuous on $(0,1]$. So $t \to \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1}[B(s)x(s)+G(s)]\frac{ds}{s}$ is continuous on $[0,1]$ by defining

$$\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1}[B(s)x(s)+G(s)]\frac{ds}{s}\Big|_{t=1}=0. \tag{3.1.21}$$

One sees that

$$\int_w^t \left(\log \frac{t}{s}\right)^{\alpha-1}\left(\log \frac{s}{w}\right)^{-\alpha}\frac{ds}{s} \text{ by } \frac{\log s-\log w}{\log t-\log w}=u$$

$$=\int_0^1 (1-u)^{\alpha-1}u^{-\alpha}du=\Gamma(1-\alpha)\Gamma(\alpha).$$

We have by Definition 2.6 and ${}^{H}I_{1^+}^{\alpha}\,{}^{CH}D_{1^+}^{\alpha}x(t)={}^{H}I_{1^+}^{\alpha}[B(t)x(t)+G(t)]$. So

$$\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1}[A(s)x(s)+F(s)]\frac{ds}{s}={}^{H}I_{1^+}^{\alpha}[B(t)x(t)+G(t)]$$

$$={}^{H}I_{1^+}^{\alpha}\,{}^{CH}D_{1^+}^{\alpha}x(t)$$

$$=\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1}\left[\frac{1}{\Gamma(1-\alpha)}\int_1^s \left(\log \frac{s}{w}\right)^{-\alpha}wx'(w)\frac{dw}{w}\right]\frac{ds}{s}$$

$$=\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1}\left[\frac{1}{\Gamma(1-\alpha)}\int_0^s \left(\log \frac{s}{w}\right)^{-\alpha}x'(w)dw\right]\frac{ds}{s}$$

$$=\frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(1-\alpha)}\int_1^t\int_w^t \left(\log \frac{t}{s}\right)^{\alpha-1}\left(\log \frac{s}{w}\right)^{-\alpha}\frac{ds}{s}x'(w)dw$$

$$=\int_1^t x'(w)dw=x(t)-\lim_{t\to 1^+}x(t)=x(t)-\eta.$$

Then $x\in C(1,e]$ is a solution of (3.1.19).

On the other hand, if $x\in C(1,e]$ is a solution of (3.1.19), together with (3.1.20) implies $\lim_{t\to 1^+}(t)=\eta$. Furthermore, we have that

$${}^{CH}D_{1^+}^{\alpha}x(t)=\frac{1}{\Gamma(1-\alpha)}\int_1^t \left(\log \frac{t}{s}\right)^{-\alpha}sx'(s)\frac{ds}{s}$$

$$=\frac{1}{\Gamma(1-\alpha)}\int_1^t \left(\log \frac{t}{s}\right)^{-\alpha}\left(\eta+\frac{1}{\Gamma(\alpha)}\int_1^s \left(\log \frac{s}{w}\right)^{\alpha-1}[B(w)x(w)+G(w)]\frac{dw}{w}\right)'ds$$

$$=\frac{1}{\Gamma(1-\alpha)}\int_1^t \left(\log \frac{t}{s}\right)^{-\alpha}\left(\frac{1}{\Gamma(\alpha)}\int_1^s \left(\log \frac{s}{w}\right)^{\alpha-1}[B(w)x(w)+G(w)]\frac{dw}{w}\right)'ds$$

$$=\frac{t}{\Gamma(2-\alpha)}\left(\int_1^t \left(\log \frac{t}{s}\right)^{1-\alpha}\left(\frac{1}{\Gamma(\alpha)}\int_1^s \left(\log \frac{s}{w}\right)^{\alpha-1}[B(w)x(w)+G(w)]\frac{dw}{w}\right)'ds\right)'$$

$$=\frac{t}{\Gamma(2-\alpha)}\left[\left(\log \frac{t}{s}\right)^{1-\alpha}\frac{1}{\Gamma(\alpha)}\int_1^s \left(\log \frac{s}{w}\right)^{\alpha-1}[B(w)x(w)+G(w)]\frac{dw}{w}\Big|_1^t\right.$$

$$\left.+(1-\alpha)\frac{1}{s}\int_1^t \left(\log \frac{t}{s}\right)^{-\alpha}\frac{1}{\Gamma(\alpha)}\int_1^s \left(\log \frac{s}{w}\right)^{\alpha-1}[B(w)x(w)+G(w)]\frac{dw}{w}ds\right]'$$

$$= \frac{t}{\Gamma(1-\alpha)}\frac{1}{\Gamma(\alpha)}\left[\frac{1}{s}\int_1^t\left(\log\frac{t}{s}\right)^{-\alpha}\int_1^s\left(\log\frac{s}{w}\right)^{\alpha-1}[B(w)x(w)+G(w)]\frac{dw}{w}ds\right]'$$

$$= \frac{t}{\Gamma(1-\alpha)}\frac{1}{\Gamma(\alpha)}\left[\int_1^t\int_w^t\left(\log\frac{t}{s}\right)^{-\alpha}\left(\log\frac{s}{w}\right)^{\alpha-1}\frac{ds}{s}[B(w)x(w)+G(w)]\frac{dw}{w}\right]' \text{ by } \frac{\log s}{\log t}=u$$

$$= \frac{t}{\Gamma(1-\alpha)}\frac{1}{\Gamma(\alpha)}\left[\int_1^t\int_0^1(1-u)^{-\alpha}u^{\alpha-1}du[B(w)x(w)+G(w)]\frac{dw}{w}\right]' \text{ by } \mathbf{B}(1-\alpha,\alpha)=\Gamma(1-\alpha)\Gamma(\alpha)$$

$$= B(t)x(t)+G(t).$$

So $x \in C(1,e]$ is a solution of IVP(3.1.4). The proof is completed. □

Theorem 3.8. *Suppose that (A4) holds. Then (3.1.4) has a unique solution. If $B(t) \equiv \lambda$ and there exists constants $k_2 > -1$, $l_2 \le 0$ with $l_2 > \{-\alpha, -\alpha - k_2\}$ and $M_G \ge 0$ such that $|G(t)| \le M_G t^{k_2}(1-t)^{l_2}$ for all $t \in (1,e)$, then following special problem*

$$\begin{cases} {}^{CH}D_{0^+}^{\mathbf{ff}}x(t) = \lambda x(t) + G(t),\ t \in (1,e], \\ \lim\limits_{t\to 1^+} x(t) = \eta \end{cases} \tag{3.1.22}$$

has a unique solution

$$x(t) = \eta E_{\alpha,1}(\lambda(\log t)^\alpha) + \int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{s}\right)^\alpha\right)G(s)\frac{ds}{s},\ t \in (1,e]. \tag{3.1.23}$$

Proof. From Claims 1, 2 and 3, Theorem 3.7, (3.1.4) has a unique solution. From the assumption and $A(t) \equiv \lambda$, one sees that (A4) holds with $k_1 = l_1 = 0$ and k_2, l_2 mentioned. Thus (3.1.22) has a unique solution. We get from the Picard function sequence that

$$\phi_n(t) = \eta + \lambda\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\phi_{n-1}(s)\frac{ds}{s} + \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}G(s)\frac{ds}{s}$$

$$= \eta + \eta\lambda\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{ds}{s} + \lambda^2\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\int_1^s\left(\log\frac{s}{w}\right)^{\alpha-1}\phi_{n-2}(w)\frac{dw}{w}\frac{ds}{s}$$

$$+\lambda\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\int_1^s\left(\log\frac{s}{w}\right)^{\alpha-1}G(w)\frac{dw}{w}\frac{ds}{s} + \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}G(s)\frac{ds}{s}$$

$$= \eta + \frac{\eta\lambda(\log t)^\alpha}{\Gamma(\alpha+1)} + \lambda^2\frac{1}{\Gamma(\alpha)}\int_1^t\int_w^t\left(\log\frac{t}{s}\right)^{\alpha-1}\left(\log\frac{s}{w}\right)^{\alpha-1}\frac{ds}{s}\phi_{n-2}(w)\frac{dw}{w}$$

$$+\lambda\frac{1}{\Gamma(\alpha)}\int_0^t\int_w^t\left(\log\frac{t}{s}\right)^{\alpha-1}\left(\log\frac{s}{w}\right)^{\alpha-1}\frac{ds}{s}F(w)\frac{dw}{w} + \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}F(s)\frac{ds}{s}$$

$$= \eta\left(1 + \frac{\lambda(\log t)^\alpha}{\Gamma(\alpha+1)}\right) + \lambda^2\frac{1}{\Gamma(2\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{2\alpha-1}\phi_{n-2}(w)\frac{dw}{w}$$

$$+\lambda\frac{1}{\Gamma(2\alpha)}\int_0^t\left(\log\frac{t}{s}\right)^{2\alpha-1}G(w)\frac{dw}{w} + \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}G(s)\frac{ds}{s}$$

$$= \cdots\cdots\cdots$$

$$\eta\sum_{j=0}^n\frac{\lambda^j(\log t)^{j\alpha}}{\Gamma(j\alpha+1)} + \int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\left(\sum_{j=0}^n\frac{\lambda^j\left(\log\frac{t}{s}\right)^{j\alpha}}{\Gamma((j+1)\alpha)}\right)G(s)ds$$

$$\to \eta E_{\alpha,1}(\lambda(\log t)^\alpha) + \int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{s}\right)^\alpha\right)G(s)\frac{ds}{s}.$$

Then we get (3.1.23). The proof is complete. □

Theorem 3.9. *(Schaefer's fixed point theorem). Let E be a Banach spaces and $T : E \mapsto E$ be a completely continuous operator. If the set $E(T) = \{x = \theta(Tx) :$ for some $\theta \in [0,1], x \in E\}$ is bounded, then T has at least a fixed point in E.*

3.2 Exact piecewise continuous solutions of FDEs

In this section, we present exact piecewise continuous solutions of the following fractional differential equations, respectively

$$ {}^{C}D_{0^+}^{\mathbf{ff}}x(t) = \lambda x(t) + F(t),\ t \in (t_i, t_{i+1}], i \in \mathbb{N}_0, \tag{3.2.1}$$

$$ {}^{RL}D_{0^+}^{\mathbf{ff}}x(t) = \lambda x(t) + F(t),\ t \in (t_i, t_{i+1}], i \in \mathbb{N}_0, \tag{3.2.2}$$

$$ {}^{RLH}D_{0^+}^{\mathbf{ff}}x(t) = \lambda x(t) + G(t),\ t \in (s_i, s_{i+1}], i \in \mathbb{N}_0, \tag{3.2.3}$$

and

$$ {}^{CH}D_{0^+}^{\mathbf{ff}}x(t) = \lambda x(t) + G(t),\ t \in (s_i, s_{i+1}], i \in \mathbb{N}_0, \tag{3.2.4}$$

where $\lambda \in \mathbb{R}$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$ in (3.2.1) and (3.2.2) and $1 = t_0 < t_1 < \cdots < t_m < t_{m+1} = e$ in (3.2.3) and (3.2.4). We say that $x : (0, 1] \mapsto \mathbb{R}$ is a piecewise solution of (3.2.1) (or (3.2.2) if $x \in P_mC(0, 1]$ (or $P_mC_{1-\alpha}(0, 1]$ and satisfies (3.2.1) or (3.2.2). We say that $x(1, e] \mapsto \mathbb{R}$ is a piecewise continuous solutions of (3.2.3) (or (3.2.4)) if $x \in LP_mC_{1-\alpha}(1, e]$, (or $LP_mC(1, e]$) and x satisfies all equations in (3.2.3) (or (3.2.4)).

Theorem 3.10. *Suppose that F is continuous on $(0, 1)$ and there exist constants $k > -1$ and $l \in (-\alpha, -\alpha - k, 0]$ such that $|F(t)| \le t^k(1-t)^l$ for all $t \in (0, 1)$. Then x is a piecewise solution of (3.2.1) if and only if x and there exists constants $c_i (i \in \mathbb{N}_0) \in \mathbb{R}$ such that*

$$x(t) = \sum_{v=0}^{j} c_v \mathbf{E}_{\alpha,1}(\lambda(t-t_v)^\alpha) + \int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^\alpha)F(s)ds, t \in (t_j, t_{j+1}], j \in \mathbb{N}_0. \tag{3.2.5}$$

Proof. Firstly, we have for $t \in (t_i, t_{i+1}]$ that

$$\left|\int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^\alpha)F(s)ds\right| \le \int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^\alpha)|F(s)|ds$$

$$\le \int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^\alpha)s^k(1-s)^l ds$$

$$= \sum_{j=0}^{+\infty} \frac{\lambda^j}{\Gamma((j+1)\alpha)} \int_0^t (t-s)^{\alpha-1}(t-s)^{\alpha j}s^k(1-s)^l ds$$

$$\le \sum_{j=0}^{+\infty} \frac{\lambda^j}{\Gamma((j+1)\alpha)} \int_0^t (t-s)^{\alpha+l-1}(t-s)^{\alpha j}s^k ds$$

$$= \sum_{j=0}^{+\infty} \frac{\lambda^j}{\Gamma((j+1)\alpha)} t^{\alpha+\alpha j+k+l} \int_0^1 (1-w)^{\alpha+\alpha j+l-1}w^k dw$$

$$\le \sum_{j=0}^{+\infty} \frac{\lambda^j t^{\alpha j}}{\Gamma((j+1)\alpha)} t^{\alpha+k+l} \int_0^1 (1-w)^{\alpha+l-1}w^k dw = t^{\alpha+k+l}\mathbf{E}_{\alpha,\alpha}(\lambda t^\alpha)\mathbf{B}(\alpha+l, k+1).$$

Then $\int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^\alpha)F(s)ds$ is convergent and is continuous on $[0, 1]$. If x is a piecewise continuous solution of (3.2.5), then we know that $x \in P_mC(0, 1]$ and $\lim_{t\to t_i^+} x(t) (i \in \mathbb{N}_0)$ exist. Now we prove that x satisfies differential equation in (3.2.1). In fact, for $t \in (t_i, t_{i+1}] (i \in \mathbb{N}_0)$, we have that

$${}^{C}D_{0^+}^{\alpha}x(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}x'(s)ds = \frac{1}{\Gamma(1-\alpha)}\left[\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}} (t-s)^{-\alpha}x'(s)ds + \int_{t_i}^t (t-s)^{-\alpha}x'(s)ds\right]$$

$$= \frac{1}{\Gamma(1-\alpha)}\left[\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}(t-s)^{-\alpha}\left(\sum_{\kappa=0}^{j}c_\kappa \mathbf{E}_{\alpha,1}(\lambda(s-t_\kappa)^\alpha)+\int_0^s(s-v)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-v)^\alpha)F(v)dv\right)'ds\right.$$

$$\left.+\int_{t_i}^{t}(t-s)^{-\alpha}\left(\sum_{\kappa=0}^{i}c_\kappa \mathbf{E}_{\alpha,1}(\lambda(s-t_\kappa)^\alpha)+\int_0^s(s-v)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-v)^\alpha)F(v)dv\right)'ds\right]$$

$$= \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}(t-s)^{-\alpha}\left[\sum_{\kappa=0}^{j}c_\kappa \mathbf{E}_{\alpha,1}(\lambda(s-t_\kappa)^\alpha)\right]'ds$$

$$+\frac{1}{\Gamma(1-\alpha)}\int_{t_i}^{t}(t-s)^{-\alpha}\left[\sum_{\kappa=0}^{i}c_\kappa \mathbf{E}_{\alpha,1}(\lambda(s-t_\kappa)^\alpha)\right]'ds$$

$$+\frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t}(t-s)^{-\alpha}\left[\int_0^s(s-v)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-v)^\alpha)F(v)dv\right]'ds$$

$$= \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{i-1}\sum_{\kappa=0}^{j}c_\kappa\int_{t_j}^{t_{j+1}}(t-s)^{-\alpha}\left[\sum_{m=0}^{+\infty}\frac{\lambda^m(s-t_\kappa)^{m\alpha}}{\Gamma(m\alpha+1)}\right]'ds$$

$$+\frac{1}{\Gamma(1-\alpha)}\sum_{\kappa=0}^{i}c_\kappa\int_{t_i}^{t}(t-s)^{-\alpha}\left[\sum_{m=0}^{+\infty}\frac{\lambda^m(s-t_\kappa)^{m\alpha}}{\Gamma(m\alpha+1)}\right]'ds$$

$$+\frac{1}{\Gamma(1-\alpha)}\sum_{m=0}^{+\infty}\frac{\lambda^m}{\Gamma(\alpha(m+1))}\int_0^t(t-s)^{-\alpha}\left[\int_0^s(s-v)^{\alpha+m\alpha-1}F(v)dv\right]'ds$$

$$= \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{i-1}\sum_{\kappa=0}^{j}c_\kappa\sum_{m=1}^{+\infty}\frac{\lambda^m m\alpha}{\Gamma(m\alpha+1)}\int_{t_j}^{t_{j+1}}(t-s)^{-\alpha}(s-t_\kappa)^{m\alpha-1}ds$$

$$+\frac{1}{\Gamma(1-\alpha)}\sum_{m=1}^{+\infty}\frac{m\alpha\lambda^m}{\Gamma(m\alpha+1)}\sum_{\kappa=0}^{i}c_\kappa\int_{t_i}^{t}(t-s)^{-\alpha}(s-t_\kappa)^{m\alpha-1}ds+\sum_{m=0}^{+\infty}\lambda^m D_{0^+}^{\alpha}I_{0^+}^{\alpha(m+1)}F(t)$$

$$= \frac{1}{\Gamma(1-\alpha)}\sum_{m=1}^{+\infty}\frac{\lambda^m m\alpha}{\Gamma(m\alpha+1)}\sum_{j=0}^{i-1}\sum_{\kappa=0}^{j}c_\kappa(t-t_\kappa)^{m\alpha-\alpha}\int_{\frac{t_j-t_\kappa}{t-t_\kappa}}^{\frac{t_{j+1}-t_\kappa}{t-t_\kappa}}(1-w)^{-\alpha}w^{m\alpha-1}dw$$

$$+\frac{1}{\Gamma(1-\alpha)}\sum_{m=1}^{+\infty}\frac{m\alpha\lambda^m}{\Gamma(m\alpha+1)}\sum_{\kappa=0}^{i}c_\kappa(t-t_\kappa)^{\alpha m-\alpha}\int_{\frac{t_i-y_\kappa}{t-t_\kappa}}^{1}(1-w)^{-\alpha}w^{m\alpha-1}dw+\sum_{m=0}^{+\infty}\lambda^m I_{0^+}^{\alpha m}F(t)$$

$$= \frac{1}{\Gamma(1-\alpha)}\sum_{m=1}^{+\infty}\frac{\lambda^m m\alpha}{\Gamma(m\alpha+1)}\sum_{\kappa=0}^{i-1}c_\kappa(t-t_\kappa)^{m\alpha-\alpha}\sum_{j=\kappa}^{i-1}\int_{\frac{t_j-t_\kappa}{t-t_\kappa}}^{\frac{t_{j+1}-t_\kappa}{t-t_\kappa}}(1-w)^{-\alpha}w^{m\alpha-1}dw$$

$$+\frac{1}{\Gamma(1-\alpha)}\sum_{m=1}^{+\infty}\frac{m\alpha\lambda^m}{\Gamma(m\alpha+1)}\sum_{\kappa=0}^{i-1}c_\kappa(t-t_\kappa)^{\alpha m-\alpha}\int_{\frac{t_i-y_\kappa}{t-t_\kappa}}^{1}(1-w)^{-\alpha}w^{m\alpha-1}dw$$

$$+\frac{c_i}{\Gamma(1-\alpha)}\sum_{m=1}^{+\infty}\frac{m\alpha\lambda^m}{\Gamma(m\alpha+1)}\int_0^1(1-w)^{-\alpha}w^{m\alpha-1}dw+f(t)+\lambda\int_0^t\sum_{m=1}^{+\infty}(t-s)^{\alpha-1}\frac{\lambda^{m-1}(t-s)^{\alpha(m-1)}}{\Gamma(\alpha m)}F(s)ds$$

$$= \frac{1}{\Gamma(1-\alpha)}\sum_{m=1}^{+\infty}\frac{\lambda^m m\alpha}{\Gamma(m\alpha+1)}\sum_{\kappa=0}^{i-1}c_\kappa(t-t_\kappa)^{m\alpha-\alpha}\int_0^1(1-w)^{-\alpha}w^{m\alpha-1}dw$$

$$+\frac{c_i}{\Gamma(1-\alpha)}\sum_{m=1}^{+\infty}\frac{m\alpha\lambda^m}{\Gamma(m\alpha+1)}\int_0^1(1-w)^{-\alpha}w^{m\alpha-1}dw+f(t)+\lambda\int_0^t\sum_{m=1}^{+\infty}(t-s)^{\alpha-1}\frac{\lambda^{m-1}(t-s)^{\alpha(m-1)}}{\Gamma(\alpha m)}F(s)ds$$

$$= \lambda x(t)+F(t).$$

We have done that x satisfies (3.2.1) if x satisfies (3.2.5).

Now, we suppose that x is a solution of (3.2.1). We will prove that x satisfies (3.2.5) by mathematical induction method. Since x is continuous on $(t_i, t_{i+1}]$ and the limit $\lim\limits_{t\to t_i^+} x(t)(i \in N_0)$ exists, then $x \in P_mC(0,1]$. For $t \in (t_0, t_1]$, we know from Theorem 3.2 that there exists $c_0 \in \mathbb{R}$ such that

$$x(t) = c_0\mathbf{E}_{\alpha,1}(\lambda t^\alpha) + \int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^\alpha)F(s)ds, t \in (t_0, t_1].$$

Then (3.2.5) holds for $j = 0$. We suppose that (3.2.5) holds for all $j = 0, 1, \cdots, i$. We derive the expression of x on $(t_{i+1}, t_{i+2}]$. Suppose that

$$x(t) = \Phi(t) + \sum_{j=0}^{i} c_j\mathbf{E}_{\alpha,1}(\lambda(t-t_j)^\alpha) + \int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^\alpha)F(s)ds, t \in (t_{i+1}, t_{i+2}]. \quad (3.2.6)$$

By ${}^CD_{0^+}^\alpha x(t) - \lambda x(t) = f(t), t \in (t_{i+1}, t_{i+2}]$, we get

$$F(t) + \lambda x(t) = {}^CD_{0^+}^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}x'(s)ds$$

$$= \frac{1}{\Gamma(1-\alpha)}\left(\sum_{j=0}^{i}\int_{t_j}^{t_{j+1}}(t-s)^{-\alpha}x'(s)ds + \int_{t_{i+1}}^t (t-s)^{-\alpha}x'(s)ds\right)$$

$$= \frac{1}{\Gamma(1-\alpha)}\left[\sum_{j=0}^{i}\int_{t_j}^{t_{j+1}}(t-s)^{-\alpha}\left(\sum_{v=0}^{j}c_v\mathbf{E}_{\alpha,1}(\lambda(s-t_v)^\alpha) + \int_0^s (s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^\alpha)F(u)du\right)' ds\right.$$

$$\left. + \int_{t_{i+1}}^t (t-s)^{-\alpha}\left(\Phi(s) + \sum_{v=0}^{i}c_v\mathbf{E}_{\alpha,1}(\lambda(s-t_v)^\alpha) + \int_0^s (s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^\alpha)F(u)du\right)' ds\right]$$

$$= {}^CD_{t_{i+1}^+}^\alpha\Phi(t) + \frac{1}{\Gamma(1-\alpha)}\left[\sum_{j=0}^{i}\int_{t_j}^{t_{j+1}}(t-s)^{-\alpha}\left(\sum_{v=0}^{j}c_v\mathbf{E}_{\alpha,1}(\lambda(s-t_v)^\alpha)\right)' ds\right.$$

$$\left. + \int_{t_{i+1}}^t (t-s)^{-\alpha}\left(\sum_{v=0}^{i}c_v\mathbf{E}_{\alpha,1}(\lambda(s-t_v)^\alpha)\right)' ds + \int_0^t (t-s)^{-\alpha}\left(\int_0^s (s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^\alpha)F(u)du\right)' ds\right]$$

$$= {}^CD_{t_{i+1}^+}^\alpha\Phi(t) + \frac{1}{\Gamma(1-\alpha)}\left[\sum_{j=0}^{i}\sum_{v=0}^{j}c_v\int_{t_j}^{t_{j+1}}(t-s)^{-\alpha}\left(\sum_{\iota=0}^{+\infty}\frac{\lambda^\iota(s-t_v)^{\iota\alpha}}{\Gamma(\alpha\iota+1)}\right)' ds\right.$$

$$\left. + \sum_{v=0}^{i}c_v\int_{t_{i+1}}^t (t-s)^{-\alpha}\left(\sum_{\iota=0}^{+\infty}\frac{\lambda^\iota(s-t_v)^{\iota\alpha}}{\Gamma(\alpha\iota+1)}\right)' ds\right]$$

$$+ \frac{1}{\Gamma(2-\alpha)}\left[\int_0^t (t-s)^{1-\alpha}\left(\int_0^s (s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^\alpha)F(u)du\right)' ds\right]'$$

$$= {}^CD_{t_{i+1}^+}^\alpha\Phi(t) + \frac{1}{\Gamma(1-\alpha)}\left[\sum_{j=0}^{i}\sum_{v=0}^{j}c_v\int_{t_j}^{t_{j+1}}(t-s)^{-\alpha}\left(\sum_{\iota=1}^{+\infty}\frac{(\iota\alpha)\lambda^\iota(s-t_v)^{\iota\alpha-1}}{\Gamma(\alpha\iota+1)}\right) ds\right.$$

$$\left. + \sum_{v=0}^{i}c_v\int_{t_{i+1}}^t (t-s)^{-\alpha}\left(\sum_{\iota=1}^{+\infty}\frac{(\alpha\iota)\lambda^\iota(s-t_v)^{\iota\alpha-1}}{\Gamma(\alpha\iota+1)}\right) ds\right]$$

$$+ \frac{1}{\Gamma(2-\alpha)}\left[(t-s)^{1-\alpha}\left(\int_0^s (s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^\alpha)F(u)du\right)\Big|_0^t\right.$$

$$\left. + (1-\alpha)\int_0^t (t-s)^{-\alpha}\int_0^s (s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^\alpha)F(u)duds\right]'$$

$$= {}^C D^{\alpha}_{t^+_{i+1}}\Phi(t) + \frac{1}{\boldsymbol{\Gamma}(1-\alpha)}\left[\sum_{j=0}^{i}\sum_{v=0}^{j} c_v \sum_{\iota=1}^{+\infty}\frac{\lambda^\iota}{\Gamma(\alpha\iota)}\int_{t_j}^{t_{j+1}}(t-s)^{-\alpha}(s-t_v)^{\iota\alpha-1}ds\right.$$

$$\left.+\sum_{v=0}^{i} c_v \sum_{\iota=1}^{+\infty}\frac{\lambda^\iota}{\Gamma(\alpha\iota)}\int_{t_{i+1}}^{t}(t-s)^{-\alpha}(s-t_v)^{\iota\alpha-1}ds\right]$$

$$+\frac{1}{\Gamma(2-\alpha)}\left[(1-\alpha)\sum_{\iota=0}^{+\infty}\frac{\lambda^\iota}{\Gamma(\alpha(j+1))}\int_0^t\int_u^t(t-s)^{-\alpha}(s-u)^{\alpha j+\alpha-1}dsF(u)du\right]'$$

$$= {}^C D^{\alpha}_{t^+_{i+1}}\Phi(t) + \frac{1}{\boldsymbol{\Gamma}(1-\alpha)}\left[\sum_{v=0}^{i} c_v \sum_{j=v}^{i}\sum_{\iota=1}^{+\infty}\frac{\lambda^\iota(t-tv)^{\alpha(\iota-1)}}{\Gamma(\alpha\iota)}\int_{\frac{t_j-t_v}{t-t_v}}^{\frac{t_{j+1}-t_v}{t-t_v}}(1-w)^{-\alpha}w^{\iota\alpha-1}dw\right.$$

$$\left.+\sum_{v=0}^{i} c_v \sum_{\iota=1}^{+\infty}\frac{\lambda^\iota(t-t_v)^{\alpha(\iota-1)}}{\Gamma(\alpha\iota)}\int_{\frac{t_{i+1}-t_v}{t-t_v}}^{1}(1-w)^{-\alpha}w^{\iota\alpha-1}dw\right]$$

$$+\frac{1}{\Gamma(2-\alpha)}\left[(1-\alpha)\sum_{\iota=0}^{+\infty}\frac{\lambda^\iota}{\Gamma(\alpha(j+1))}\int_0^t(t-u)^{\alpha j}\int_0^1(1-w)^{-\alpha}w^{\alpha j+\alpha-1}dwF(u)du\right]'$$

$$= {}^C D^{\alpha}_{t^+_{i+1}}\Phi(t) + \frac{1}{\boldsymbol{\Gamma}(1-\alpha)}\left[\sum_{v=0}^{i} c_v \sum_{\iota=1}^{+\infty}\frac{\lambda^\iota(t-tv)^{\alpha(\iota-1)}}{\Gamma(\alpha\iota)}\int_0^{\frac{t_{i+1}-t_v}{t-t_v}}(1-w)^{-\alpha}w^{\iota\alpha-1}dw\right.$$

$$\left.+\sum_{v=0}^{i} c_v \sum_{\iota=1}^{+\infty}\frac{\lambda^\iota(t-t_v)^{\alpha(\iota-1)}}{\Gamma(\alpha\iota)}\int_{\frac{t_{i+1}-t_v}{t-t_v}}^{1}(1-w)^{-\alpha}w^{\iota\alpha-1}dw\right]+\left[\sum_{\iota=0}^{+\infty}\frac{\lambda^\iota}{\Gamma(\alpha j+1)}\int_0^t(t-u)^{\alpha j}F(u)du\right]'$$

$$= {}^C D^{\alpha}_{t^+_{i+1}}\Phi(t) + \frac{1}{\boldsymbol{\Gamma}(1-\alpha)}\left[\sum_{v=0}^{i} c_v \sum_{\iota=1}^{+\infty}\frac{\lambda^\iota(t-tv)^{\alpha(\iota-1)}}{\Gamma(\alpha\iota)}\int_0^1(1-w)^{-\alpha}w^{\iota\alpha-1}dw\right]$$

$$+\left[\sum_{\iota=0}^{+\infty}\frac{\lambda^\iota}{\Gamma(\alpha j+1)}\int_0^t(t-u)^{\alpha j}F(u)du\right]'$$

$$= {}^C D^{\alpha}_{t^+_{i+1}}\Phi(t) + \sum_{v=0}^{i} c_v \sum_{\iota=1}^{+\infty}\frac{\lambda^\iota(t-tv)^{\alpha(\iota-1)}}{\Gamma(\alpha(\iota-1)+1)} + \sum_{\iota=1}^{+\infty}\frac{\lambda^\iota}{\Gamma(\alpha j)}\int_0^t(t-u)^{\alpha j-1}F(u)du$$

$$= F(t)+\lambda x(t) + {}^c D^{\alpha}_{t^+_{i+1}}\Phi(t) - \lambda\Phi(t).$$

It follows that ${}^C D^{\alpha}_{t^+_{i+1}}\Phi(t) - \lambda\Phi(t) = 0$ for all $t \in (t_{i+1}, t_{i+2}]$. By Theorem 3.2, we know that there exists $c_{i+1} \in R$ such that $\Phi(t) = c_{i+1}\mathbf{E}_{\alpha,1}(\lambda(t-t_{i+1})^\alpha)$ for $t \in (t_{i+1}, t_{i+2}]$. Substituting Φ into (3.2.6), we get that (3.2.5) holds for $j = i+1$. Now suppose that (3.2.5) holds for all $j \in \mathbb{N}_0$. By the mathematical induction method, we know that x satisfies (3.2.5) and $x|_{(t_i,t_{i+1}]}$ is continuous and $\lim_{t\to t_i^+} x(t)$ exists. The proof is complete. □

Theorem 3.11. *Suppose that F is continuous on $(0,1)$ and there exist constants $k > -1$ and $l \in (-\alpha, -1-k, 0]$ such that $|F(t)| \le t^k(1-t)^l$ for all $t \in (0,1)$. Then x is a solution of (3.2.2.) if and only if there exists constants $c_i (i \in \mathbb{N}_0) \in \mathbb{R}$ such that*

$$x(t) = \sum_{v=0}^{j} c_v(t-t_v)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-t_v)^\alpha) + \int_0^t(t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^\alpha)F(s)ds, t \in (t_j, t_{j+1}], j \in \mathbb{N}_0. \quad (3.2.7)$$

Proof. For $t \in (t_j, t_{j+1}] (j \in N_0)$, similarly to the beginning of the proof of Theorem 3.10 we know that

$$t^{1-\alpha}\left|\int_0^t(t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^\alpha)F(s)ds\right| \le \int_0^t(t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^\alpha)|F(s)|ds$$

$$\le t^{1-\alpha}\int_0^t(t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^\alpha)s^k(1-s)^l ds$$

$$= t^{1-\alpha}\sum_{j=0}^{+\infty}\frac{\lambda^j}{\mathbf{\Gamma}((j+1)\alpha)}\int_0^t(t-s)^{\alpha-1}(t-s)^{\alpha j}s^k(1-s)^l ds$$

$$\leq t^{1-\alpha}\sum_{j=0}^{+\infty}\frac{\lambda^j}{\mathbf{\Gamma}((j+1)\alpha)}\int_0^t(t-s)^{\alpha+l-1}(t-s)^{\alpha j}s^k ds$$

$$= t^{1-\alpha}\sum_{j=0}^{+\infty}\frac{\lambda^j}{\mathbf{\Gamma}((j+1)\alpha)}t^{\alpha+\alpha j+k+l}\int_0^1(1-w)^{\alpha+\alpha j+l-1}w^k dw$$

$$\leq t^{1-\alpha}\sum_{j=0}^{+\infty}\frac{\lambda^j t^{\alpha j}}{\mathbf{\Gamma}((j+1)\alpha)}t^{\alpha+k+l}\int_0^1(1-w)^{\alpha+l-1}w^k dw = t^{1+k+l}\mathbf{E}_{\alpha,\alpha}(\lambda t^\alpha)\mathbf{B}(\alpha+l,k+1).$$

So $t^{1-\alpha}\int_0^t(t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^\alpha)F(s)ds$ is convergent and is continuous on $[0,1]$.

If x is a solution of (3.2.7), we have $x\in P_mC_{1-\alpha}(0,1]$. It follows for $t\in(t_i,t_{i+1}]$ that

$${}^{RL}D_{0^+}^\alpha x(t) = \frac{1}{\mathbf{\Gamma}(1-\alpha)}\left[\int_0^t(t-s)^{-\alpha}x(s)ds\right]'$$

$$= \frac{1}{\mathbf{\Gamma}(1-\alpha)}\left[\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}(t-s)^{-\alpha}\left(\sum_{\kappa=0}^{j}c_\kappa(s-t_\kappa)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-t_\kappa)^\alpha)\right.\right.$$

$$\left.\left.+\int_0^s(s-v)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-v)^\alpha)f(v)dv\right)ds\right]'$$

$$+\frac{1}{\mathbf{\Gamma}(1-\alpha)}\left[\int_{t_i}^t(t-s)^{-\alpha}\left(\sum_{\kappa=0}^{i}c_\kappa(t-t_\kappa)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-t_\kappa)^\alpha)\right.\right.$$

$$\left.\left.+\int_0^s(s-v)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-v)^\alpha)F(v)dv\right)ds\right]'$$

$$= \frac{1}{\mathbf{\Gamma}(1-\alpha)}\left[\sum_{j=0}^{i-1}\sum_{\kappa=0}^{j}c_\kappa\int_{t_j}^{t_{j+1}}(t-s)^{-\alpha}(s-t_\kappa)^{\alpha-1}\sum_{m=0}^{+\infty}\frac{\lambda^m(s-t_\kappa)^{\alpha m}}{\mathbf{\Gamma}(\alpha(m+1))}ds\right]'$$

$$+\frac{1}{\mathbf{\Gamma}(1-\alpha)}\left[\sum_{\kappa=0}^{i}c_\kappa\int_{t_i}^t(t-s)^{-\alpha}(t-t_\kappa)^{\alpha-1}\sum_{m=0}^{+\infty}\frac{\lambda^m(s-t_\kappa)^{\alpha m}}{\mathbf{\Gamma}(\alpha(m+1))}ds\right]'$$

$$+\frac{1}{\mathbf{\Gamma}(1-\alpha)}\left[\int_0^t(t-s)^{-\alpha}\int_0^s(s-v)^{\alpha-1}\sum_{m=0}^{+\infty}\frac{\lambda^m(s-v)^{\alpha m}}{\mathbf{\Gamma}(\alpha(m+1))}F(v)dvds\right]'$$

$$= \frac{1}{\mathbf{\Gamma}(1-\alpha)}\left[\sum_{m=0}^{+\infty}\frac{\lambda^m}{\mathbf{\Gamma}(\alpha(m+1))}\sum_{j=0}^{i-1}\sum_{\kappa=0}^{j}c_\kappa\int_{t_j}^{t_{j+1}}(t-s)^{-\alpha}(s-t_\kappa)^{\alpha+\alpha m-1}ds\right]'$$

$$+\frac{1}{\mathbf{\Gamma}(1-\alpha)}\left[\sum_{\kappa=0}^{i}c_\kappa\sum_{m=0}^{+\infty}\frac{\lambda^m}{\mathbf{\Gamma}(\alpha(m+1))}\int_{t_i}^t(t-s)^{-\alpha}(t-t_\kappa)^{\alpha+\alpha m-1}ds\right]'$$

$$+\frac{1}{\mathbf{\Gamma}(1-\alpha)}\left[\sum_{m=0}^{+\infty}\lambda^m\int_0^t(t-s)^{-\alpha}\int_0^s\frac{(s-v)^{\alpha+\alpha m-1}}{\mathbf{\Gamma}(\alpha(m+1))}F(v)dvds\right]'$$

$$= \frac{1}{\mathbf{\Gamma}(1-\alpha)}\left[\sum_{m=0}^{+\infty}\frac{\lambda^m}{\mathbf{\Gamma}(\alpha(m+1))}\sum_{\kappa=0}^{i-1}c_\kappa(t-t_\kappa)^{\alpha m}\sum_{j=\kappa}^{i-1}\int_{\frac{t_j-t_\kappa}{t-t_\kappa}}^{\frac{t_{j+1}-t_\kappa}{t-t_\kappa}}(1-w)^{-\alpha}w^{\alpha+\alpha m-1}dw\right]'$$

$$+\frac{1}{\mathbf{\Gamma}(1-\alpha)}\left[\sum_{m=0}^{+\infty}\sum_{\kappa=0}^{i}c_\kappa(t-t_\kappa)^{\alpha m}\frac{\lambda^m}{\mathbf{\Gamma}(\alpha(m+1))}\int_{\frac{t_i-t_\kappa}{t-t_\kappa}}^{1}(1-w)^{-\alpha}w^{\alpha+\alpha m-1}dw\right]'$$

$$+\frac{1}{\Gamma(1-\alpha)}\left[\sum_{m=0}^{+\infty}\lambda^m\int_0^t\int_v^t(t-s)^{-\alpha}\frac{(s-v)^{\alpha+\alpha m-1}}{\Gamma(\alpha(m+1))}dsF(v)dv\right]'$$

$$=\frac{1}{\Gamma(1-\alpha)}\left[\sum_{m=0}^{+\infty}\frac{\lambda^m}{\Gamma(\alpha(m+1))}\sum_{\kappa=0}^{i-1}c_\kappa(t-t_\kappa)^{\alpha m}\int_0^{\frac{t_i-t_\kappa}{t-t_\kappa}}(1-w)^{-\alpha}w^{\alpha+\alpha m-1}dw\right]'$$

$$+\frac{1}{\Gamma(1-\alpha)}\left[\sum_{m=0}^{+\infty}\sum_{\kappa=0}^{i-1}c_\kappa(t-t_\kappa)^{\alpha m}\frac{\lambda^m}{\Gamma(\alpha(m+1))}\int_{\frac{t_i-t_\kappa}{t-t_\kappa}}^1(1-w)^{-\alpha}w^{\alpha+\alpha m-1}dw\right.$$

$$\left.+\sum_{m=0}^{+\infty}c_i(t-t_i)^{\alpha m}\frac{\lambda^m}{\Gamma(\alpha(m+1))}\int_0^1(1-w)^{-\alpha}w^{\alpha+\alpha m-1}dw\right]'$$

$$+\frac{1}{\Gamma(1-\alpha)}\left[\sum_{m=0}^{+\infty}\lambda^m\int_0^t(t-v)^{\alpha m}\int_0^1(1-w)^{-\alpha}\frac{w^{\alpha+\alpha m-1}}{\Gamma(\alpha(m+1))}dwF(v)dv\right]'$$

$$=\frac{1}{\Gamma(1-\alpha)}\left[\sum_{\kappa=0}^{i-1}c_\kappa\sum_{m=0}^{+\infty}(t-t_\kappa)^{\alpha m}\frac{\lambda^m}{\Gamma(\alpha(m+1))}\int_0^1(1-w)^{-\alpha}w^{\alpha+\alpha m-1}dw\right]'$$

$$+\frac{1}{\Gamma(1-\alpha)}\left[c_i\sum_{m=0}^{+\infty}(t-t_i)^{\alpha m}\frac{\lambda^m}{\Gamma(\alpha(m+1))}\int_0^1(1-w)^{-\alpha}w^{\alpha+\alpha m-1}dw\right]'$$

$$+\frac{1}{\Gamma(1-\alpha)}\left[\sum_{m=0}^{+\infty}\lambda^m\int_0^t(t-v)^{\alpha m}\int_0^1(1-w)^{-\alpha}\frac{w^{\alpha+\alpha m-1}}{\Gamma(\alpha(m+1))}dwF(v)dv\right]'$$

$$=\frac{1}{\Gamma(1-\alpha)}\left[\sum_{\kappa=0}^{i-1}c_\kappa\sum_{m=0}^{+\infty}(t-t_\kappa)^{\alpha m}\frac{\lambda^m}{\Gamma(\alpha(m+1))}\frac{\Gamma(1-\alpha)\Gamma(\alpha(m+1))}{\Gamma(\alpha m)}\right]'$$

$$+\frac{1}{\Gamma(1-\alpha)}\left[c_i\sum_{m=0}^{+\infty}(t-t_i)^{\alpha m}\frac{\lambda^m}{\Gamma(\alpha(m+1))}\frac{\Gamma(1-\alpha)\Gamma(\alpha(m+1))}{\Gamma(\alpha m)}\right]'$$

$$+\frac{1}{\Gamma(1-\alpha)}\left[\sum_{m=0}^{+\infty}\frac{\lambda^m}{\Gamma(\alpha(m+1))}\frac{\Gamma(1-\alpha)\Gamma(\alpha(m+1))}{\Gamma(\alpha m)}\int_0^t(t-v)^{\alpha m}F(v)dv\right]'$$

$$=\sum_{\kappa=0}^{i-1}c_\kappa\sum_{m=1}^{+\infty}(t-t_\kappa)^{\alpha m-1}\frac{(\alpha m)\lambda^m}{\Gamma(\alpha m)}+c_i\sum_{m=1}^{+\infty}(t-t_i)^{\alpha m-1}\frac{(\alpha m)\lambda^m}{\Gamma(\alpha m)}$$

$$+\sum_{m=1}^{+\infty}\frac{(\alpha m)\lambda^m}{\Gamma(\alpha m)}\int_0^t(t-v)^{\alpha m-1}F(v)dv+F(t)$$

$$=F(t)+\lambda\sum_{\kappa=0}^{i}c_\kappa(t-t_\kappa)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-t_\kappa)^\alpha)+\sum_{m=1}^{+\infty}\frac{(\alpha m)\lambda^m}{\Gamma(\alpha(m+1))}\int_0^t(t-v)^{\alpha m-1}F(v)dv$$

$$=F(t)+\lambda\sum_{\kappa=0}^{i}c_\kappa(t-t_\kappa)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-t_\kappa)^\alpha)+\lambda\int_0^t(t-v)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-v)^\alpha)F(v)dv$$

$$=\lambda x(t)+F(t),t\in(t_i,t_{i+1}].$$

It follows that x is a solution of (3.2.2).

Now we prove that if x is a solution of (3.2.2), then x satisfies (3.2.7) and $x\in P_mC_{1-\alpha}(0,1]$ by mathematical induction method. By Theorem 3.4, we know that there exists a constant $c_0\in\mathbb{R}$ such that

$$x(t)=c_0t^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda t^\alpha)+\int_0^t(t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^\alpha)F(s)ds,t\in(t_0,t_1].$$

Hence (3.2.7) holds for $j=0$. Assume that (3.2.7) holds for $j=0,1,2,\cdots,i\le m$, we will prove that (3.2.7) holds for $j = i+1$. Suppose that

$$x(t) = \Phi(t) + \sum_{j=0}^{i} c_j(t-t_j)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-t_j)^{\alpha}) + \int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^{\alpha})f(s)ds, t \in (t_{i+1}, t_{i+2}].$$

Then for $t \in (t_{i+1}, t_{i+2}]$ we have

$$F(t) + \lambda x(t) = {}^{RL}D_{0^+}^{\alpha} x(t) = \frac{1}{\Gamma(1-\alpha)} \left[\sum_{j=0}^{i} \int_{t_j}^{t_{j+1}} (t-s)^{-\alpha} x(s)ds + \int_{t_{i+1}}^{t} (t-s)^{-\alpha} x(s)ds\right]'$$

$$= \frac{1}{\Gamma(1-\alpha)} \left[\sum_{j=0}^{i} \int_{t_j}^{t_{j+1}} (t-s)^{-\alpha} \left(\sum_{\kappa=0}^{j} c_\kappa (s-t_\kappa)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-t_\kappa)^{\alpha})\right.\right.$$

$$\left. + \int_0^s (s-v)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-v)^{\alpha})F(v)dv\right) ds$$

$$+ \int_{t_{i+1}}^{t} (t-s)^{-\alpha} \left(\Phi(s) + \sum_{\kappa=0}^{k} c_\kappa (t-t_\kappa)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-t_\kappa)^{\alpha})\right.$$

$$\left.\left. + \int_0^s (s-v)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-v)^{\alpha})F(v)dv\right) ds\right]'$$

$$= {}^{RL}D_{t_{i+1}^+}^{\alpha}\Phi(t) + \frac{1}{\Gamma(1-\alpha)} \left[\sum_{j=0}^{i} \int_{t_j}^{t_{j+1}} (t-s)^{-\alpha} \sum_{\kappa=0}^{j} c_\kappa (s-t_\kappa)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-t_\kappa)^{\alpha})ds\right.$$

$$+ \int_{t_{i+1}}^{t} (t-s)^{-\alpha} \sum_{\kappa=0}^{k} c_\kappa (t-t_\kappa)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-t_\kappa)^{\alpha})ds$$

$$\left. + \int_0^t (t-s)^{-\alpha} \int_0^s (s-v)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-v)^{\alpha})F(v)dvds\right]'.$$

Similarly to the proof of Theorem 3.10 we can get that

$$F(t) + \lambda x(t) = {}^{RL}D_{0^+}^{\alpha} x(t) = F(t) + \lambda x(t) + {}^{RL}D_{t_{i+1}^+}^{\alpha}\Phi(t) - \lambda\Phi(t).$$

So ${}^{RL}D_{t_{i+1}^+}^{\alpha}\Phi(t) - \lambda\Phi(t) = 0$ on $(t_{i+1}, t_{i+2}]$. Then Theorem 3.4 implies that there exists a constant $c_{i+1} \in \mathbb{R}$ such that $\Phi(t) = c_{i+1}(t-t_{i+1})^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-t_{i+1})^{\alpha})$ on $(t_{i+1}, t_{i+2}]$. Hence

$$x(t) = \sum_{j=0}^{i+1} c_j(t-t_j)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-t_j)^{\alpha}) + \int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^{\alpha})f(s)ds, t \in (t_{i+1}, t_{i+2}].$$

By mathematical induction method, we know that (3.2.7) holds for $j \in \mathbb{N}_0$. The proof is complete. □

Theorem 3.12. *Suppose that G is continuous on $(1, e)$ and there exist constants $k > -1$ and $l \in (-\alpha, -1-k, 0]$ such that $|G(t)| \leq (\log t)^k (1-\log t)^l$ for all $t \in (1, e)$. Then x is a solution of (3.2.3) if and only if there exists constants $c_i (i \in \mathbb{N}_0) \in \mathbb{R}$ such that*

$$x(t) = \sum_{v=0}^{j} c_v \Gamma(\alpha) \left(\log \frac{t}{t_v}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda \left(\log \frac{t}{t_v}\right)^{\alpha}\right)$$
$$+ \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda \left(\log \frac{t}{s}\right)^{\alpha}\right) G(s)\frac{ds}{s}, t \in (t_j, t_{j+1}], j \in \mathbb{N}_0. \quad (3.2.8)$$

Proof. For $t \in (t_j, t_{j+1}] (j \in \mathbb{N}_0)$, similarly to the beginning of the proof of Theorem 3.10 we know that

$$(\log t)^{1-\alpha} \left|\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda \left(\log \frac{t}{s}\right)^{\alpha}\right) G(s)\frac{ds}{s}\right|$$

$$\leq (\log t)^{1-\alpha} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda \left(\log \frac{t}{s}\right)^{\alpha}\right) (\log s)^k (1-\log s)^l \frac{ds}{s}$$

$$\leq (\log t)^{1-\alpha} \sum_{\iota=0}^{+\infty} \frac{\lambda^{\iota}}{\Gamma(\alpha(\iota+1))} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha\iota+\alpha+l-1} (\log s)^k \frac{ds}{s} \text{ by } \frac{\log s}{\log t} = w$$

$$= (\log t)^{1-\alpha} \sum_{\iota=0}^{+\infty} \frac{\lambda^\iota}{\Gamma(\alpha(\iota+1))} (\log t)^{\alpha\iota+\alpha+k+l} \int_0^1 (1-w)^{\alpha\iota+\alpha+l-1} w^k dw$$

$$\leq (\log t)^{1+k+l} E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{s}\right)^\alpha\right) \int_0^1 (1-w)^{\alpha+l-1} w^k dw$$

$$= (\log t)^{1+k+l} E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{s}\right)^\alpha\right) \mathbf{B}(\alpha+l, k+1).$$

So $\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{s}\right)^\alpha\right) G(s)\frac{ds}{s}$ is convergent for all $t \in (1, e]$ and

$$\lim_{t\to 1^+} (\log t)^{1-\alpha} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{s}\right)^\alpha\right) G(s)\frac{ds}{s} \text{ exists.}$$

If x is a solution of (3.2.8), we have $x \in LP_mC_{1-\alpha}(1, e]$. By using Definition 2.5, it follows for $t \in (t_i, t_{i+1}]$ that

$$^{RLH}D_{1^+}^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} t \left[\int_1^t \left(\log\frac{t}{s}\right)^{-\alpha} x(s)\frac{ds}{s}\right]'$$

$$= \frac{1}{\Gamma(1-\alpha)} t \left[\sum_{j=0}^{i-1} \int_{t_i}^{t_{i+1}} \left(\log\frac{t}{s}\right)^{-\alpha} \left(\sum_{v=0}^{j} c_v\Gamma(\alpha)\left(\log\frac{s}{t_v}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{t_v}\right)^\alpha\right)\right.\right.$$

$$\left. + \int_1^s \left(\log\frac{s}{u}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{u}\right)^\alpha\right) G(u)\frac{du}{u}\right)\frac{ds}{s}$$

$$+ \int_{t_i}^t \left(\log\frac{t}{s}\right)^{-\alpha} \left(\sum_{v=0}^{i} c_v\Gamma(\alpha)\left(\log\frac{s}{t_v}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{t_v}\right)^\alpha\right)\right.$$

$$\left.\left. + \int_1^s \left(\log\frac{s}{u}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{u}\right)^\alpha\right) G(u)\frac{du}{u}\right)\frac{ds}{s}\right]'$$

$$= \frac{1}{\Gamma(1-\alpha)} t \left[\Gamma(\alpha)\sum_{j=0}^{i-1}\sum_{v=0}^{j} c_v \int_{t_i}^{t_{i+1}} \left(\log\frac{t}{s}\right)^{-\alpha} \left(\log\frac{s}{t_v}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{t_v}\right)^\alpha\right)\frac{ds}{s}\right.$$

$$+ \Gamma(\alpha)\sum_{v=0}^{i} c_v \int_{t_i}^t \left(\log\frac{t}{s}\right)^{-\alpha} \left(\log\frac{s}{t_v}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{t_v}\right)^\alpha\right)\frac{ds}{s}$$

$$\left. + \int_1^t \left(\log\frac{t}{s}\right)^{-\alpha} \int_1^s \left(\log\frac{s}{u}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{u}\right)^\alpha\right) G(u)\frac{du}{u}\frac{ds}{s}\right]'.$$

One sees that

$$\int_{t_i}^t \left(\log\frac{t}{s}\right)^{-\alpha} \left(\log\frac{s}{t_v}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{t_v}\right)^\alpha\right)\frac{ds}{s}$$

$$= \sum_{\kappa=0}^{+\infty} \frac{\lambda^\iota}{\Gamma(\alpha(\iota+1))} \int_{t_i}^t \left(\log\frac{t}{s}\right)^{-\alpha} \left(\log\frac{s}{t_v}\right)^{\alpha\kappa+\alpha-1} \frac{ds}{s} \text{ by } \frac{\log s - \log t_v}{\log t - \log t_v} = w$$

$$= \sum_{\kappa=0}^{+\infty} \frac{\lambda^\kappa}{\Gamma(\alpha(\kappa+1))} \left(\log\frac{t}{t_v}\right)^{\alpha\kappa} \int_{\frac{\log t_i - \log t_v}{\log t - \log t_v}}^1 (1-w)^{-\alpha} w^{\alpha\kappa+\alpha-1} dw$$

and

$$\int_{t_i}^{t_{i+1}} \left(\log\frac{t}{s}\right)^{-\alpha} \left(\log\frac{s}{t_v}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{t_v}\right)^\alpha\right)\frac{ds}{s}$$

$$= \sum_{\kappa=0}^{+\infty} \frac{\lambda^\kappa}{\Gamma(\alpha(\kappa+1))} \left(\log\frac{t}{t_v}\right)^{\alpha\kappa} \int_{\frac{\log t_i - \log t_v}{\log t - \log t_v}}^{\frac{\log t_{i+1} - \log t_v}{\log t - \log t_v}} (1-w)^{-\alpha} w^{\alpha\kappa+\alpha-1} dw.$$

Similarly

$$\int_1^t \left(\log\frac{t}{s}\right)^{-\alpha} E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{s}\right)^{\alpha}\right)\int_1^s \left(\log\frac{s}{u}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{u}\right)^{\alpha}\right) G(u)\frac{du}{u}\frac{ds}{s}$$

$$=\int_1^t\int_u^t \left(\log\frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{s}\right)^{\alpha}\right)\left(\log\frac{s}{u}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{u}\right)^{\alpha}\right)\frac{ds}{s}G(u)\frac{du}{u}$$

$$=\sum_{\kappa=0}^{+\infty}\int_1^t \left(\log\frac{t}{u}\right)^{\alpha\kappa}\frac{\lambda^\kappa}{\Gamma(\alpha(\kappa+1))}\int_0^1 (1-w)^{-\alpha} w^{\alpha\kappa+\alpha-1}dwG(u)\frac{du}{u}.$$

So

$$^{RLH}D_{1^+}^{\alpha}x(t)=\frac{1}{\Gamma(1-\alpha)}t\left[\Gamma(\alpha)\sum_{v=0}^{i-1}c_v\sum_{\kappa=0}^{+\infty}\frac{\lambda^\kappa}{\Gamma(\alpha(\kappa+1))}\left(\log\frac{t}{t_v}\right)^{\alpha\kappa}\sum_{j=v}^{i-1}\int_{\frac{\log t_j-\log t_v}{\log t-\log t_v}}^{\frac{\log t_{i+1}-\log t_v}{\log t-\log t_v}}(1-w)^{-\alpha}w^{\alpha\kappa+\alpha-1}dw\right.$$

$$+\Gamma(\alpha)\sum_{v=0}^{i}c_v\sum_{\kappa=0}^{+\infty}\frac{\lambda^\kappa}{\Gamma(\alpha(\kappa+1))}\left(\log\frac{t}{t_v}\right)^{\alpha\kappa}\int_{\frac{\log t_i-\log t_v}{\log t-\log t_v}}^{1}(1-w)^{-\alpha}w^{\alpha\kappa+\alpha-1}dw$$

$$\left.+\sum_{\kappa=0}^{+\infty}\int_1^t\left(\log\frac{t}{u}\right)^{\alpha\kappa}\frac{\lambda^\kappa}{\Gamma(\alpha(\kappa+1))}\int_0^1(1-w)^{-\alpha}w^{\alpha\kappa+\alpha-1}dwG(u)\frac{du}{u}\right]'$$

$$=t\left[\Gamma(\alpha)\sum_{v=0}^{i}c_v\sum_{\kappa=0}^{+\infty}\frac{\lambda^\kappa}{\Gamma(\alpha\kappa+1)}\left(\log\frac{t}{t_v}\right)^{\alpha\kappa}+\sum_{\kappa=0}^{+\infty}\int_1^t\left(\log\frac{t}{u}\right)^{\alpha\kappa}\frac{\lambda^\kappa}{\Gamma(\alpha\kappa+1)}G(u)\frac{du}{u}\right]'$$

$$=F(t)+t\left[\frac{\Gamma(\alpha)}{t}\sum_{v=0}^{i}c_v\sum_{\kappa=1}^{+\infty}\frac{\lambda^\kappa}{\Gamma(\alpha\kappa)}\left(\log\frac{t}{t_v}\right)^{\alpha\kappa-1}+\sum_{\kappa=1}^{+\infty}\int_1^t\left(\log\frac{t}{u}\right)^{\alpha\kappa-1}\frac{\lambda^\kappa}{\Gamma(\alpha\kappa)}G(u)\frac{du}{u}\right]$$

$$=\lambda x(t)+F(t),t\in(t_i,t_{i+1}].$$

It follows that x is a solution of (3.2.3).

Now we prove that if x is a solution of (3.2.3), then x satisfies (3.2.8) and $x\in LP_mC_{1-\alpha}(1,e]$ by mathematical induction method. By Theorem 3.6, we know that there exists a constant $c_0\in\mathbb{R}$ such that

$$x(t)=c_0(\log t)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(\log t)^{\alpha})+\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\left(\lambda\left(\frac{t}{s}\right)^{\alpha}\right)F(s)ds,t\in(t_0,t_1].$$

Hence (3.2.8) holds for $j=0$. Assume that (3.2.8) holds for $j=0,1,2,\cdots,i\le m$, we will prove that (3.2.8) holds for $j=i+1$. Suppose that

$$x(t)=\Phi(t)+\sum_{v=0}^{i}c_v\Gamma(\alpha)\left(\log\frac{t}{t_v}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{t_v}\right)^{\alpha}\right)$$

$$+\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{s}\right)^{\alpha}\right)G(s)\frac{ds}{s},t\in(t_{i+1},t_{i+2}].$$

Then for $t\in(t_{i+1},t_{i+2}]$ we have

$$F(t)+\lambda x(t)={}^{RLH}D_{1^+}^{\alpha}x(t)=\frac{1}{\Gamma(1-\alpha)}\left[\sum_{j=0}^{i}\int_{t_j}^{t_{j+1}}\left(\log\frac{t}{s}\right)^{-\alpha}x(s)\frac{ds}{s}+\int_{t_{i+1}}^{t}\left(\log\frac{t}{s}\right)^{-\alpha}x(s)\frac{ds}{s}\right]'$$

$$= \frac{1}{\Gamma(1-\alpha)} \left[\sum_{j=0}^{i} \int_{t_j}^{t_{j+1}} \left(\log \frac{t}{s}\right)^{-\alpha} \left(\sum_{v=0}^{i} c_v \Gamma(\alpha) \left(\log \frac{s}{t_v}\right)^{\alpha-1} E_{\alpha,\alpha} \left(\lambda \left(\log \frac{s}{t_v}\right)^{\alpha}\right) \right.\right.$$

$$+ \int_1^s \left(\log \frac{s}{u}\right)^{\alpha-1} E_{\alpha,\alpha} \left(\lambda \left(\log \frac{s}{u}\right)^{\alpha}\right) G(u) \frac{du}{u} \Big) \frac{ds}{s}$$

$$+ \int_{t_{i+1}}^{t} \left(\log \frac{t}{s}\right)^{-\alpha} \left(\Phi(s) + \sum_{v=0}^{i} c_v \Gamma(\alpha) \left(\log \frac{s}{t_v}\right)^{\alpha-1} E_{\alpha,\alpha} \left(\lambda \left(\log \frac{s}{t_v}\right)^{\alpha}\right) \right.$$

$$\left.\left. + \int_1^s \left(\log \frac{s}{u}\right)^{\alpha-1} E_{\alpha,\alpha} \left(\lambda \left(\log \frac{s}{u}\right)^{\alpha}\right) G(u) \frac{du}{u} \right) \frac{ds}{s} \right]'$$

$$= {}^{RL}D_{t_{i+1}^+}^{\alpha} \Phi(t) + \frac{1}{\Gamma(1-\alpha)} \left[\sum_{j=0}^{i} \int_{t_j}^{t_{j+1}} \left(\log \frac{t}{s}\right)^{-\alpha} \left(\sum_{v=0}^{i} c_v \Gamma(\alpha) \left(\log \frac{s}{t_v}\right)^{\alpha-1} E_{\alpha,\alpha} \left(\lambda \left(\log \frac{s}{t_v}\right)^{\alpha}\right) \right.\right.$$

$$+ \int_1^s \left(\log \frac{s}{u}\right)^{\alpha-1} E_{\alpha,\alpha} \left(\lambda \left(\log \frac{s}{u}\right)^{\alpha}\right) G(u) \frac{du}{u} \Big) \frac{ds}{s}$$

$$+ \int_{t_{i+1}}^{t} \left(\log \frac{t}{s}\right)^{-\alpha} \left(\sum_{v=0}^{i} c_v \Gamma(\alpha) \left(\log \frac{s}{t_v}\right)^{\alpha-1} E_{\alpha,\alpha} \left(\lambda \left(\log \frac{s}{t_v}\right)^{\alpha}\right) \right.$$

$$\left.\left. + \int_1^s \left(\log \frac{s}{u}\right)^{\alpha-1} E_{\alpha,\alpha} \left(\lambda \left(\log \frac{s}{u}\right)^{\alpha}\right) G(u) \frac{du}{u} \right) \frac{ds}{s} \right]'.$$

Similarly to above discussion we can get that

$$F(t) + \lambda x(t) = {}^{RLH}D_{1^+}^{\alpha} x(t) = F(t) + \lambda x(t) + {}^{RLH}D_{t_{i+1}^+}^{\alpha} \Phi(t) - \lambda \Phi(t).$$

So ${}^{RLH}D_{t_{i+1}^+}^{\alpha} \Phi(t) - \lambda \Phi(t) = 0$ on $(t_{i+1}, t_{i+2}]$. Then Theorem 3.6 implies that there exists a constant $c_{i+1} \in \mathbb{R}$ such that $\Phi(t) = c_{i+1} \Gamma(\alpha) \left(\log \frac{t}{t_{i+1}}\right)^{\alpha-1} \mathbf{E}_{\alpha,\alpha} \left(\lambda \left(\frac{t}{t_{i+1}}\right)^{\alpha}\right)$ on $(t_{i+1}, t_{i+2}]$. Hence

$$x(t) = \sum_{v=0}^{i+1} c_v \Gamma(\alpha) \left(\log \frac{t}{t_v}\right)^{\alpha-1} E_{\alpha,\alpha} \left(\lambda \left(\log \frac{t}{t_v}\right)^{\alpha}\right)$$

$$+ \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha} \left(\lambda \left(\log \frac{t}{s}\right)^{\alpha}\right) G(s) \frac{ds}{s}, t \in (t_{i+1}, t_{i+2}].$$

By mathematical induction method, we know that (3.2.8) holds for $j \in \mathbb{N}_0$. The proof is complete. □

Theorem 3.13. *Suppose that G is continuous on $(1, e)$ and there exist constants $k > -1$ and $l \in (-\alpha, -\alpha + k, 0]$ such that $|G(t)| \le (\log t)^k (1 - \log t)^l$ for all $t \in (1, e)$. Then x is a piecewise solution of (3.2.4) if and only if x and there exists constants $c_i (i \in \mathbb{N}_0) \in \mathbb{R}$ such that*

$$x(t) = \sum_{v=0}^{j} c_v E_{\alpha,1} \left(\lambda \left(\log \frac{t}{t_v}\right)^{\alpha}\right) + \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha} \left(\lambda \left(\log \frac{t}{s}\right)^{\alpha}\right) G(s) \frac{ds}{s}, t \in (t_j, t_{j+1}], j \in \mathbb{N}_0. \quad (3.2.9)$$

Proof. For $t \in (t_j, t_{j+1}] (j \in \mathbb{N}_0)$, similarly to the beginning of the proof of Theorem 3.12 we know that

$$\left| \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha} \left(\lambda \left(\log \frac{t}{s}\right)^{\alpha}\right) G(s) \frac{ds}{s} \right|$$

$$\le (\log t)^{\alpha+k+l} E_{\alpha,\alpha} \left(\lambda \left(\log \frac{t}{s}\right)^{\alpha}\right) \mathbf{B}(\alpha + l, k + 1).$$

So $\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha} \left(\lambda \left(\log \frac{t}{s}\right)^{\alpha}\right) G(s) \frac{ds}{s}$ is convergent for all $t \in (1, e]$ and

$$\lim_{t \to 1^+} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha} \left(\lambda \left(\log \frac{t}{s}\right)^{\alpha}\right) G(s) \frac{ds}{s} \text{ exists.}$$

If x is a solution of (3.2.9), we have $x \in LP_mC(1,e]$. By using Definition 2.6, it follows for $t \in (t_i, t_{i+1}]$ that

$$\begin{aligned}
&{}^{CH}D_{1^+}^{\alpha}x(t) = \frac{1}{\Gamma(1-\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{-\alpha} sx'(s)\frac{ds}{s}\\
&= \frac{1}{\Gamma(1-\alpha)}\left[\sum_{j=0}^{i-1}\int_{t_i}^{t_{i+1}}\left(\log\frac{t}{s}\right)^{-\alpha}\left(\sum_{v=0}^{j}c_v\Gamma(\alpha)E_{\alpha,1}\left(\lambda\left(\log\frac{s}{t_v}\right)^{\alpha}\right)\right.\right.\\
&+\left.\int_1^s\left(\log\frac{s}{u}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{u}\right)^{\alpha}\right)G(u)\frac{du}{u}\right)'ds\\
&+\int_{t_i}^t\left(\log\frac{t}{s}\right)^{-\alpha}\left(\sum_{v=0}^{i}c_v\Gamma(\alpha)E_{\alpha,1}\left(\lambda\left(\log\frac{s}{t_v}\right)^{\alpha}\right)\right.\\
&+\left.\left.\int_1^s\left(\log\frac{s}{u}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{u}\right)^{\alpha}\right)G(u)\frac{du}{u}\right)'ds\right]\\
&= \frac{1}{\Gamma(1-\alpha)}t\left[\sum_{j=0}^{i-1}\sum_{v=0}^{j}c_v\int_{t_i}^{t_{i+1}}\left(\log\frac{t}{s}\right)^{-\alpha}\left(E_{\alpha,1}\left(\lambda\left(\log\frac{s}{t_v}\right)^{\alpha}\right)\right)'ds\right.\\
&+\sum_{v=0}^{i}c_v\int_{t_i}^t\left(\log\frac{t}{s}\right)^{-\alpha}\left(E_{\alpha,1}\left(\lambda\left(\log\frac{s}{t_v}\right)^{\alpha}\right)\right)'ds\\
&+\left.\int_1^t\left(\log\frac{t}{s}\right)^{-\alpha}\left(\int_1^s\left(\log\frac{s}{u}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{u}\right)^{\alpha}\right)G(u)\frac{du}{u}\right)'ds\right].
\end{aligned}$$

One sees that

$$\begin{aligned}
&\int_{t_i}^t\left(\log\frac{t}{s}\right)^{-\alpha}\left(E_{\alpha,1}\left(\lambda\left(\log\frac{s}{t_v}\right)^{\alpha}\right)\right)'ds = \int_{t_i}^t\left(\log\frac{t}{s}\right)^{-\alpha}\left(\sum_{\kappa=0}^{+\infty}\frac{\lambda^\kappa}{\Gamma(\alpha\kappa+1)}\left(\log\frac{s}{t_v}\right)^{\kappa\alpha}\right)'ds\\
&= \sum_{\kappa=1}^{+\infty}\frac{(\kappa\alpha)\lambda^\kappa}{\Gamma(\alpha\kappa+1)}\int_{t_i}^t\left(\log\frac{t}{s}\right)^{-\alpha}\left(\log\frac{s}{t_v}\right)^{\kappa\alpha-1}ds = \sum_{\kappa=1}^{+\infty}\frac{(\kappa\alpha)\lambda^\kappa}{\Gamma(\alpha\kappa+1)}\left(\log\frac{t}{t_v}\right)^{\alpha(\kappa-1)}\int_{\frac{t_i-t_v}{t-t_v}}^1(1-w)^{-\alpha}w^{\kappa\alpha-1}dw
\end{aligned}$$

and

$$\int_{t_i}^{t_{i+1}}\left(\log\frac{t}{s}\right)^{-\alpha}\left(E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{t_v}\right)^{\alpha}\right)\right)'ds = \sum_{\kappa=1}^{+\infty}\frac{(\alpha\kappa)\lambda^\kappa}{\Gamma(\alpha\kappa+1)}\left(\log\frac{t}{t_v}\right)^{\alpha\kappa-1}\int_{\frac{\log t_i-\log t_v}{\log t-\log t_v}}^{\frac{\log t_{i+1}-\log t_v}{\log t-\log t_v}}(1-w)^{-\alpha}w^{\alpha\kappa-1}dw.$$

Similarly

$$\begin{aligned}
&\int_1^t\left(\log\frac{t}{s}\right)^{-\alpha}\left(\int_1^s\left(\log\frac{s}{u}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{u}\right)^{\alpha}\right)G(u)\frac{du}{u}\right)'ds\\
&= \frac{t}{1-\alpha}\left[\int_1^t\left(\log\frac{t}{s}\right)^{1-\alpha}\left(\int_1^s\left(\log\frac{s}{u}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{u}\right)^{\alpha}\right)G(u)\frac{du}{u}\right)'ds\right]'\\
&= \frac{t}{1-\alpha}\left[\left(\log\frac{t}{s}\right)^{1-\alpha}\left(\int_1^s\left(\log\frac{s}{u}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{u}\right)^{\alpha}\right)G(u)\frac{du}{u}\right)\Big|_1^t\right.\\
&+\left.(1-\alpha)\int_1^t\left(\log\frac{t}{s}\right)^{-\alpha}\int_1^s\left(\log\frac{s}{u}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{u}\right)^{\alpha}\right)G(u)\frac{du}{u}\frac{ds}{s}\right]'\\
&= t\left[\sum_{\kappa=0}^{+\infty}\frac{\lambda^\kappa}{\Gamma(\alpha(\kappa+1))}\int_1^t\int_u^t\left(\log\frac{t}{s}\right)^{-\alpha}\left(\log\frac{s}{u}\right)^{\alpha\kappa+\alpha-1}\frac{ds}{s}G(u)\frac{du}{u}\right]'\\
&= t\left[\sum_{\kappa=0}^{+\infty}\frac{\lambda^\kappa}{\Gamma(\alpha(\kappa+1))}\int_1^t\left(\log\frac{t}{u}\right)^{\alpha\kappa}\int_0^1(1-w)^{-\alpha}w^{\alpha\kappa+\alpha-1}dwG(u)\frac{du}{u}\right]'\\
&= F(t)+t\sum_{\kappa=1}^{+\infty}\frac{(\alpha\kappa)\lambda^\kappa}{\Gamma(\alpha(\kappa+1))}\int_1^t\left(\log\frac{t}{u}\right)^{\alpha\kappa-1}\int_0^1(1-w)^{-\alpha}w^{\alpha\kappa+\alpha-1}dwG(u)\frac{du}{u}.
\end{aligned}$$

So

$$^{CH}D_{1^+}^{\alpha}x(t)=\frac{1}{\Gamma(1-\alpha)}t\left[\sum_{j=0}^{i-1}\sum_{v=0}^{j}c_v\sum_{\kappa=1}^{+\infty}\frac{(\alpha\kappa)\lambda^\kappa}{\Gamma(\alpha\kappa+1)}\left(\log\frac{t}{t_v}\right)^{\alpha\kappa-1}\int_{\frac{\log t_i-\log t_v}{\log t-\log t_v}}^{\frac{\log t_{i+1}-\log t_v}{\log t-\log t_v}}(1-w)^{-\alpha}w^{\alpha\kappa-1}dw\right.$$

$$+\sum_{v=0}^{i}c_v\sum_{\kappa=1}^{+\infty}\frac{(\kappa\alpha)\lambda^\kappa}{\Gamma(\alpha\kappa+1)}\left(\log\frac{t}{t_v}\right)^{\alpha(\kappa-1)}\int_{\frac{t_i-t_v}{t-t_v}}^{1}(1-w)^{-\alpha}w^{\kappa\alpha-1}dw$$

$$+F(t)+t\sum_{\kappa=1}^{+\infty}\frac{(\alpha\kappa)\lambda^\kappa}{\Gamma(\alpha(\kappa+1))}\int_1^t\left(\log\frac{t}{u}\right)^{\alpha\kappa-1}\int_0^1(1-w)^{-\alpha}w^{\alpha\kappa+\alpha-1}dwG(u)\frac{du}{u}$$

$$=\lambda x(t)+F(t),t\in(t_i,t_{i+1}].$$

It follows that x is a solution of (3.2.4).

Now we prove that if x is a solution of (3.2.4), then x satisfies (3.2.9) and $x\in LP_mC(1,e]$ by mathematical induction method. By Theorem 3.8, we know that there exists a constant $c_0\in\mathbb{R}$ such that

$$x(t)=c_0\mathbf{E}_{\alpha,\alpha}(\lambda(\log t)^\alpha)+\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}\left(\lambda\left(\frac{t}{s}\right)^\alpha\right)F(s)ds,t\in(t_0,t_1].$$

Hence (3.2.9) holds for $j=0$. Assume that (3.2.9) holds for $j=0,1,2,\cdots,i\le m$, we will prove that (3.2.9) holds for $j=i+1$. Suppose that

$$x(t)=\Phi(t)+\sum_{v=0}^{i}c_vE_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{t_v}\right)^\alpha\right)+\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{s}\right)^\alpha\right)G(s)\frac{ds}{s},t\in(t_{i+1},t_{i+2}].$$

Then for $t\in(t_{i+1},t_{i+2}]$ we have

$$F(t)+\lambda x(t)={}^{CH}D_{1^+}^{\alpha}x(t)=\frac{1}{\Gamma(1-\alpha)}\left[\sum_{j=0}^{i}\int_{t_j}^{t_{j+1}}\left(\log\frac{t}{s}\right)^{-\alpha}x'(s)ds+\int_{t_{i+1}}^{t}\left(\log\frac{t}{s}\right)^{-\alpha}x'(s)ds\right]'$$

$$=\frac{1}{\Gamma(1-\alpha)}\left[\sum_{j=0}^{i}\int_{t_j}^{t_{j+1}}\left(\log\frac{t}{s}\right)^{-\alpha}\left(\sum_{v=0}^{i}c_vE_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{t_v}\right)^\alpha\right)+\int_1^s\left(\log\frac{s}{u}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{u}\right)^\alpha\right)G(u)\frac{du}{u}\right)'ds\right.$$

$$\left.+\int_{t_{i+1}}^{t}\left(\log\frac{t}{s}\right)^{-\alpha}\left(\Phi(s)+\sum_{v=0}^{i}c_vE_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{t_v}\right)^\alpha\right)+\int_1^s\left(\log\frac{s}{u}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{u}\right)^\alpha\right)G(u)\frac{du}{u}\right)'ds\right].$$

Similarly to above discussion we can get that

$$F(t)+\lambda x(t)={}^{CH}D_{1^+}^{\alpha}x(t)=F(t)+\lambda x(t)+{}^{CH}D_{t_{i+1}^+}^{\alpha}\Phi(t)-\lambda\Phi(t).$$

So ${}^{CH}D_{t_{i+1}^+}^{\alpha}\Phi(t)-\lambda\Phi(t)=0$ on $(t_{i+1},t_{i+2}]$. Then Theorem 3.8 implies that there exists a constant $c_{i+1}\in\mathbb{R}$ such that $\Phi(t)=c_{i+1}\mathbf{E}_{\alpha,\alpha}\left(\lambda\left(\frac{t}{t_{i+1}}\right)^\alpha\right)$ on $(t_{i+1},t_{i+2}]$. Hence

$$x(t)=\sum_{v=0}^{i+1}c_vE_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{t_v}\right)^\alpha\right)+\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{s}\right)^\alpha\right)G(s)\frac{ds}{s},t\in(t_{i+1},t_{i+2}].$$

By mathematical induction method, we know that (3.2.9) holds for $j\in\mathbb{N}_0$. The proof is complete. □

3.3 Preliminary for BVP(1.7)

In this section, we present some preliminary results that can be used in next sections for get solutions of BVP(1.7). For ease expression, denote

$$\delta_{\alpha,\lambda}(t,s)=(t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha),$$

$$\Lambda=-1+\Gamma(\alpha)\delta_{\alpha,\lambda}(1,0)+\Gamma(\alpha)^2\delta_{\alpha,\lambda}(t_1,0)\delta_{\alpha,\lambda}(1,t_1).$$

Lemma 3.1. *Suppose that $\Lambda \neq 0$ and $\sigma : (0,1) \to R$ is continuous and satisfies that there exist numbers $k > -1$ and $\max\{-\alpha, -k-1\} < l \leq 0$ such that $|\sigma(t)| \leq t^k(1-t)^l$ for all $t \in (0,1)$. The x is a solutions of*

$$\begin{cases} {}^{RL}D_{0^+}^{\alpha} x(t) - \lambda x(t) = \sigma(t), t \in (t_i, t_{i+1}], i = 0, 1, \\ x(1) - \lim\limits_{t \to 0} t^{1-\alpha} x(t) = a_0, \quad \lim\limits_{t \to t_1^+} (t - t_1)^{1-\alpha} x(t) - x(t_1) = I_0 \end{cases} \tag{3.3.1}$$

if and only if $x \in P_1C_{1-\alpha}(0,1]$ and

$$x(t) = \begin{cases} \frac{\Gamma(\alpha)\delta_{\alpha,\lambda}(t,0)}{\Lambda} \Big[a_0 - \Gamma(\alpha)\delta_{\alpha,\lambda}(1,t_1)I_0 \ - \int_0^1 \delta_{\alpha,\lambda}(1,s)\sigma(s)ds \\ -\Gamma(\alpha)\delta_{\alpha,\lambda}(1,t_1) \int_0^{t_1} \delta_{\alpha,\lambda}(t_1,s)\sigma(s)ds\Big] + \int_0^t \delta_{\alpha,\lambda}(t,s)\sigma(s)ds, t \in (0,t_1], \\ \frac{\Gamma(\alpha)\delta_{\alpha,\lambda}(t,0)}{\Lambda} \Big[a_0 - \Gamma(\alpha)\delta_{\alpha,\lambda}(1,t_1)I_0 \ - \int_0^1 \delta_{\alpha,\lambda}(1,s)\sigma(s)ds \\ -\Gamma(\alpha)\delta_{\alpha,\lambda}(1,t_1) \int_0^{t_1} \delta_{\alpha,\lambda}(t_1,s)\sigma(s)ds\Big] + \frac{\Gamma(\alpha)\delta_{\alpha,\lambda}(t,t_1)}{\Lambda} \big[\Gamma(\alpha)\delta_{\alpha,\lambda}(t_1,0)a_0 \\ +(\Gamma(\alpha)\delta_{\alpha,\lambda}(1,0)-1)I_0 \ -\Gamma(\alpha)\delta_{\alpha,\lambda}(t_1,0) \int_0^1 \delta_{\alpha,\lambda}(1,s)\sigma(s)ds \\ +(\Gamma(\alpha)\delta_{\alpha,\lambda}(1,0)-1) \int_0^{t_1} \delta_{\alpha,\lambda}(t_1,s)\sigma(s)ds\Big] + \int_0^t \delta_{\alpha,\lambda}(t,s)\sigma(s)ds, t \in (t_1,1]. \end{cases} \tag{3.3.2}$$

Proof. Let x be a solution of (3.3.1). By Theorem 3.11, we know that there exist numbers $A_0, A_1 \in \mathbb{R}$ such that

$$x(t) = A_0\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha}) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^{\alpha})\sigma(s)ds, t \in (t_0,t_1] \tag{3.3.3}$$

and

$$\begin{aligned} x(t) &= A_0\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha}) + A_1\Gamma(\alpha)E_{\alpha,\alpha}(\lambda(t-t_1)^{\alpha})(t-t_1)^{\alpha-1} \\ &+ \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^{\alpha})\sigma(s)ds, t \in (t_1,t_2]. \end{aligned} \tag{3.3.4}$$

Note $E_{\alpha,\alpha}(0) = \frac{1}{\Gamma(\alpha)}$. It follows from the boundary conditions and the impulse assumption in (3.3.1) that

$$\begin{aligned} &A_0\Gamma(\alpha)E_{\alpha,\alpha}(\lambda) + A_1\Gamma(\alpha)E_{\alpha,\alpha}(\lambda(1-t_1)^{\alpha})(1-t_1)^{\alpha-1} \\ &+ \int_0^1 (1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(1-s)^{\alpha})\sigma(s)ds - A_0 = a_0, \\ &A_1 - \Big[A_0\Gamma(\alpha)t_1^{\alpha-1}E_{\alpha,\alpha}(\lambda t_1^{\alpha}) + \int_0^{t_1}(t_1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t_1-s)^{\alpha})\sigma(s)ds\Big] = I_0. \end{aligned}$$

Then

$$\begin{aligned} A_0 &= \frac{1}{\Lambda}\Big[a_0 - \Gamma(\alpha)(1-t_1)^{\alpha}E_{\alpha,\alpha}(\lambda(1-t_1)^{\alpha})I_0 - \int_0^1 (1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(1-s)^{\alpha})\sigma(s)ds \\ &-\Gamma(\alpha)(1-t_1)^{\alpha}E_{\alpha,\alpha}(\lambda(1-t_1)^{\alpha}) \int_0^{t_1}(t_1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t_1-s)^{\alpha})\sigma(s)ds\Big], \\ A_1 &= \frac{1}{\Lambda}\Big[\Gamma(\alpha)t_1^{\alpha-1}E_{\alpha,\alpha}(\lambda t_1^{\alpha})a_0 + (\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)-1)I_0 \\ &-\Gamma(\alpha)t_1^{\alpha-1}E_{\alpha,\alpha}(\lambda t_1^{\alpha}) \int_0^1 (1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(1-s)^{\alpha})\sigma(s)ds \\ &+(\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)-1) \int_0^{t_1}(t_1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t_1-s)^{\alpha})\sigma(s)ds\Big]. \end{aligned} \tag{3.3.5}$$

Substituting A_0, A_1 into (3.3.3) and (3.3.4), we get (3.3.2) obviously.

On the other hand, if x satisfies (3.3.2), then $x|_{(0,t_1]}$ and $x|_{(t_1,1]}$ are continuous and the limits $\lim\limits_{t \to 0} t^{1-\alpha}x(t)$

and $\lim_{t\to t_1}(1-t_1)^{1-\alpha}x(t)$ exist. So $x \in P_1C_{1-\alpha}(0,1]$. Using (3.3.5), we rewrite x by

$$x(t) = \begin{cases} A_0\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha)\sigma(s)ds, t \in (0,t_1], \\ A_0\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha) + A_1\Gamma(\alpha)E_{\alpha,\alpha}(\lambda(t-t_1)^\alpha)(t-t_1)^{\alpha-1} \\ +\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha)\sigma(s)ds, t \in (t_1,1]. \end{cases}$$

Since σ is continuous on $(0,1)$ and $|\sigma(t)| \le t^k(1-t)^l$, one can show easily that x is continuous on $(t_i,t_{i+1}](i=0,1)$ and using the method at the beginning of the proof of this lemma, we know that both the limits $\lim_{t\to 0^+} t^{1-\alpha}x(t)$ and $\lim_{t\to t_1^+}(t-t_1)^{1-\alpha}x(t)$ exist. So $x \in P_1C_{1-\alpha}(0,1]$. Furthermore, by direct computation, we have $x(1) - \lim_{t\to 0} t^{1-\alpha}x(t) = a_0$, and $\lim_{t\to t_1^+}(t-t_1)^{1-\alpha}x(t) - x(t_1) = I_0$. One have from Theorem 3.12 easily for $t \in (t_0,t_1]$ that $D_{0^+}^\alpha x(t) = \lambda x(t) + \sigma(t)$ and for $t \in (t_1,t_2]$ that

$${}^{RL}D_{0^+}^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)}\left[\int_0^t (t-s)^{-\alpha}x(s)ds\right]'$$

$$= \frac{1}{\Gamma(1-\alpha)}\left[\int_0^{t_1}(t-s)^{-\alpha}\left(A_0\Gamma(\alpha)s^{\alpha-1}E_{\alpha,\alpha}(\lambda s^\alpha) + \int_0^s (s-w)^{\alpha-1}E_{\alpha,\alpha}(\lambda(s-w)^\alpha)\sigma(w)dw\right)ds\right.$$

$$+\int_{t_1}^t (t-s)^{-\alpha}\left(A_0\Gamma(\alpha)s^{\alpha-1}E_{\alpha,\alpha}(\lambda s^\alpha) + A_1\Gamma(\alpha)E_{\alpha,\alpha}(\lambda(s-t_1)^\alpha)(s-t_1)^{\alpha-1}\right.$$

$$\left.\left.+\int_0^s (s-w)^{\alpha-1}E_{\alpha,\alpha}(\lambda(s-w)^\alpha)\sigma(w)dw\right)ds\right]'$$

$$= \frac{1}{\Gamma(1-\alpha)}\left[A_0\Gamma(\alpha)\int_0^t (t-s)^{-\alpha}s^{\alpha-1}E_{\alpha,\alpha}(\lambda s^\alpha)ds + A_1\Gamma(\alpha)\int_{t_1}^t (t-s)^{-\alpha}E_{\alpha,\alpha}(\lambda(s-t_1)^\alpha)(s-t_1)^{\alpha-1}ds\right.$$

$$\left.+\int_0^t (t-s)^{-\alpha}\int_0^s (s-w)^{\alpha-1}E_{\alpha,\alpha}(\lambda(s-w)^\alpha)\sigma(w)dwds\right]'.$$

One has by variable transformation $w = \frac{s-t_1}{t-t_1}$ and $\mathbf{B}(1-\alpha, j\alpha+\alpha) = \frac{\Gamma(1-\alpha)\Gamma(j\alpha+\alpha)}{\Gamma(j\alpha+1)}$ that

$$\int_{t_1}^t (t-s)^{-\alpha}E_{\alpha,\alpha}(\lambda(s-t_1)^\alpha)(s-t_1)^{\alpha-1}ds = \sum_{j=0}^{+\infty}\frac{\lambda^j}{\Gamma(\alpha(j+1))}\int_{t_1}^t (t-s)^{-\alpha}(s-t_1)^{j\alpha+\alpha-1}ds$$

$$= \sum_{j=0}^{+\infty}\frac{\lambda^j}{\Gamma(\alpha(j+1))}(t-t_1)^{j\alpha}\int_0^1 (1-w)^{-\alpha}w^{j\alpha+\alpha-1}dw = \sum_{j=0}^{+\infty}\frac{\lambda^j(t-t_1)^{j\alpha}}{\Gamma(j\alpha+1)}\Gamma(1-\alpha).$$

Then

$${}^{RL}D_{0^+}^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)}\left[A_0\Gamma(\alpha)\sum_{j=0}^{+\infty}\frac{\lambda^j t^{j\alpha}}{\Gamma(j\alpha+1)}\Gamma(1-\alpha) + A_1\Gamma(\alpha)\sum_{j=0}^{+\infty}\frac{\lambda^j(t-t_1)^{j\alpha}}{\Gamma(j\alpha+1)}\Gamma(1-\alpha)\right.$$

$$\left.+\sum_{j=0}^{+\infty}\frac{\lambda^j}{\Gamma(\alpha(j+1))}\int_0^t\int_w^t (t-s)^{-\alpha}(s-w)^{j\alpha+\alpha-1}ds\sigma(w)dw\right]'$$

$$= \frac{1}{\Gamma(1-\alpha)}\left[A_0\Gamma(\alpha)\sum_{j=0}^{+\infty}\frac{\lambda^j t^{j\alpha}}{\Gamma(j\alpha+1)}\Gamma(1-\alpha) + A_1\Gamma(\alpha)\sum_{j=0}^{+\infty}\frac{\lambda^j(t-t_1)^{j\alpha}}{\Gamma(j\alpha+1)}\Gamma(1-\alpha)\right.$$

$$+\sum_{j=0}^{+\infty}\frac{\lambda^j}{\Gamma(\alpha(j+1))}\int_0^t(t-w)^{j\alpha}\int_0^1(1-u)^{-\alpha}u^{j\alpha+\alpha-1}du\sigma(w)dw\Big]' \text{ by } \frac{s-w}{t-w}=u$$

$$=\frac{1}{\Gamma(1-\alpha)}\left[A_0\Gamma(\alpha)\sum_{j=0}^{+\infty}\frac{\lambda^j t^{j\alpha}}{\Gamma(j\alpha+1)}\Gamma(1-\alpha)+A_1\Gamma(\alpha)\sum_{j=0}^{+\infty}\frac{\lambda^j(t-t_1)^{j\alpha}}{\Gamma(j\alpha+1)}\Gamma(1-\alpha)\right.$$

$$\left.+\sum_{j=0}^{+\infty}\frac{\lambda^j}{\Gamma(j\alpha+1)}\int_0^t(t-w)^{j\alpha}\sigma(w)dw\Gamma(1-\alpha)\right]'$$

$$=\left[A_0\Gamma(\alpha)\sum_{j=0}^{+\infty}\frac{\lambda^j t^{j\alpha}}{\Gamma(j\alpha+1)}+A_1\Gamma(\alpha)\sum_{j=0}^{+\infty}\frac{\lambda^j(t-t_1)^{j\alpha}}{\Gamma(j\alpha+1)}+\sum_{j=0}^{+\infty}\frac{\lambda^j}{\Gamma(j\alpha+1)}\int_0^t(t-w)^{j\alpha}\sigma(w)dw\right]'$$

$$=\lambda x(t)+\sigma(t).$$

So x is a solution of (3.3.1). The proof is completed. □

For ease expression, denote for a function $H:(0,1)\times R\to R$ that $H_x(t)=H(t,x(t))$. Define the nonlinear operator T on $P_1C_{1-\alpha}(0,1]$ for $x\in P_1C_{1-\alpha}(0,1]$ by

$$(Tx)(t)=\begin{cases}\frac{\Gamma(\alpha)\delta_{\alpha,\lambda}(t,0)}{\Lambda}\Big[\int_0^1\phi(s)G_x(s)ds-\Gamma(\alpha)\delta_{\alpha,\lambda}(1,t_1)I_x(t_1)\ -\int_0^1\delta_{\alpha,\lambda}(1,s)p(s)f_x(s)ds\\ -\Gamma(\alpha)\delta_{\alpha,\lambda}(1,t_1)\int_0^{t_1}\delta_{\alpha,\lambda}(t_1,s)p(s)f_x(s)ds\Big]+\int_0^t\delta_{\alpha,\lambda}(t,s)p(s)f_x(s))ds,t\in(0,t_1],\\ \frac{\Gamma(\alpha)\delta_{\alpha,\lambda}(t,0)}{\Lambda}\Big[\int_0^1\phi(s)G_x(s)ds-\Gamma(\alpha)\delta_{\alpha,\lambda}(1,t_1)I_x(t_1)\ -\int_0^1\delta_{\alpha,\lambda}(1,s)p(s)f_x(s)ds\\ -\Gamma(\alpha)\delta_{\alpha,\lambda}(1,t_1)\int_0^{t_1}\delta_{\alpha,\lambda}(t_1,s)p(s)f_x(s)ds\Big]+\frac{\Gamma(\alpha)\delta_{\alpha,\lambda}(t,t_1)}{\Lambda}\Big[\Gamma(\alpha)\delta_{\alpha,\lambda}(t_1,0)\int_0^1\phi(s)G_x(s)ds\\ +(\Gamma(\alpha)\delta_{\alpha,\lambda}(1,0)-1)I_x(t_1)\ -\Gamma(\alpha)\delta_{\alpha,\lambda}(t_1,0)\int_0^1\delta_{\alpha,\lambda}(1,s)p(s)f_x(s)ds\\ +(\Gamma(\alpha)\delta_{\alpha,\lambda}(1,0)-1)\int_0^{t_1}\delta_{\alpha,\lambda}(t_1,s)p(s)f_x(s)ds\Big]+\int_0^t\delta_{\alpha,\lambda}(t,s)p(s)f_x(s)ds,t\in(t_1,1].\end{cases}$$

Lemma 3.2. *Suppose that (a)-(e) hold, $\Lambda\neq 0$, and f,G are* **impulsive II-Carathéodory functions**, *I a* **discrete II-Carathódory functions**. *Then $x\in P_1C_{1-\alpha}(0,]$ is a solution of BVP(1.7) if and only if $x\in P_1C_{1-\alpha}(0,1]$ is a fixed point of T, $T:P_1C_{1-\alpha}(0,1]\to P_1C_{1-\alpha}(0,1]$ is well defined and is completely continuous.*

Proof. **Step (i)** Prove that $T:P_mC_{1-\alpha}(0,1]\to P_mC_{1-\alpha}(0,1]$ is well defined.

It comes from the method in Theorem 3.12 that $Tx|_{(0,t_1]}$, $Tx|_{(t_1,1]}(i=0,1)$ are continuous and the limits $\lim_{t\to 0}t^{1-\alpha}(Tx)(t)$ and $\lim_{t\to t_1^+}(t-t_1)^{1-\alpha}(Tx)(t)$ exist. We see from Lemma 3.3.1 that $x\in P_1C_{1-\alpha}(0,1]$ is a solution of BVP(1.7) if and only if $x\in P_1C_{1-\alpha}(0,1]$ is a fixed point of T in $P_1C_{1-\alpha}(0,1]$.

Step (ii) We prove that T is continuous.

Let $x_n\in P_1C_{1-\alpha}(0,1]$ with $x_n\to x_0$ as $n\to+\infty$. We can show that $Tx_n\to Tx_0$ as $n\to+\infty$ by using the dominant convergence theorem. We refer the readers to the papers [65, 77, 81].

Step (iii) Prove that T is compact, i.e., prove that $T(\overline{\Omega})$ is relatively compact for every bounded subset $\Omega\subset P_1C_{1-\alpha}(0,1]$.

Let Ω be a bounded open nonempty subset of $P_1C_{1-\alpha}(0,1]$. We have

$$||x||=\max\left\{\sup_{t\in(t_i,t_{i+1}]}(t-t_i)^{1-\alpha}|x(t)|:i=0,1\right\}\le r<+\infty,(x,y)\in\overline{\Omega}. \quad (3.3.6)$$

Since f,G are **impulsive II-Carathéodory functions**, I ia a **discrete II-Carathéodory function**, then there exists constants $M_f, M_I, M_G \geq 0$ such that

$$
\begin{aligned}
&|f(t,x(t)))| = \left|f\left(t,(t-t_i)^{\alpha-1}(t-t_i)^{1-\alpha}x(t)\right)\right| \leq M_f, t \in (t_i,t_{i+1}], i=0,1,\\
&|G(t,x(t))| \leq M_G, t \in (t_i,t_{i+1}], i=0,1,\\
&|I(t_1,x(t_1))| = |I(t_1,t^{\alpha-1}t_1^{1-\alpha}x(t_1))| \leq M_I.
\end{aligned} \tag{3.3.7}
$$

This step is done by three sub-steps:

Sub-step (iii1) Prove that $T(\overline{\Omega})$ is uniformly bounded.

Using (3.3.2) and (3.3.7), we have for $t \in (0,t_1]$ that

$$
\begin{aligned}
t^{1-\alpha}|(Tx)(t)| &\leq \tfrac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda t^\alpha)}{|\Lambda|}\left[||\phi||_1 M_G + \Gamma(\alpha)\delta_{\alpha,\lambda}(1,t_1)|M_I\right.\\
&+\int_0^1(1-s)^{\alpha-1}\sum_{v=0}^{+\infty}\tfrac{\lambda^v(1-s)^{\alpha v}}{\Gamma(\alpha(v+1))}M_f s^k(1-s)^l ds\\
&\left.+\Gamma(\alpha)\delta_{\alpha,\lambda}(1,t_1)\int_0^{t_1}(t_1-s)^{\alpha-1}\sum_{v=0}^{+\infty}\tfrac{\lambda^v(t_1-s)^{\alpha v}}{\Gamma(\alpha(v+1))}M_f s^k(1-s)^l ds\right]\\
&+M_f t^{1-\alpha}\sum_{v=0}^{+\infty}\tfrac{\lambda^v(t-s)^{\alpha v}}{\Gamma(\alpha(v+1))}\int_0^t(t-s)^{\alpha+l-1}s^k ds\\
&\leq \tfrac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda t^\alpha)}{|\Lambda|}\left[||\phi||_1 M_G + \Gamma(\alpha)\delta_{\alpha,\lambda}(1,t_1)|M_I\right.\\
&+M_f\sum_{v=0}^{+\infty}\tfrac{\lambda^v}{\Gamma(\alpha(v+1))}\int_0^1(1-s)^{\alpha v+\alpha+l-1}s^k ds\\
&\left.+\Gamma(\alpha)M_f\delta_{\alpha,\lambda}(1,t_1)\sum_{v=0}^{+\infty}\tfrac{\lambda^v}{\Gamma(\alpha(v+1))}\int_0^{t_1}(t_1-s)^{\alpha v+\alpha+l-1}s^k ds\right]\\
&+M_f t^{1-\alpha}\sum_{v=0}^{+\infty}\tfrac{\lambda^v}{\Gamma(\alpha(v+1))}\int_0^t(t-s)^{\alpha v+\alpha+l-1}s^k ds\\
&\leq \tfrac{\Gamma(\alpha)||\phi||_1 E_{\alpha,\alpha}(|\lambda|)}{|\Lambda|}M_G + \tfrac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^2(1-t_1)^{\alpha-1}}{|\Lambda|}|M_I\\
&+\left(\tfrac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)^2}{|\Lambda|}+\tfrac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^3(1-t_1)^{\alpha-1}t_1^{\alpha+k+l}}{|\Lambda|}+E_{\alpha,\lambda}(|\lambda|)\right)\mathbf{B}(\alpha+l,k+1).
\end{aligned}
$$

For $t \in (t_1,t_2]$, we have similarly that

$$
\begin{aligned}
(t-t_1)^{1-\alpha}|(Tx)(t)| &\leq \tfrac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{|\Lambda|}\left[||\phi||_1 M_G + \Gamma(\alpha)(1-t_1)^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)M_I\right.\\
&+\sum_{v=0}^{+\infty}\tfrac{\lambda^v}{\Gamma(\alpha(v+1))}\int_0^1(1-s)^{\alpha v+\alpha+l-1}s^k ds M_f\\
&\left.+\Gamma(\alpha)(1-t_1)^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)\sum_{v=0}^{+\infty}\tfrac{\lambda^v}{\Gamma(\alpha(v+1))}\int_0^{t_1}(t_1-s)^{\alpha v+\alpha+l-1}s^k ds M_f\right]\\
&+\tfrac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda(t-t_1)^\alpha)}{|\Lambda|}\left[\Gamma(\alpha)t_1^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)||\phi||_1 M_G\right.\\
&+(\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)+1)M_I \ +\Gamma(\alpha)t_1^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)\sum_{v=0}^{+\infty}\tfrac{\lambda^v}{\Gamma(\alpha(v+1))}\int_0^1(1-s)^{\alpha v+\alpha+l-1}s^k ds M_f\\
&\left.+(\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)+1)\sum_{v=0}^{+\infty}\tfrac{\lambda^v}{\Gamma(\alpha(v+1))}\int_0^{t_1}(t_1-s)^{\alpha v+\alpha+l-1}s^k ds M_f\right]
\end{aligned}
$$

$$+M_f(t-t_1)^{1-\alpha}\sum_{v=0}^{+\infty}\frac{\lambda^v}{\Gamma(\alpha(v+1))}\int_0^t(t-s)^{\alpha v+\alpha+l-1}s^kds$$

$$\leq \frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{|\Lambda|}\left[||\phi||_1M_G+\Gamma(\alpha)(1-t_1)^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)M_I\right.$$

$$+E_{\alpha,\alpha}(|\lambda|)\mathbf{B}(\alpha+l,k+1)M_f$$

$$\left.+\Gamma(\alpha)(1-t_1)^{\alpha-1}t_1^{\alpha+k+l}E_{\alpha,\alpha}(|\lambda|)^2\mathbf{B}(\alpha+l,k+1)M_f\right]$$

$$+\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{|\Lambda|}\left[\Gamma(\alpha)t_1^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)||\phi||_1M_G\right.$$

$$+(\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)+1)M_I\ +\Gamma(\alpha)t_1^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)^2\mathbf{B}(\alpha+l,k+1)M_f$$

$$\left.+(\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)+1)t_1^{\alpha+k+l}E_{\alpha,\alpha}(|\lambda|)\mathbf{B}(\alpha+l,k+1)M_f\right]$$

$$+M_f(t-t_1)^{1-\alpha}t^{\alpha+k+l}E_{\alpha,\alpha}(|\lambda|)\mathbf{B}(\alpha+l,k+1)$$

$$\leq\left(\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)||\phi||_1}{|\Lambda|}+\frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^2||\phi||_1t_1^{\alpha-1}}{|\Lambda|}\right)M_G$$

$$+\left(\frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^2(1-t_1)^{\alpha-1}}{|\Lambda|}+\frac{\Gamma(\alpha)(\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)+1)E_{\alpha,\alpha}(|\lambda|)}{|\Lambda|}\right)M_I$$

$$+\left(\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)^2}{|\Lambda|}+\frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^3(1-t_1)^{\alpha-1}t_1^{\alpha+k+l}}{|\Lambda|}+\frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^3t_1^{\alpha-1}}{|\Lambda|}\right.$$

$$\left.+\frac{\Gamma(\alpha)(\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)+1)E_{\alpha,\alpha}(|\lambda|)^2t_1^{\alpha+k+l}}{|\Lambda|}+E_{\alpha,\alpha}(|\lambda|)\right)\mathbf{B}(\alpha+l,k+1)M_f.$$

From above discussion, we get

$$\begin{aligned}||Tx||\leq&\left[\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)||\phi||_1}{|\Lambda|}+\frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^2||\phi||_1}{|\Lambda|t_1^{1-\alpha}}\right]M_G\\&+\left[\frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^2}{|\Lambda|(1-t_1)^{1-\alpha}}+\frac{\Gamma(\alpha)(\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)+1)E_{\alpha,\alpha}(|\lambda|)}{|\Lambda|}\right]M_I\\&+\left(\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)^2}{|\Lambda|}+\frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^3(1-t_1)^{\alpha-1}t_1^{\alpha+k+l}}{|\Lambda|}+\frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^3t_1^{\alpha-1}}{|\Lambda|}\right.\\&\left.+\frac{\Gamma(\alpha)(\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)+1)E_{\alpha,\alpha}(|\lambda|)^2t_1^{\alpha+k+l}}{|\Lambda|}+E_{\alpha,\alpha}(|\lambda|)\right)\mathbf{B}(\alpha+l,k+1)M_f.\end{aligned}\tag{3.3.8}$$

From above discussion, $T(\overline{\Omega})$ is uniformly bounded.

Sub-step (iii2) Prove that $t\to(t-t_i)^{1-\alpha}T(\overline{\Omega})$ is equi-continuous on $(t_i,t_{i+1}](i=0,1)$.
Let

$$(t-t_i)^{1-\alpha}\overline{(T_1(x,y))}(t)=\begin{cases}(t-t_i)^{1-\alpha}(T_1(x,y))(t),t\in(t_i,t_{i+1}],\\ \lim\limits_{t\to t_i^+}(t-t_i)^{1-\alpha}(T_1(x,y))(t),t=t_i.\end{cases}$$

Then $t\to(t-t_i)^{1-\alpha}\overline{(Tx)}(t)$ is continuous on $[t_i,t_{i+1}]$. Let $s_2\leq s_1$ and $s_1,s_2\in[t_0,t_1]$. By AscoliCArzela theorem on the closed interval, We can prove that

$$\left|s_1^{1-\alpha}\overline{(Tx)}(s_1)-s_2^{1-\alpha}\overline{(Tx)}(s_2)\right|\to 0\text{ unoformly as }s_1\to s_2$$

and for $s_2\leq s_1$ and $s_1,s_2\in(t_1,t_2]$, we have

$$\left|(s_1-t_1)^{1-\alpha}\overline{(Tx)}(s_1)-(s_2-t_1)^{1-\alpha}\overline{(Tx)}(s_2)\right|\to 0\text{ unoformly as }s_1\to s_2.$$

Then $t \to (t-t_i)^{1-\alpha}T(\overline{\Omega})$ is equi-continuous on $(t_i, t_{i+1}](i=0,1)$. So $T(\overline{\Omega})$ is relatively compact. Then T is completely continuous. The proofs are completed. □

3.4 Preliminary for BVP(1.8)

In this section, we present some preliminary results that can be used in next sections for get solutions of BVP(1.8).

Lemma 3.3. *Suppose that $E_{\alpha,1}(\lambda)-1 \neq 0$ and $\sigma:(0,1)\mapsto \mathbb{R}$ is continuous and satisfies that there exist numbers $k>-1$ and $l \leq 0$ with $l \in (\max\{-\alpha, -\alpha-k\}, 0]$ such that $|\sigma(t)| \leq t^k(1-t)^l$ for all $t \in (0,1)$. The x is a solutions of*

$$\begin{cases} {}^C D_{0^+}^{\alpha} x(t) - \lambda x(t) = \sigma(t), t \in (t_i, t_{i+1}], i = 0,1, \\ x(1) - \lim\limits_{t\to 0} x(t) = a_0, \quad \lim\limits_{t\to t_1^+} x(t) - x(t_1) = I_0 \end{cases} \tag{3.4.1}$$

if and only if x and

$$x(t) = \begin{cases} \frac{E_{\alpha,1}(\lambda t^\alpha)}{E_{\alpha,1}(\lambda)-1}\left[a_0 - E_{\alpha,1}(\lambda(1-t_1)^\alpha)I_0 - \int_0^1 \delta_{\alpha,\lambda}(1,s)\sigma(s)ds\right] \\ + \int_0^t \delta_{\alpha,\lambda}(t,s)\sigma(s)ds, t \in (0,t_1], \\ \frac{E_{\alpha,1}(\lambda t^\alpha)}{E_{\alpha,1}(\lambda)-1}\left[a_0 - E_{\alpha,1}(\lambda(1-t_1)^\alpha)I_0 - \int_0^1 \delta_{\alpha,\lambda}(1,s)\sigma(s)ds\right] \\ + E_{\alpha,1}(\lambda(t-t_1)^\alpha)I_0 + \int_0^t \delta_{\alpha,\lambda}(t,s)\sigma(s)ds, t \in (t_1,1]. \end{cases} \tag{3.4.2}$$

Proof. Let x be a solution of (3.4.1). We know by Theorem 3.10 that there exist numbers $A_0, A_1 \in \mathbb{R}$ such that

$$x(t) = A_0 E_{\alpha,1}(\lambda t^\alpha) + \int_0^t \delta_{\alpha,\lambda}(t,s)\sigma(s)ds, t \in (t_0, t_1] \tag{3.4.3}$$

and

$$x(t) = A_0 E_{\alpha,1}(\lambda t^\alpha) + A_1 E_{\alpha,1}(\lambda(t-t_1)^\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha)\sigma(s)ds, t \in (t_1, t_2]. \tag{3.4.4}$$

Note $E_{\alpha,1}(0)=1$. It follows from (3.4.3), (3.4.4), the boundary conditions and the impulse assumption in (3.4.1) that

$$A_0 E_{\alpha,1}(\lambda) + A_1 E_{\alpha,1}(\lambda(1-t_1)^\alpha) + \int_0^1 (1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(1-s)^\alpha)\sigma(s)ds - A_0 = a_0,$$

$$A_1 = I_0.$$

Then

$$A_0 = \frac{1}{E_{\alpha,1}(\lambda)-1}\left[a_0 - E_{\alpha,1}(\lambda(1-t_1)^\alpha)I_0 - \int_0^1 (1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(1-s)^\alpha)\sigma(s)ds\right]. \tag{3.4.5}$$

Substituting A_0, A_1 into (3.4.3) and (3.4.4), we get (3.3.2) obviously.

On the other hand, if x satisfies (3.4.2), then both $x|_{(0,t_1]}$ and $x|_{(t_1,1]}$ are continuous and the limits $\lim\limits_{t\to 0} x(t)$ and $\lim\limits_{t\to t_1} x(t)$ exist. So $x \in P_1C(0,1]$. Using (3.4.5) and $A_1 = I_0$, we rewrite x by

$$x(t) = \begin{cases} A_0 E_{\alpha,1}(\lambda t^\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha)\sigma(s)ds, t \in (0,t_1], \\ A_0 E_{\alpha,1}(\lambda t^\alpha) + A_1 E_{\alpha,1}(\lambda(t-t_1)^\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha)\sigma(s)ds, t \in (t_1,1]. \end{cases}$$

Since σ is continuous on $(0,1)$ and $|\sigma(t)| \leq t^k(1-t)^l$, one can show easily that x is continuous on $(t_i, t_{i+1}](i=0,1)$ and using the method at the beginning of the proof of this lemma, we know that both the limits $\lim\limits_{t\to 0^+} x(t)$ and $\lim\limits_{t\to t_1^+} x(t)$ exist. So $x \in P_1C(0,1]$. Furthermore, by direct computation, we have $x(1) - \lim\limits_{t\to 0} x(t) = a_0$, and

$\lim_{t\to t_1^+} x(t)-x(t_1)=I_0$. One have easily from Theorem 3.2.1 for $t\in(t_0,t_1]$ that ${}^CD_{0^+}^\alpha x(t)=\lambda x(t)+\sigma(t)$ and for $t\in(t_1,t_2]$ that

$$
\begin{aligned}
&{}^CD_{0^+}^\alpha x(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-s)^{-\alpha}x'(s)ds\\
&=\frac{1}{\Gamma(1-\alpha)}\int_0^{t_1}(t-s)^{-\alpha}\left(A_0E_{\alpha,1}(\lambda s^\alpha)+\int_0^s(s-w)^{\alpha-1}E_{\alpha,\alpha}(\lambda(s-w)^\alpha)\sigma(w)dw\right)'ds\\
&+\frac{1}{\Gamma(1-\alpha)}\int_{t_1}^t(t-s)^{-\alpha}\left(A_0E_{\alpha,1}(\lambda s^\alpha)+A_1E_{\alpha,1}(\lambda(s-t_1)^\alpha)\ +\int_0^s(s-w)^{\alpha-1}E_{\alpha,\alpha}(\lambda(s-w)^\alpha)\sigma(w)dw\right)'ds\\
&=\frac{A_0}{\Gamma(1-\alpha)}\int_0^t(t-s)^{-\alpha}\left(E_{\alpha,1}(\lambda s^\alpha)\right)'ds+\frac{A_1}{\Gamma(1-\alpha)}\int_{t_1}^t(t-s)^{-\alpha}\left(E_{\alpha,1}(\lambda(s-t_1)^\alpha)\right)'ds\\
&+\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-s)^{-\alpha}\left(\int_0^s(s-w)^{\alpha-1}E_{\alpha,\alpha}(\lambda(s-w)^\alpha)\sigma(w)dw\right)'ds\\
&=\frac{A_0}{\Gamma(1-\alpha)}\int_0^t(t-s)^{-\alpha}\left(\sum_{j=0}^{+\infty}\frac{\lambda^j}{\Gamma(j\alpha+1)}s^{j\alpha}\right)'ds+\frac{A_1}{\Gamma(1-\alpha)}\int_{t_1}^t(t-s)^{-\alpha}\left(\sum_{j=0}^{+\infty}\frac{\lambda^j}{\Gamma(j\alpha+1)}(s-t_1)^{j\alpha}\right)'ds\\
&+\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-s)^{-\alpha}\left(\int_0^s(s-w)^{\alpha-1}\sum_{j=0}^{+\infty}\frac{\lambda^j}{\Gamma(\alpha(j+1))}(s-w)^{j\alpha}\sigma(w)dw\right)'ds\\
&=\frac{A_0}{\Gamma(1-\alpha)}\sum_{j=1}^{+\infty}\frac{(\alpha j)\lambda^j}{\Gamma(j\alpha+1)}\int_0^t(t-s)^{-\alpha}s^{j\alpha-1}ds+\frac{A_1}{\Gamma(1-\alpha)}\sum_{j=1}^{+\infty}\frac{(\alpha j)\lambda^j}{\Gamma(j\alpha+1)}\int_{t_1}^t(t-s)^{-\alpha}(s-t_1)^{j\alpha-1}ds\\
&+\frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{+\infty}\frac{\lambda^j}{\Gamma(\alpha(j+1))}\int_0^t(t-s)^{-\alpha}\left(\int_0^s(s-w)^{\alpha-1}(s-w)^{j\alpha}\sigma(w)dw\right)'ds\\
&=\frac{A_0}{\Gamma(1-\alpha)}\sum_{j=1}^{+\infty}\frac{(\alpha j)\lambda^j}{\Gamma(j\alpha+1)}t^{\alpha j-\alpha}\int_0^1(1-w)^{-\alpha}w^{j\alpha-1}dw+\frac{A_1}{\Gamma(1-\alpha)}\sum_{j=1}^{+\infty}\frac{(\alpha j)\lambda^j}{\Gamma(j\alpha+1)}(t-t_1)^{\alpha j-\alpha}\int_0^1(1-u)^{-\alpha}u^{j\alpha-1}du\\
&+\frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{+\infty}\frac{\lambda^j}{\Gamma(\alpha(j+1))}\Gamma(j\alpha+\alpha)\int_0^t(t-s)^{-\alpha}\left(I_{0^+}^{\alpha j+\alpha}\sigma(s)\right)'ds
\end{aligned}
$$

One has by $\mathbf{B}(1-\alpha,j\alpha)=\frac{\Gamma(1-\alpha)\Gamma(j\alpha)}{\Gamma((j-1)\alpha+1)}$ that

$$
\begin{aligned}
{}^CD_{0^+}^\alpha x(t)&=\frac{A_0}{\Gamma(1-\alpha)}\sum_{j=1}^{+\infty}\frac{(\alpha j)\lambda^j}{\Gamma(j\alpha+1)}t^{\alpha j-\alpha}\frac{\Gamma(1-\alpha)\Gamma(j\alpha)}{\Gamma((j-1)\alpha+1)}\\
&+\frac{A_1}{\Gamma(1-\alpha)}\sum_{j=1}^{+\infty}\frac{(\alpha j)\lambda^j}{\Gamma(j\alpha+1)}(t-t_1)^{\alpha j-\alpha}\frac{\Gamma(1-\alpha)\Gamma(j\alpha)}{\Gamma((j-1)\alpha+1)}+\sum_{j=0}^{+\infty}\lambda^j{}^CD_{0^+}^\alpha\left(I_{0^+}^{\alpha j+\alpha}\sigma(t)\right)
\end{aligned}
$$

$$
\begin{aligned}
&=\lambda A_0E_{\alpha,1}(\lambda t^\alpha)+\lambda A_1E_{\alpha,1}(\lambda(t-t_1)^\alpha)+\sigma(t)+\sum_{j=1}^{+\infty}\lambda^jI_{0^+}^{\alpha j}\sigma(t)\\
&=\lambda A_0E_{\alpha,1}(\lambda t^\alpha)+\lambda A_1E_{\alpha,1}(\lambda(t-t_1)^\alpha)+\sigma(t)+\lambda\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha)\sigma(s)ds\\
&=\lambda x(t)+\sigma(t).
\end{aligned}
$$

So x is a solution of (3.4.1). The proof is completed. □

Define the nonlinear operator Q on $P_1C(0,1]$ by Qx for $x \in P_1C(0,1]$ with

$$(Qx)(t) = \begin{cases} \frac{E_{\alpha,1}(\lambda t^\alpha)}{E_{\alpha,1}(\lambda)-1}\left[\int_0^1 \phi(s)G_x(s)ds - E_{\alpha,1}(\lambda(1-t_1)^\alpha)I_x(t_1) - \int_0^1 \delta_{\alpha,\lambda}(1,s)p(s)f_x(s)ds\right] \\ + \int_0^t \delta_{\alpha,\lambda}(t,s)p(s)f_x(s)ds, t \in (0,t_1], \\ \frac{E_{\alpha,1}(\lambda t^\alpha)}{E_{\alpha,1}(\lambda)-1}\left[\int_0^1 \phi(s)G_x(s)ds - E_{\alpha,1}(\lambda(1-t_1)^\alpha)I_x(t_1) - \int_0^1 \delta_{\alpha,\lambda}(1,s)p(s)f_x(s)ds\right] \\ + E_{\alpha,1}(\lambda(t-t_1)^\alpha)I_0 + \int_0^t \delta_{\alpha,\lambda}(t,s)p(s)f_x(s)ds, t \in (t_1,1]. \end{cases}$$

Lemma 3.4. *Suppose that (b), (c), (f)-(g) hold, $E_{\alpha,1}(\lambda) - 1 \neq 0$ and f, G are* **impulsive I-Carathéodory functions**, *I a* **discrete I-Carathódory function**. *Then $x \in P_1C(),1]$ is a solution of BVP(1.0.8) if and only if $x \in P_1C(0,1]$ is a fixed point of Q, $Q : P_1C(0,1] \to P_1C(0,1]$ is well defined and is completely continuous.*

Proof. The proof is similar to that of Lemma 3.2 and is omitted. □

3.5 Preliminary for BVP(1.9)

In this section, we present some preliminary results that can be used in next sections for get solutions of BVP(1.9). For ease expression, denote

$$\varrho_{\alpha,\lambda}(t,s) = \left(\log\frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{s}\right)^\alpha\right),$$

$$\Lambda_1 = -1 + \Gamma(\alpha)\varrho_{\alpha,\lambda}(e,t_1) + \Gamma(\alpha)^2\varrho_{\alpha,\lambda}(t_1,1)\varrho_{\alpha,\lambda}(e,1).$$

Lemma 3.5. *Suppose that $\Lambda_1 \neq 0$ and $\sigma : (0,1) \mapsto \mathbb{R}$ is continuous and satisfies that there exist numbers $k > -1$ and $l \leq 0$ with $l > \max\{-\alpha, -k-1\}$ such that $|\sigma(t)| \leq (\log t)^k (1-\log t)^l$ for all $t \in (1,e)$. The x is a solutions of*

$$\begin{cases} {}^{RLH}D_{1^+}^\alpha x(t) - \lambda x(t) = \sigma(t), t \in (t_i,t_{i+1}], i = 0,1, \\ x(e) - \lim_{t\to 1^+}(\log t)^{1-\alpha}x(t) = a_0, \ \lim_{t\to t_1^+}\left(\log\frac{t}{t_1}\right)^{1-\alpha}x(t) - x(t_1) = I_0 \end{cases} \tag{3.5.1}$$

if and only if $x \in LPC_{1-\alpha}(1,e]$ and

$$x(t) = \begin{cases} \frac{\Gamma(\alpha)\varrho_{\alpha,\lambda}(t,1)}{\Lambda_1}\left[a_0 - \Gamma(\alpha)\varrho_{\alpha,\lambda}(e,1)I_0 - \int_1^e \varrho_{\alpha,\lambda}(e,s)\sigma(s)\frac{ds}{s}\right. \\ \left.+\Gamma(\alpha)\varrho_{\alpha,\lambda}(e,1)\int_1^{t_1}\varrho_{\alpha,\lambda}(t_1,s)\sigma(s)\frac{ds}{s}\right] + \int_0^t \varrho_{\alpha,\lambda}(t,s)\sigma(s)\frac{ds}{s}, t \in (1,t_1], \\ \frac{\Gamma(\alpha)\varrho_{\alpha,\lambda}(t,1)}{\Lambda_1}\left[a_0 - \Gamma(\alpha)\varrho_{\alpha,\lambda}(e,1)I_0 - \int_1^e \varrho_{\alpha,\lambda}(e,s)\sigma(s)\frac{ds}{s}\right. \\ \left.+\Gamma(\alpha)\varrho_{\alpha,\lambda}(e,1)\int_1^{t_1}\varrho_{\alpha,\lambda}(t_1,s)\sigma(s)\frac{ds}{s}\right] \\ +\frac{\Gamma(\alpha)\varrho_{\alpha,\lambda}(t,t_1)}{\Lambda_1}\left[\Gamma(\alpha)\varrho_{\alpha,\lambda}(t_1,1)a_0 - (1-\Gamma(\alpha)\varrho_{\alpha,\lambda}(e,t_1))I_0\right. \\ \left.-\Gamma(\alpha)\varrho_{\alpha,\lambda}(t_1,1)\int_1^e \varrho_{\alpha,\lambda}(e,s)\sigma(s)\frac{ds}{s} + (1-\Gamma(\alpha)\varrho_{\alpha,\lambda}(e,t_1))\int_1^{t_1}\varrho_{\alpha,\lambda}(t_1,s)\sigma(s)\frac{ds}{s}\right] \\ + \int_0^t \varrho_{\alpha,\lambda}(t,s)\sigma(s)\frac{ds}{s}, t \in (1,e]. \end{cases} \tag{3.5.2}$$

Proof. Let x be a solution of (3.5.1). We know from Theorem 3.12 that there exist numbers $A_0, A_1 \in R$ such that

$$x(t) = A_0\Gamma(\alpha)\varrho_{\alpha,\lambda}(t,1) + \int_1^t \varrho_{\alpha,\lambda}(t,s)\sigma(s)\frac{ds}{s}, t \in (t_0,t_1] \tag{3.5.3}$$

and

$$x(t) = A_0\Gamma(\alpha)\varrho_{\alpha,\lambda}(t,1) + A_1\Gamma(\alpha)\varrho_{\alpha,\lambda}(t,t_1) + \int_1^t \varrho_{\alpha,\lambda}(t,s)\sigma(s)\frac{ds}{s}, t \in (t_1,t_2]. \tag{3.5.4}$$

Note $E_{\alpha,\alpha}(0)=\frac{1}{\Gamma(\alpha)}$. It follows from (3.5.3), (3.5.4), the boundary conditions and the impulse assumption in (3.5.1) that

$$(-1+\Gamma(\alpha)\varrho_{\alpha,\lambda}(e,t_1))\,A_0+\Gamma(\alpha)\varrho_{\alpha,\lambda}(e,1)A_1=a_0-\int_1^e\varrho_{\alpha,\lambda}(e,s)\sigma(s)\frac{ds}{s},$$

$$-\Gamma(\alpha)\varrho_{\alpha,\lambda}(t_1,1)A_0+A_1=I_0-\int_1^{t_1}\varrho_{\alpha,\lambda}(t_1,s)\sigma(s)\frac{ds}{s}.$$

Then

$$A_0=\frac{1}{\Lambda_1}\left[a_0-\Gamma(\alpha)\varrho_{\alpha,\lambda}(e,1)I_0-\int_1^e\varrho_{\alpha,\lambda}(e,s)\sigma(s)\frac{ds}{s}+\Gamma(\alpha)\varrho_{\alpha,\lambda}(e,1)\int_1^{t_1}\varrho_{\alpha,\lambda}(t_1,s)\sigma(s)\frac{ds}{s}\right],$$

$$A_1=\frac{1}{\Lambda_1}\left[\Gamma(\alpha)\varrho_{\alpha,\lambda}(t_1,1)a_0-(1-\Gamma(\alpha)\varrho_{\alpha,\lambda}(e,t_1))\,I_0\right.\tag{3.5.5}$$

$$\left.-\Gamma(\alpha)\varrho_{\alpha,\lambda}(t_1,1)\int_1^e\varrho_{\alpha,\lambda}(e,s)\sigma(s)\frac{ds}{s}+(1-\Gamma(\alpha)\varrho_{\alpha,\lambda}(e,t_1))\int_1^{t_1}\varrho_{\alpha,\lambda}(t_1,s)\sigma(s)\frac{ds}{s}\right].$$

Substituting A_0,A_1 into (3.5.3) and (3.5.4), we get (3.5.2) obviously.

On the other hand, if x satisfies (3.5.2), then $x|_{(1,t_1]}$ and $x|_{(t_1,e]}$ are continuous and the limits $\lim_{t\to1^+}(\log t)^{1-\alpha}x(t)$ and $\lim_{t\to t_1^+}(\log t-\log t_1)^{1-\alpha}x(t)$ exist. So $x\in LP_1C_{1-\alpha}(1,e]$. Using (3.5.5), we rewrite x by

$$x(t)=\begin{cases}A_0\Gamma(\alpha)\varrho_{\alpha,\lambda}(t,1)+\int_1^t\varrho_{\alpha,\lambda}(t,s)\sigma(s)\frac{ds}{s},\ t\in(0,t_1],\\ A_0\Gamma(\alpha)\varrho_{\alpha,\lambda}(t,1)+A_1\Gamma(\alpha)\varrho_{\alpha,\lambda}(t,t_1)+\int_1^t\varrho_{\alpha,\lambda}(t,s)\sigma(s)\frac{ds}{s},\ t\in(t_1,1].\end{cases}$$

Furthermore, by direct computation, we have $x(e)-\lim_{t\to1^+}(\log t)^{1-\alpha}x(t)=a_0$, and $\lim_{t\to t_1^+}(\log t-\log t_1)^{1-\alpha}x(t)-x(t_1)=I_0$. One have from Theorem 3.2.3 easily for $t\in(t_0,t_1]$ that ${}^{RLH}D_{0^+}^{\alpha}x(t)=\lambda x(t)+\sigma(t)$ for $t\in(t_0,t_1]$. For $t\in(t_1,t_2]$, we have by Definition 2.5 that

$${}^{RLH}D_{1^+}^{\alpha}x(t)=\frac{t}{\Gamma(1-\alpha)}\left(\int_1^t\left(\log\frac{t}{s}\right)^{-\alpha}x(s)\frac{ds}{s}\right)'$$

$$=\frac{t}{\Gamma(1-\alpha)}\left[\int_1^{t_1}\left(\log\frac{t}{s}\right)^{-\alpha}\left(A_0\Gamma(\alpha)\varrho_{\alpha,\lambda}(s,1)+\int_1^s\varrho_{\alpha,\lambda}(s,u)\sigma(u)\frac{du}{u}\right)\frac{ds}{s}\right.$$

$$\left.+\int_{t_1}^t\left(\log\frac{t}{s}\right)^{-\alpha}\left(A_0\Gamma(\alpha)\varrho_{\alpha,\lambda}(s,1)+A_1\Gamma(\alpha)\varrho_{\alpha,\lambda}(s,t_1)+\int_1^s\varrho_{\alpha,\lambda}(s,u)\sigma(u)\frac{du}{u}\right)\frac{ds}{s}\right]'$$

$$=\frac{t}{\Gamma(1-\alpha)}\left[\Gamma(\alpha)A_0\int_1^t\left(\log\frac{t}{s}\right)^{-\alpha}\varrho_{\alpha,\lambda}(s,1)\frac{ds}{s}+\int_1^t\left(\log\frac{t}{s}\right)^{-\alpha}\int_1^s\varrho_{\alpha,\lambda}(s,u)\sigma(u)\frac{du}{u}\frac{ds}{s}\right.$$

$$\left.+\Gamma(\alpha)A_1\int_{t_1}^t\left(\log\frac{t}{s}\right)^{-\alpha}\varrho_{\alpha,\lambda}(s,t_1)\frac{ds}{s}\right]'$$

$$=\frac{t}{\Gamma(1-\alpha)}\left[\Gamma(\alpha)A_0\int_1^t\left(\log\frac{t}{s}\right)^{-\alpha}\varrho_{\alpha,\lambda}(s,1)\frac{ds}{s}+\int_1^t\int_u^t\left(\log\frac{t}{s}\right)^{-\alpha}\varrho_{\alpha,\lambda}(s,u)\frac{ds}{s}\sigma(u)\frac{du}{u}\right.$$

$$\left.+\Gamma(\alpha)A_1\int_{t_1}^t\left(\log\frac{t}{s}\right)^{-\alpha}\varrho_{\alpha,\lambda}(s,t_1)\frac{ds}{s}\right]'$$

One finds that

$$\int_u^t\left(\log\frac{t}{s}\right)^{-\alpha}\varrho_{\alpha,\lambda}(s,u)\frac{ds}{s}=\int_u^t\left(\log\frac{t}{s}\right)^{-\alpha}\left(\log\frac{s}{u}\right)^{\alpha-1}\sum_{j=0}^{+\infty}\frac{\lambda^j(\log s-\log u)^{j\alpha}}{\Gamma((j+1)\alpha)}\frac{ds}{s}$$

$$=\sum_{j=0}^{+\infty}\frac{\lambda^j}{\Gamma((j+1)\alpha)}\int_u^t\left(\log\frac{t}{s}\right)^{-\alpha}\left(\log\frac{s}{u}\right)^{\alpha j+\alpha-1}\frac{ds}{s}\text{ by }\frac{logs-\log u}{\log t-\log u}=w$$

$$=\sum_{j=0}^{+\infty}\frac{\lambda^j}{\Gamma((j+1)\alpha)}\left(\log\frac{t}{u}\right)^{\alpha j}\int_0^1(1-w)^{-\alpha}w^{\alpha j+\alpha-1}dw$$

$$=\Gamma(1-\alpha)\sum_{j=0}^{+\infty}\frac{\lambda^j}{\Gamma(j\alpha+1)}\left(\log\frac{t}{u}\right)^{\alpha j}\text{ by }\mathbf{B}(1-\alpha,\alpha(j+1))=\frac{\Gamma(1-\alpha)\Gamma(\alpha(j+1))}{\Gamma(\alpha j-1))}.$$

So

$$^{RLH}D_{1^+}^{\alpha}x(t)=\frac{t}{\Gamma(1-\alpha)}\left[\Gamma(\alpha)A_0\Gamma(1-\alpha)\sum_{j=0}^{+\infty}\frac{\lambda^j}{\Gamma(j\alpha+1)}(\log t)^{\alpha j}\right.$$

$$\left.+\int_1^t\Gamma(1-\alpha)\sum_{j=0}^{+\infty}\frac{\lambda^j}{\Gamma(j\alpha+1)}\left(\log\frac{t}{u}\right)^{\alpha j}\sigma(u)\frac{du}{u}\;+\Gamma(\alpha)A_1\Gamma(1-\alpha)\sum_{j=0}^{+\infty}\frac{\lambda^j}{\Gamma(j\alpha+1)}\left(\log\frac{t}{t_1}\right)^{\alpha j}\right]'$$

$$=t\left[\frac{\Gamma(\alpha)A_0}{t}\sum_{j=1}^{+\infty}\frac{(\alpha j)\lambda^j}{\Gamma(j\alpha+1)}(\log t)^{\alpha j-1}\;+\int_1^t\sigma(u)\frac{du}{u}+\frac{1}{t}\int_1^t\sum_{j=1}^{+\infty}\frac{(\alpha j)\lambda^j}{\Gamma(j\alpha+1)}\left(\log\frac{t}{u}\right)^{\alpha j-1}\sigma(u)\frac{du}{u}\right.$$

$$\left.+\frac{\Gamma(\alpha)A_1}{t}\sum_{j=1}^{+\infty}\frac{(\alpha j)\lambda^j}{\Gamma(j\alpha+1)}\left(\log\frac{t}{t_1}\right)^{\alpha j-1}\right]=\lambda x(t)+\sigma(t).$$

So x is a solution of (3.5.1). The proof is completed. □

Define the nonlinear operator R on $LP_1C_{1-\alpha}(0,1]$ for $x\in LP_1C_{1-\alpha}(0,1]$ by $(Rx)(t)$ by

$$(Rx)(t)=\begin{cases}\frac{\Gamma(\alpha)\varrho_{\alpha,\lambda}(t,1)}{\Lambda_1}\left[\int_0^1\phi(s)G_x(s)ds-\Gamma(\alpha)\varrho_{\alpha,\lambda}(e,1)I_x(t_1)-\int_1^e\varrho_{\alpha,\lambda}(e,s)p(s)f_x(s)\frac{ds}{s}\right.\\ \left.+\Gamma(\alpha)\varrho_{\alpha,\lambda}(e,1)\int_1^{t_1}\varrho_{\alpha,\lambda}(t_1,s)p(s)f_x(s)\frac{ds}{s}\right]\\ \quad+\int_1^t\varrho_{\alpha,\lambda}(t,s)p(s)f_x(s)\frac{ds}{s},t\in(1,t_1],\\ \frac{\Gamma(\alpha)\varrho_{\alpha,\lambda}(t,1)}{\Lambda_1}\left[\int_0^1\phi(s)G_x(s)ds-\Gamma(\alpha)\varrho_{\alpha,\lambda}(e,1)I_x(t_1)-\int_1^e\varrho_{\alpha,\lambda}(e,s)p(s)f_x(s)\frac{ds}{s}\right.\\ \left.+\Gamma(\alpha)\varrho_{\alpha,\lambda}(e,1)\int_1^{t_1}\varrho_{\alpha,\lambda}(t_1,s)p(s)f_x(s)\frac{ds}{s}\right]\\ +\frac{\Gamma(\alpha)\varrho_{\alpha,\lambda}(t,t_1)}{\Lambda_1}\left[\Gamma(\alpha)\varrho_{\alpha,\lambda}(t_1,1)\int_0^1\phi(s)G_x(s)ds-(1-\Gamma(\alpha)\varrho_{\alpha,\lambda}(e,t_1))I_x(t_1)\right.\\ \left.-\Gamma(\alpha)\varrho_{\alpha,\lambda}(t_1,1)\int_1^e\varrho_{\alpha,\lambda}(e,s)p(s)f_x(s)\frac{ds}{s}+(1-\Gamma(\alpha)\varrho_{\alpha,\lambda}(e,t_1))\int_1^{t_1}\varrho_{\alpha,\lambda}(t_1,s)p(s)f_x(s)\frac{ds}{s}\right]\\ \quad+\int_1^t\varrho_{\alpha,\lambda}(t,s)p(s)f_x(s)\frac{ds}{s},t\in(1,e].\end{cases}$$

Lemma 3.6. *Suppose that (h), (i) and (j) hold, $\Lambda_1\neq 0$, and f,G are* **impulsive III-Carathéodory functions***, I* **a discrete III-Carathódory function***. Then $R:LP_1C_{1-\alpha}(1,e]\to LP_1C_{1-\alpha}(1,e]$ is well defined and is completely continuous.*

Proof. The proof is similar to that of the proof of Lemma 3.2 and is omitted. □

3.6 Preliminary for BVP(1.10)

In this section, we present some preliminary results that can be used in next sections for get solutions of BVP(1.10).

Lemma 3.7. *Suppose that $E_{\alpha,1}(\lambda)-1\neq 0$ and $\sigma:(0,1)\mapsto\mathbb{R}$ is continuous and satisfies that there exist numbers $k>-1$ and $l\leq 0$ with $l\in(\max\{-\alpha,-\alpha-k\},0]$ such that $|\sigma(t)|\leq(\log t)^k(1-\log t)^l$ for all $t\in(0,1)$. The x is a solutions of*

$$\begin{cases}^{CH}D_{1^+}^{\alpha}x(t)-\lambda x(t)=\sigma(t),t\in(t_i,t_{i+1}],i=0,1,\\ x(e)-\lim_{t\to1^+}x(t)=a_0,\;\;\lim_{t\to t_1^+}x(t)-x(t_1)=I_0\end{cases}\tag{3.6.1}$$

if and only if $x \in PC(1,e]$ *and*

$$x(t)=\begin{cases}\frac{E_{\alpha,1}(\lambda(\log t)^{\alpha})}{E_{\alpha,1}(\lambda)-1}\left[a_0-E_{\alpha,1}(\lambda(1-\log t_1)^{\alpha})I_0-\int_1^e\delta_{\alpha,\lambda}(e,s)\sigma(s)\frac{ds}{s}\right]\\ \quad+\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{s}\right)^{\alpha}\right)\sigma(s)\frac{ds}{s},t\in(0,t_1],\\ \frac{E_{\alpha,1}(\lambda(\log t)^{\alpha})}{E_{\alpha,1}(\lambda)-1}\left[a_0-E_{\alpha,1}(\lambda(1-\log t_1)^{\alpha})I_0-\int_1^e\delta_{\alpha,\lambda}(e,s)\sigma(s)\frac{ds}{s}\right]\\ +E_{\alpha,1}\left(\lambda\left(\log\frac{t}{t_1}\right)^{\alpha}\right)I_0+\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{s}\right)^{\alpha}\right)\sigma(s)\frac{ds}{s},t\in(t_1,e].\end{cases}\tag{3.6.2}$$

Proof. Let x be a solution of (3.6.1). We know by Theorem 3.13 that there exist numbers $A_0, A_1 \in \mathbb{R}$ such that

$$x(t)=A_0E_{\alpha,1}(\lambda(\log t)^{\alpha})+\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{s}\right)^{\alpha}\right)\sigma(s)\frac{ds}{s},t\in(t_0,t_1]\tag{3.6.3}$$

$$x(t)=A_0E_{\alpha,1}(\lambda(\log t)^{\alpha})+A_1E_{\alpha,1}(\lambda(\log t-\log t_1)^{\alpha})+\int_1^t\delta_{\alpha,\lambda}(t,s)\sigma(s)\frac{ds}{s},t\in(t_1,t_2].\tag{3.6.4}$$

Note $E_{\alpha,1}(0)=1$. It follows from (3.6.3), (3.6.4), the boundary conditions and the impulse assumption in (3.6.1) that

$$A_0E_{\alpha,1}(\lambda)+A_1E_{\alpha,1}(\lambda(1-\log t_1)^{\alpha})+\int_1^e\delta_{\alpha,\lambda}(e,s)\sigma(s)ds-A_0=a_0,\ \ A_1=I_0.$$

Then

$$A_0=\frac{1}{E_{\alpha,1}(\lambda)-1}\left[a_0-E_{\alpha,1}(\lambda(1-\log t_1)^{\alpha})I_0-\int_1^e\delta_{\alpha,\lambda}(e,s)\sigma(s)\frac{ds}{s}\right].\tag{3.6.5}$$

Substituting A_0, A_1 into (3.6.3) and (3.6.4), we get (3.6.2) obviously.

On the other hand, if x satisfies (3.6.2), then $x|_{(1,t_1]}$ and $x|_{(t_1,1]}$ are continuous and the limits $\lim_{t\to 1^+}x(t)$ and $\lim_{t\to t_1^+}x(t)$ exist. So $x\in P_1C(1,e]$. Using (3.6.5) and $A_1=I_0$, we rewrite x by

$$x(t)=\begin{cases}A_0E_{\alpha,1}(\lambda(\log t)^{\alpha})+\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{s}\right)^{\alpha}\right)\sigma(s)\frac{ds}{s},t\in(1,t_1],\\ A_0E_{\alpha,1}(\lambda(\log t)^{\alpha})+A_1E_{\alpha,1}(\lambda(\log t-\log t_1)^{\alpha})+\int_1^t\delta_{\alpha,\lambda}(t,s)\sigma(s)\frac{ds}{s},t\in(t_1,e].\end{cases}$$

One have from Theorem 3.2.4 easily for $t\in(t_0,t_1]$ that ${}^{CH}D_{1^+}^{\alpha}x(t)=\lambda x(t)+\sigma(t)$ and for $t\in(t_1,t_2]$ that

$$\begin{aligned}&{}^{CH}D_{1^+}^{\alpha}x(t)=\frac{1}{\Gamma(1-\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{-\alpha}x'(s)\frac{ds}{s}\\ &=\frac{1}{\Gamma(1-\alpha)}\int_1^{t_1}\left(\log\frac{t}{s}\right)^{-\alpha}\left(A_0E_{\alpha,1}(\lambda(\log s)^{\alpha})+\int_1^s\left(\log\frac{s}{u}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{s}{u}\right)^{\alpha}\right)\sigma(u)\frac{du}{u}\right)'\frac{ds}{s}\\ &+\frac{1}{\Gamma(1-\alpha)}\int_{t_1}^t\left(\log\frac{t}{s}\right)^{-\alpha}\left(A_0E_{\alpha,1}(\lambda(\log s)^{\alpha})+A_1E_{\alpha,1}\left(\lambda\left(\log\frac{s}{t_1}\right)^{\alpha}\right)+\int_1^s\delta_{\alpha,\lambda}(s,u)\sigma(u)\frac{du}{u}\right)'\frac{ds}{s}\\ &=\lambda x(t)+\sigma(t).\end{aligned}$$

So x is a solution of (3.6.1). The proof is completed. □

Define the nonlinear operator J on $LP_1C(1,e]$ by (Jx) by

$$(Jx)(t)=\begin{cases}\frac{E_{\alpha,1}(\lambda(\log t)^{\alpha})}{E_{\alpha,1}(\lambda)-1}\left[\int_0^1\phi(s)G_x(s)ds-E_{\alpha,1}(\lambda(1-\log t_1)^{\alpha})I_x(t_1)-\int_1^e\delta_{\alpha,\lambda}(e,s)p(s)f_x(s)\frac{ds}{s}\right]\\ \quad+\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{s}\right)^{\alpha}\right)p(s)f_x(s)\frac{ds}{s},t\in(0,t_1],\\ \frac{E_{\alpha,1}(\lambda(\log t)^{\alpha})}{E_{\alpha,1}(\lambda)-1}\left[\int_0^1\phi(s)G_x(s)ds-E_{\alpha,1}(\lambda(1-\log t_1)^{\alpha})I_x(t_1)-\int_1^e\delta_{\alpha,\lambda}(e,s)p(s)f_x(s)\frac{ds}{s}\right]\\ +E_{\alpha,1}\left(\lambda\left(\log\frac{t}{t_1}\right)^{\alpha}\right)I_x(t_1)+\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{s}\right)^{\alpha}\right)p(s)f_x(s)\frac{ds}{s},t\in(t_1,e].\end{cases}$$

Lemma 3.8. *Suppose that (k), (l) and (m) hold,* $E_{\alpha,1}(\lambda)-1\neq 0$, *and* f, G *are* **impulsive I-Carathéodory functions**, I *a* **discrete I-Carathódory function**. *Then* $R: LP_1C(1,e]\to LP_1C(1,e]$ *is well defined and is completely continuous.*

Proof. The proof is similar to that of the proof of Lemma3.2 and is omitted. □

4 Solvability of BVP(1.7)-BVP(1.10)

Now, we prove that main theorems in this paper by using the Schaefer's fixed point theorem [57].

(B1) there exists nonnegative a constant I_0, nondecreasing functions $b, B, \overline{B} : [0,+\infty) \mapsto \mathbb{R}$, bounded continuous functions $\phi, \psi : (0,1) \mapsto \mathbb{R}$ such that

$$|f(t,(t-t_i)^{\alpha-1}x) - \phi(t)| \le b(|x|), t \in (t_i, t_{i+1}], i = 0,1, x \in \mathbb{R},$$

$$|G(t,(t-t_i)^{\alpha-1}x) - \psi(t)| \le B(|x|), t \in (t_i, t_{i+1}], i = 0,1, x \in \mathbb{R},$$

$$|I(t_1, t_1^{\alpha-1}x) - I_0| \le \overline{B}(|x|), x \in \mathbb{R}.$$

Let

$$\Phi(t) = \begin{cases} \frac{\Gamma(\alpha)\delta_{\alpha,\lambda}(t,0)}{\Lambda}\Big[\int_0^1 \phi(s)\psi(s)ds - \Gamma(\alpha)\delta_{\alpha,\lambda}(1,t_1)I_0 \ - \int_0^1 \delta_{\alpha,\lambda}(1,s)p(s)\phi(s)ds \\ -\Gamma(\alpha)\delta_{\alpha,\lambda}(1,t_1)\int_0^{t_1}\delta_{\alpha,\lambda}(t_1,s)p(s)\phi(s)ds\Big] + \int_0^t \delta_{\alpha,\lambda}(t,s)p(s)\phi(s))ds, t \in (0,t_1], \\ \frac{\Gamma(\alpha)\delta_{\alpha,\lambda}(t,0)}{\Lambda}\Big[\int_0^1 \phi(s)\psi(s)ds - \Gamma(\alpha)\delta_{\alpha,\lambda}(1,t_1)I_0 \ - \int_0^1 \delta_{\alpha,\lambda}(1,s)p(s)\phi(s)ds \\ -\Gamma(\alpha)\delta_{\alpha,\lambda}(1,t_1)\int_0^{t_1}\delta_{\alpha,\lambda}(t_1,s)p(s)\phi(s)ds\Big] + \frac{\Gamma(\alpha)\delta_{\alpha,\lambda}(t,t_1)}{\Lambda}\Big[\Gamma(\alpha)\delta_{\alpha,\lambda}(t_1,0)\int_0^1 \phi(s)\psi(s)ds \\ +(\Gamma(\alpha)\delta_{\alpha,\lambda}(1,0)-1)I_0 \ -\Gamma(\alpha)\delta_{\alpha,\lambda}(t_1,0)\int_0^1 \delta_{\alpha,\lambda}(1,s)p(s)\phi(s)ds \\ +(\Gamma(\alpha)\delta_{\alpha,\lambda}(1,0)-1)\int_0^{t_1}\delta_{\alpha,\lambda}(t_1,s)p(s)\phi(s)ds\Big] + \int_0^t \delta_{\alpha,\lambda}(t,s)p(s)\phi(s)ds, t \in (t_1,1]. \end{cases}$$

Theorem 4.14. *Suppose that (a)-(e), (B1) hold, $\Lambda \neq 0$. Then BVP(1.7) has at least one solution if there exists a $r_0 > 0$ such that*

$$\frac{A_1B(r_0+||\Phi||)+A_2\overline{B}(r_0+||\Phi||)+A_3b(r_0+||\Phi||)}{r_0} < 1, \tag{4.1}$$

where

$$A_1 = \frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)||\phi||_1}{|\Lambda|} + \frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^2||\phi||_1}{|\Lambda|t_1^{1-\alpha}},$$

$$A_2 = \frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^2}{|\Lambda|(1-t_1)^{1-\alpha}} + \frac{\Gamma(\alpha)(\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)+1)E_{\alpha,\alpha}(|\lambda|)}{|\Lambda|},$$

$$A_3 = \Bigg(\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)^2}{|\Lambda|} + \frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^3(1-t_1)^{\alpha-1}t_1^{\alpha+k+l}}{|\Lambda|} + \frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^3t_1^{\alpha-1}}{|\Lambda|}$$

$$+\frac{\Gamma(\alpha)(\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)+1)E_{\alpha,\alpha}(|\lambda|)^2t_1^{\alpha+k+l}}{|\Lambda|} + E_{\alpha,\alpha}(|\lambda|)\Bigg)\mathbf{B}(\alpha+l,k+1).$$

Proof. From Lemma 3.1, Lemma 3.2, the definition of T, $x \in P_1C_{1-\alpha}(0,1]$ is a solution of BVP(1.7) if and only if $x \in P_1C_{1-\alpha}(0,1]$ is a fixed point of T in $P_1C_{1-\alpha}(0,1]$. Lemma 3.2 implies that T is a completely continuous operator.

For $r > 0$, denote $\Omega = \{x \in P_1C_{1-\alpha}(0,1] : ||x-\Phi|| \le r\}$. For $x \in \Omega$, we get $||x|| \le ||x-\Phi|| + ||\Phi|| \le r + ||\Phi||$ Then (B1) implies that

$$|f(t,x(t)) - \phi(t)| = |f(t,(t-t_i)^{\alpha-1}(t-t_i)^{1-\alpha}x(t)) - \phi(t)| \le b(|(t-t_i)^{1-\alpha}x(t)|)$$

$$\le b(||x||) \le b(r+||\Phi||), t \in (t_i, t_{i+1}], i = 0,1,$$

$$|G(t,x(t)) - \psi(t)| \le B(||x||) \le B(r+||\Phi||), t \in (t_i, t_{i+1}], i = 0,1,$$

$$|I(t_1,x(t_1)) - I_0| \le \overline{B}(||x||) \le \overline{B}(r+||\Phi||), t \in (0,1).$$

By the definition of T and the method used in **Step (iii1)** in the proof of Lemma 3.2, we have

$$||Tx|| \leq \left[\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)||\phi||_1}{|\Lambda|} + \frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^2||\phi||_1}{|\Lambda|t_1^{1-\alpha}}\right]B(r+||\Phi||)$$

$$+\left[\frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^2}{|\Lambda|(1-t_1)^{1-\alpha}} + \frac{\Gamma(\alpha)(\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)+1)E_{\alpha,\alpha}(|\lambda|)}{|\Lambda|}\right]\overline{B}(r+||\Phi||)$$

$$+\left(\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)^2}{|\Lambda|} + \frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^3(1-t_1)^{\alpha-1}t_1^{\alpha+k+l}}{|\Lambda|} + \frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^3t_1^{\alpha-1}}{|\Lambda|}\right.$$

$$\left.+\frac{\Gamma(\alpha)(\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)+1)E_{\alpha,\alpha}(|\lambda|)^2t_1^{\alpha+k+l}}{|\Lambda|} + E_{\alpha,\alpha}(|\lambda|)\right)\mathbf{B}(\alpha+l,k+1)b(r+||\Phi||)$$

$$= A_1B(r+||\Phi||) + A_2\overline{B}(r+||\Phi||) + A_3b(r+||\Phi||).$$

From (4.1), there exists a constant $r_0 > 0$ such that $A_1B(r_0+||\Phi||) + A_2\overline{B}(r_0+||\Phi||) + A_3b(r_0+||\Phi||) < r_0$. Choose $\Omega = \{x \in P_1C_{1-\alpha}(0,1] : ||x-\Phi|| \leq r_0\}$. For $x \in \partial\Omega$, we see easily that $x =\neq \lambda(Tx)$ for all $\lambda \in [0,1]$. In fact, if $x = Tx$ for some $x \in \partial\Omega$ and $\lambda \in [0,1]$, then $r_0 = ||x|| = \lambda||Tx|| \leq ||Tx|| \leq A_1B(r_0+||\Phi||) + A_2\overline{B}(r_0+||\Phi||) + A_3b(r_0+||\Phi||) < r_0$, a contradiction. So Lemma 3.9 implies that T has at least one fixed point in Ω. Then BVP(1.7) has at least one solution. The proof is complete. □

Theorem 4.15. *Suppose that $\Lambda \neq 0$, $\theta \geq 0$, $|\psi(t)| \leq B_\psi$, $|\phi(t)| \leq b_\phi$ for all $t \in (0,1)$ and $b(x) = b_1x^\theta$, $B(x) = B_1x^\theta$ and $\overline{B}(x) = \overline{B}_1x^\theta$ in (B1). Then BVP(1.7) has at least one solution if one of the following item holds:*

(i) $\theta \in [0,1)$;

(ii) $\theta = 1$ *with* $A_1B + A_2\overline{B}r + A_3b < 1$;

(iii) $\theta > 1$ *with* $(A_1B_1 + A_2\overline{B}_1 + A_3b_1)(A_1B_\psi + A_2|I_0| + A_3b_\phi) \leq \frac{(\theta-1)^{\theta-1}}{\theta^\theta}$.

Proof. It is easy to see that $\Phi \in P_1C_{1-\alpha}(0,1]$. By using the method in **Step (iii2)** in the proof of Lemma 3.2, we get $||\Phi|| \leq A_1B_\psi + A_2|I_0| + A_3b_\phi$. By Theorem 4.1, we know that BVP(1.0.7) has at least one solution if there exists $r_0 > 0$ such that (4.1) holds.

When $\theta \in [0,1)$, we have

$$\inf_{r\in(0,+\infty)} \frac{A_1B(r+||\Phi||)+A_2\overline{B}(r+||\Phi||)+A_3b(r+||\Phi||)}{r}$$

$$= \inf_{r\in(0,+\infty)} \frac{A_1B_1[r+||\Phi||]^\theta+A_2\overline{B}_1[r+||\Phi||]^\theta+A_3b_1[r+||\Phi||]^\theta}{r} = 0 < 1.$$

Then Theorem 4.1 implies that BVP(1.7) has at least one solution.

When $\theta = 1$, we have from $A_1B + A_2\overline{B}r + A_3b < 1$ that

$$\inf_{r\in(0,+\infty)} \frac{A_1B[r+||\Phi||]+A_2\overline{B}[r+||\Phi||]+A_3b[r+||\Phi||]}{r} < 1.$$

Then there $r_0 > 0$ such that (4.1) holds. Then Theorem 4.1 implies that BVP(1.7) has at least one solution.

When $\theta > 1$, it is easy to see that $(A_1B_1 + A_2\overline{B}_1 + A_3b_1)(A_1B_\psi + A_2|I_0| + A_3b_\phi) \leq \frac{(\theta-1)^{\theta-1}}{\theta^\theta}$ implies that $(A_1B_1 + A_2\overline{B}_1 + A_3b_1)||\Phi|| \leq \frac{(\theta-1)^{\theta-1}}{\theta^\theta}$. Choose $r_0 = \frac{||\Phi||}{\theta-1}$. It is easy to check that

$$\frac{A_1B_1(r_0+||\Phi||)^\theta+A_2\overline{B}_1(r_0+||\Phi||)^\theta+A_3b_1(r_0+||\Phi||)^\theta}{r_0} \leq 1.$$

Then Theorem 4.1 implies that BVP(1.7) has at least one solution. The proof of Theorem 4.2 is complete. □

Remark 4.1. **(i)** *When $\alpha \in (0,1)$, $G(t,x) \equiv 0$, and replace $I(t_1,x)$ in (1.0.7) by $I(x) - x$, we see that BVP(1.7) becomes BVP(1.3). According Theorem 4.2, BVP(1.3) has at leat one solution if both f and I are bounded.*

(ii) *When $\alpha \in (0,1)$, one chooses $G(t,x) \equiv 0$, $f(t,x) = 1 + t^2 + (t-t_i)^{1-\alpha}x$ for $t \in (t_i,t_{i+1}](i = 0,1)$, $0 = t_0 < t_1 = \frac{1}{2} < t_2 = 1$, and $I(t_1,x) \equiv 0$, then BVP(1.7) becomes BVP(1.3) with $I(x) = x$. According Theorem 4.2, BVP(1.3) has at least one solution. But The results in [69] can not be applied.*

Theorem 4.16. *Suppose that (b), (c), (f), (g) hold, $E_{\alpha,1}(\lambda)-1\neq 0$, and*

(B2) *there exist nondecreasing functions $b, B, \overline{B}:[0,+\infty)\mapsto \mathbb{R}$ such that*

$$|f(t,x)|\leq b(|x|), t\in(0,1), x\in\mathbb{R},$$

$$|G(t,x)|\leq B(|x|), t\in(0,1), x\in\mathbb{R},$$

$$|I(t_1,x)|\leq \overline{B}(|x|), x\in\mathbb{R}.$$

Then BVP(1.8) has at least one solution if there exists $r_0>0$ such that

$$\begin{aligned}&\tfrac{E_{\alpha,1}(|\lambda|)||\phi||_1}{|E_{\alpha,1}(\lambda)-1|}B(r_0)+\left(\tfrac{E_{\alpha,1}(|\lambda|)E_{\alpha,1}(|\lambda|)}{|E_{\alpha,1}(\lambda)-1|}+E_{\alpha,1}(|\lambda|)\right)\overline{B}(r_0)\\ &+\left(\tfrac{E_{\alpha,1}(|\lambda|)E_{\alpha,\alpha}(|\lambda|)}{|E_{\alpha,1}(\lambda)-1|}+E_{\alpha,\alpha}(|\lambda|)\right)\mathbf{B}(\alpha+l,k+1)b(r_0)<r_0.\end{aligned}\tag{4.2}$$

Proof. From Lemma 3.3, Lemma 3.4 and the definition of Q, $x\in P_1C(0,1]$ is a solution of BVP(1.8) if and only if $x\in P_1C(0,1]$ is a fixed point of Q. Lemma 3.4 implies that Q is a completely continuous operator. From (B2), we have for $x\in P_1C(0,1]$ that

$$|f(t,x(t))|\leq b(|x(t)|)\leq b(||x||), t\in(0,1),$$

$$|G(t,x(t))|\leq B(||x||), t\in(0,1),$$

$$|I(t_1,x(t_1))|\leq \overline{B}(||x||).$$

We consider the set $\Omega=\{x\in P_1C(0,1]: x=\lambda(Tx),\ \text{for some}\ \lambda\in[0,1]\}$. For $x\in\Omega$, we have for $t\in(t_0,t_1]$ that

$$\begin{aligned}|(Qx)(t)|\leq\ &\tfrac{E_{\alpha,1}(|\lambda|)}{|E_{\alpha,1}(\lambda)-1|}\Big[||\phi||_1B(||x||)+E_{\alpha,1}(|\lambda|)\overline{B}(||x||)\\ &+\sum_{v=0}^{+\infty}\tfrac{\lambda^v}{\Gamma(\alpha(v+1))}\int_0^1(1-s)^{\alpha v+\alpha-1}s^k(1-s)^l ds b(||x||)\Big]\\ &+\sum_{v=0}^{+\infty}\tfrac{\lambda^v}{\Gamma(\alpha(v+1))}\int_0^t(t-s)^{\alpha v+\alpha-1}s^k(1-s)^l ds b(||x||)\\ \leq\ &\tfrac{E_{\alpha,1}(|\lambda|)}{|E_{\alpha,1}(\lambda)-1|}\left[||\phi||_1B(||x||)+E_{\alpha,1}(|\lambda|)\overline{B}(||x||)\right.\\ &\left.+E_{\alpha,\alpha}(|\lambda|)\mathbf{B}(\alpha+l,k+1)b(||x||)\right]+E_{\alpha,\alpha}(|\lambda|)\mathbf{B}(\alpha+l,k+1)b(||x||)\\ =\ &\tfrac{E_{\alpha,1}(|\lambda|)||\phi||_1}{|E_{\alpha,1}(\lambda)-1|}B(||x||)+\tfrac{E_{\alpha,1}(|\lambda|)E_{\alpha,1}(|\lambda|)}{|E_{\alpha,1}(\lambda)-1|}\overline{B}(||x||)\\ &+\left(\tfrac{E_{\alpha,1}(|\lambda|)}{|E_{\alpha,1}(\lambda)-1|}E_{\alpha,\alpha}(|\lambda|)+E_{\alpha,\alpha}(|\lambda|)\right)\mathbf{B}(\alpha+l,k+1)b(||x||).\end{aligned}$$

For $t\in(t_1,t_2]$, one has that

$$\begin{aligned}|(Qx)(t)|\leq\ &\tfrac{E_{\alpha,1}(|\lambda|)||\phi||_1}{|E_{\alpha,1}(\lambda)-1|}B(||x||)+\left(\tfrac{E_{\alpha,1}(|\lambda|)E_{\alpha,1}(|\lambda|)}{|E_{\alpha,1}(\lambda)-1|}+E_{\alpha,1}(|\lambda|)\right)\overline{B}(||x||)\\ &+\left(\tfrac{E_{\alpha,1}(|\lambda|)E_{\alpha,\alpha}(|\lambda|)}{|E_{\alpha,1}(\lambda)-1|}+E_{\alpha,\alpha}(|\lambda|)\right)\mathbf{B}(\alpha+l,k+1)b(||x||).\end{aligned}$$

It follows that

$$\begin{aligned}||x||=\lambda||Tx||\leq||Tx||\leq\ &\tfrac{E_{\alpha,1}(|\lambda|)||\phi||_1}{|E_{\alpha,1}(\lambda)-1|}B(||x||)+\left(\tfrac{E_{\alpha,1}(|\lambda|)E_{\alpha,1}(|\lambda|)}{|E_{\alpha,1}(\lambda)-1|}+E_{\alpha,1}(|\lambda|)\right)\overline{B}(||x||)\\ &+\left(\tfrac{E_{\alpha,1}(|\lambda|)E_{\alpha,\alpha}(|\lambda|)}{|E_{\alpha,1}(\lambda)-1|}+E_{\alpha,\alpha}(|\lambda|)\right)\mathbf{B}(\alpha+l,k+1)b(||x||).\end{aligned}$$

From (4.2), we choose $\Omega = \{x \in P_1C(),1] : ||x|| \le r_0\}$. For $x \in \Omega$, we get $x \neq \lambda(Tx)$ for any $\lambda \in [0,1]$ and $x \in \partial\Omega$.

As a consequence of Schaefer's fixed point theorem, we deduce that Q has a fixed point which is a solution of the problem BVP(1.8). The proof is completed. The proof of Theorem 4.3 is complete. □

Theorem 4.17. *Suppose that (h), (i), (j) hold, $\Lambda_1 \neq 0$, and*

(B3) *there exist nondecreasing functions $b, B, \overline{B} : [0,+\infty) \mapsto \mathbb{R}$ such that*

$$|f(t,t^{\alpha-1}x)| \le b(|x|), t \in (t_i,t_{i+1}], i = 0,1, x \in \mathbb{R},$$

$$|G(t,t^{\alpha-1}x)| \le B(|x|), t \in (t_i,t_{i1}], i = 0,1, x \in \mathbb{R},$$

$$|I(t_1,t_1^{\alpha-1}x)| \le \overline{B}(|x|), x \in \mathbb{R}.$$

Then BVP(1.9) has at least one solution if there exists a constant $r_0 > 0$ such that

$$B_1B(r_0) + B_2\overline{B}(r_0) + B_3b(r_0) < r_0, \tag{4.3}$$

where

$$B_1 = \frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)||\phi||_1}{|\Lambda_1|} + \frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^2||\phi||_1(\log t_1)^{\alpha-1}}{|\Lambda_1|},$$

$$B_2 = \frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^2}{|\Lambda_1|} + \frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)\left|1-\Gamma(\alpha)(1-\log t_1)^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)\right|}{|\Lambda_1|},$$

$$B_3 = \left(\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)^2}{|\Lambda_1|} + \frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^3(\log t_1)^{\alpha+k+l}}{|\Lambda_1|} + \frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^3(\log t_1)^{\alpha-1}}{|\Lambda_1|}\right.$$
$$\left.+\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)^2\left|1-\Gamma(\alpha)(1-\log t_1)^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)\right|(\log t_1)^{\alpha+k+l}}{|\Lambda_1|} + E_{\alpha,\alpha}(|\lambda|)\right)\mathbf{B}(\alpha+l,k+1).$$

Proof. From Lemma 3.5, Lemma 3.6 and the definition of R, $x \in LP_1C_{1-\alpha}(1,e]$ is a solution of BVP(1.9) if and only if $x \in LP_1C_{1-\alpha}(1,e]$ is a fixed point of R. Lemma 3.6 implies that R is a completely continuous operator.

From (B3), we have for $x \in LP_1C_{1-\alpha}(1,e]$ that

$$|f(t,x(t))| = \left|f\left(t,\left(\log\tfrac{t}{s}\right)^{\alpha-1}\left(\log\tfrac{t}{s}\right)^{1-\alpha}x(t)\right)\right| \le b\left(\left|\left(\log\tfrac{t}{s}\right)^{-\alpha}x(t)\right|\right) \le b(||x||), t \in (t_i,t_{i+1}], i = 0,1,$$

$$|G(t,x(t))| \le B(||x||), t \in (1,e),$$

$$|I(t_1,x(t_1))| \le \overline{B}(||x||).$$

We consider the set $\Omega = \{x \in LP_1C_{1-\alpha}(0,1] : x = \lambda(Rx)$, for some $\lambda \in [0,1]\}$. For $x \in \Omega$, we have for $t \in (t_0,t_1]$ that

$$\left|(\log t)^{1-\alpha}(Rx)(t)\right| \le \frac{\Gamma(\alpha)E_{\alpha,\alpha}\left(\lambda(\log t)^{\alpha}\right)}{|\Lambda_1|}\left[||\phi||_1B(||x||) + \Gamma(\alpha)E_{\alpha,\alpha}(\lambda)\overline{B}(||x||)\right.$$
$$+\int_1^e\left(\log\tfrac{e}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\tfrac{e}{s}\right)^{\alpha}\right)(\log s)^k(1-\log s)^l\tfrac{ds}{s}b(||x||)$$
$$\left.+\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)\int_1^{t_1}\left(\log\frac{t_1}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t_1}{s}\right)^{\alpha}\right)(\log s)^k(1-\log s)^l\tfrac{ds}{s}b(||x||)\right]$$
$$+(\log t)^{1-\alpha}\int_1^t\left(\log\tfrac{t}{s}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\log\tfrac{t}{s}\right)^{\alpha}\right)(\log s)^k(1-\log s)^l\tfrac{ds}{s}b(||x||)$$

$$\leq \frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{|\Lambda_1|}\Big[||\phi||_1 B(||x||)+\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)\overline{B}(||x||)$$

$$+\sum_{v=0}^{+\infty}\frac{\lambda^v}{\Gamma(\alpha(v+1))}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha v+\alpha+l-1}(\log s)^k\frac{ds}{s}b(||x||)$$

$$+\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)\sum_{v=0}^{+\infty}\frac{\lambda^v}{\Gamma(\alpha(v+1))}\int_1^{t_1}\left(\log\frac{t_1}{s}\right)^{\alpha v+\alpha+l-1}(\log s)^k\frac{ds}{s}b(||x||)\Big]$$

$$+(\log t)^{1-\alpha}\sum_{v=0}^{+\infty}\frac{\lambda^v}{\Gamma(\alpha(v+1))}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha v+\alpha+l-1}(\log s)^k\frac{ds}{s}b(||x||)$$

$$\leq \frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{|\Lambda_1|}\Big[||\phi||_1 B(||x||)+\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)\overline{B}(||x||)$$

$$+\sum_{v=0}^{+\infty}\frac{\lambda^v}{\Gamma(\alpha(v+1))}\int_0^1(1-w)^{\alpha v+\alpha+l-1}w^k dw b(||x||)$$

$$+\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)\sum_{v=0}^{+\infty}\frac{\lambda^v}{\Gamma(\alpha(v+1))}(\log t_1)^{\alpha v+\alpha+k+l}\int_0^1(1-w)^{\alpha v+\alpha+l-1}w^k dw b(||x||)\Big]$$

$$+(\log t)^{1-\alpha}\sum_{v=0}^{+\infty}\frac{\lambda^v}{\Gamma(\alpha(v+1))}(\log t)^{\alpha v+\alpha+k+l}\int_0^1(1-w)^{\alpha v+\alpha+l-1}w^k dw b(||x||)$$

$$\leq \frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)||\phi||_1}{|\Lambda_1|}B(||x||)+\frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^2}{|\Lambda_1|}\overline{B}(||x||)$$

$$+\left(\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)^2}{|\Lambda_1|}+\frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^3(\log t_1)^{\alpha+k+l}}{|\Lambda_1|}+E_{\alpha,\alpha}(|\lambda|)\right)\mathbf{B}(\alpha+l,k+1)b(||x||).$$

For $t\in(t_1,t_2]$, one has that

$$\left|\left(\log\frac{t}{t_1}\right)^{1-\alpha}(Rx)(t)\right|\leq \frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{|\Lambda_1|}\Big[||\phi||_1B(||x||)+\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)\overline{B}(||x||)$$

$$+E_{\alpha,\alpha}(|\lambda|)\mathbf{B}(\alpha+l,k+1)b(||x||)+\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)^2(\log t_1)^{\alpha+k+l}\mathbf{B}(\alpha+l,k+1)\Big]$$

$$+\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{|\Lambda_1|}\Big[\Gamma(\alpha)(\log t_1)^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)||\phi||_1B(||x||)+\left|1-\Gamma(\alpha)(1-\log t_1)^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)\right|\overline{B}(||x||)$$

$$+\Gamma(\alpha)(\log t_1)^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)^2\mathbf{B}(\alpha+l,k+1)b(||x||)$$

$$+\left|1-\Gamma(\alpha)(1-\log t_1)^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)\right|(\log t_1)^{\alpha+k+l}E_{\alpha,\alpha}(|\lambda|)\mathbf{B}(\alpha+l,k+1)b(||x||)\Big]$$

$$+\left(\log\frac{t}{t_1}\right)^{1-\alpha}(\log t)^{\alpha+k+l}E_{\alpha,\alpha}(|\lambda|)\mathbf{B}(\alpha+l,k+1)b(||x||)$$

$$\leq\left(\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)||\phi||_1}{|\Lambda_1|}+\frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^2||\phi||_1(\log t_1)^{\alpha-1}}{|\Lambda_1|}\right)B(||x||)$$

$$+\left(\frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^2}{|\Lambda_1|}+\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)\left|1-\Gamma(\alpha)(1-\log t_1)^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)\right|}{|\Lambda_1|}\right)\overline{B}(||x||)$$

$$+\Bigg(\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)^2}{|\Lambda_1|}+\frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^3(\log t_1)^{\alpha+k+l}}{|\Lambda_1|}+\frac{\Gamma(\alpha)^2E_{\alpha,\alpha}(|\lambda|)^3(\log t_1)^{\alpha-1}}{|\Lambda_1|}$$

$$+\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)^2\left|1-\Gamma(\alpha)(1-\log t_1)^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)\right|(\log t_1)^{\alpha+k+l}}{|\Lambda_1|}+E_{\alpha,\alpha}(|\lambda|)\Bigg)\mathbf{B}(\alpha+l,k+1)b(||x||)$$

$$=B_1B(||x||)+B_2\overline{B}(||x||)+B_3b(||x||).$$

It follows that

$$||x|| = \lambda||Rx|| \le ||Rx|| \le B_1B(||x||) + B_2\overline{B}(||x||) + B_3b(||x||).$$

From (4.3), we choose $\Omega = \{x \in LP_1C_{1-\alpha}(0,1] : ||x|| \le r_0\}$. For $x \in \partial\Omega$, we get $x \ne \lambda(Rx)$ for any $\lambda \in [0,1]$. In fact, if there exists $x \in \partial\Omega$ such that $x = \lambda(Rx)$ for some $\lambda \in [0,1]$. Then $r_0 = ||x|| = \lambda||Rx|| \le ||Rx|| \le B_1B(r_0) + B_2\overline{B}(r_0) + B_3b(r_0) < r_0$, a contradiction.

As a consequence of Schaefer's fixed point theorem, we deduce that R has a fixed point which is a solution of the problem BVP(1.9). The proof is completed. The proof of Theorem 4.4 is complete. □

Theorem 4.18. *Suppose that (k), (l), (m) hold,* $E_{\alpha,1}(\lambda) - 1 \ne 0$, *and*

(B4) *there exist nondecreasing functions* $b, B, \overline{B} : [0,+\infty) \mapsto \mathbb{R}$ *such that*

$$|f(t,x)| \le b(|x|), t \in (t_i, t_{i+1}], i = 0,1, x \in \mathbb{R},$$

$$|G(t,x)| \le B(|x|), t \in (t_i, t_{i1}], i = 0,1, x \in \mathbb{R},$$

$$|I(t_1,x)| \le \overline{B}(|x|), x \in \mathbb{R}.$$

Then BVP(1.10) has at least one solution if there exists a constant $r_0 > 0$ *such that*

$$\begin{aligned}&\frac{E_{\alpha,1}(|\lambda|)||\phi||_1}{|E_{\alpha,1}(\lambda)-1|}B(r_0) + \left(\frac{E_{\alpha,1}(|\lambda|)^2}{|E_{\alpha,1}(\lambda)-1|} + E_{\alpha,1}(|\lambda|)\right)\overline{B}(r_0)\\ &+\left(\frac{E_{\alpha,1}(|\lambda|)^2}{|E_{\alpha,1}(\lambda)-1|} + E_{\alpha,\alpha}(|\lambda|)\right)\mathbf{B}(\alpha+l,k+1)b(r_0) < r_0.\end{aligned} \tag{4.4}$$

Proof. From Lemma 3.7, Lemma 3.8 and the definition of J, $x \in LP_1C(1,e]$ is a solution of BVP(1.10) if and only if $x \in LP_1C(1,e]$ is a fixed point of R. Lemma 3.8 implies that J is a completely continuous operator.

From (B4), we have for $x \in LP_1C(1,e]$ that

$$|f(t,x(t))| \le b(|x(t)|) \le b(||x||), t \in (t_i, t_{i+1}], i = 0,1,$$

$$|G(t,x(t))| \le B(||x||), t \in (1,e),$$

$$|I(t_1,x(t_1))| \le \overline{B}(||x||).$$

We consider the set $\Omega = \{x \in LP_1C(0,1] : x = \lambda(Jx),$ for some $\lambda \in [0,1]\}$. For $x \in \Omega$, we have for $t \in (t_0,t_1]$ that

$$\begin{aligned}|(Jx)(t)| \le& \frac{E_{\alpha,1}(|\lambda|)}{|E_{\alpha,1}(\lambda)-1|}\Big[||\phi||_1B(||x||) + E_{\alpha,1}(|\lambda|)\overline{B}(||x||)\\ &+\sum_{v=0}^{+\infty}\frac{\lambda^v}{\Gamma(\alpha(v+1))}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha v+\alpha+l-1}(\log s)^k\frac{ds}{s}b(||x||)\Big]\\ &+\sum_{v=0}^{+\infty}\frac{\lambda^v}{\Gamma(\alpha(v+1))}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha v+\alpha+l-1}(\log s)^k\frac{ds}{s}b(||x||)\\ \le& \frac{E_{\alpha,1}(|\lambda|)||\phi||_1}{|E_{\alpha,1}(\lambda)-1|}B(||x||) + \frac{E_{\alpha,1}(|\lambda|)^2}{|E_{\alpha,1}(\lambda)-1|}\overline{B}(||x||)\\ &+\left(\frac{E_{\alpha,1}(|\lambda|)^2}{|E_{\alpha,1}(\lambda)-1|} + E_{\alpha,\alpha}(|\lambda|)\right)\mathbf{B}(\alpha+l,k+1)b(||x||)\end{aligned}$$

For $t \in (t_1,t_2]$, one has that

$$\begin{aligned}|(Jx)(t)| \le& \frac{E_{\alpha,1}(|\lambda|)||\phi||_1}{|E_{\alpha,1}(\lambda)-1|}B(||x||) + \left(\frac{E_{\alpha,1}(|\lambda|)^2}{|E_{\alpha,1}(\lambda)-1|} + E_{\alpha,1}(|\lambda|)\right)\overline{B}(||x||)\\ &+\left(\frac{E_{\alpha,1}(|\lambda|)^2}{|E_{\alpha,1}(\lambda)-1|} + E_{\alpha,\alpha}(|\lambda|)\right)\mathbf{B}(\alpha+l,k+1)b(||x||).\end{aligned}$$

It follows that

$$||x|| = \lambda||Rx|| \leq ||Rx|| \leq \frac{E_{\alpha,1}(|\lambda|)||\phi||_1}{|E_{\alpha,1}(\lambda)-1|}B(||x||) + \left(\frac{E_{\alpha,1}(|\lambda|)^2}{|E_{\alpha,1}(\lambda)-1|} + E_{\alpha,1}(|\lambda|)\right)\overline{B}(||x||)$$

$$+ \left(\frac{E_{\alpha,1}(|\lambda|)^2}{|E_{\alpha,1}(\lambda)-1|} + E_{\alpha,\alpha}(|\lambda|)\right)\mathbf{B}(\alpha+l,k+1)b(||x||).$$

From (4.4), we choose $\Omega = \{x \in LP_1C(0,1] : ||x|| \leq r_0\}$. For $x \in \partial\Omega$, we get $x \neq \lambda(Jx)$ for any $\lambda \in [0,1]$. In fact, if there exists $x \in \partial\Omega$ such that $x = \lambda(Jx)$ for some $\lambda \in [0,1]$. Then

$$r_0 = ||x|| = \lambda||Jx|| \leq ||Jx|| \leq \frac{E_{\alpha,1}(|\lambda|)||\phi||_1}{|E_{\alpha,1}(\lambda)-1|}B(r_0) + \left(\frac{E_{\alpha,1}(|\lambda|)^2}{|E_{\alpha,1}(\lambda)-1|} + E_{\alpha,1}(|\lambda|)\right)\overline{B}(r_0)$$

$$+ \left(\frac{E_{\alpha,1}(|\lambda|)^2}{|E_{\alpha,1}(\lambda)-1|} + E_{\alpha,\alpha}(|\lambda|)\right)\mathbf{B}(\alpha+l,k+1)b(r_0) < r_0,$$

a contradiction.

As a consequence of Schaefer's fixed point theorem, we deduce that R has a fixed point which is a solution of the problem BVP(1.10). The proof is completed. The proof of Theorem 4.5 is complete. □

5 Applications

In [33, 78, 89], authors studied the existence and uniqueness of solutions of BVP(1.1). It was proved in [33] that if f is a jointly continuous function and there is a constant $\overline{\lambda} \in \left[0, 1-\frac{1}{p}\right]$ for some $p \in \left(1, \frac{1}{1-q}\right]$ and $L > 0$ such that $|f(t,x)| \leq L(1+|x|^{\overline{\lambda}}]$ for each $t \in [0,T]$ and $x \in R$. Then BVP(1.1) has at least one solution. It seems that the solvability of BVP(1.1) is not related to the impulse function I_k. For periodic boundary value problem, this is not true. For example, the following problem

$${}^{C}D_{0^+}^{\frac{1}{2}}x(t) = 1, t \in (t_i, t_{i+1}], i \in N_0,\ x(0) = x(T), \Delta x(t_i) = I_i > 0, i \in \mathbb{N}_0,$$

has no solution. In fact, from Theorem 3.10 and ${}^{C}D_{0^+}^{\frac{1}{2}}x(t) = 1, t \in (t_i, t_{i+1}]$, we get that there exist constants $c_0(i \in \mathbb{N}_0)$ such that $x(t) = \sum_{j=0}^{i} c_j + \frac{2t^{1/2}}{\Gamma(1/2)}$. By $x(0) = x(T)$, we know $c_0 = \sum_{j=0}^{m} c_j + \frac{2T^{1/2}}{\Gamma(1/2)}$. From $\Delta x(t_i) = I_i$, we see that $c_i = I_i$. Thus $\sum_{j=1}^{m} I_j + \frac{2T^{1/2}}{\Gamma(1/2)} = 0$, a contradiction.

Consider the following periodic boundary value problem of fractional differential equation

$$\begin{cases} {}^{C}D_{0^+}^{\alpha}x(t) = p(t)f(t,x(t)), t \in (t_i, t_{i+1}], i = 0, 1, \\ \Delta x(t_1) = b_1 x(t_1) + I(t_1, x(t_1)), \\ x(0) = x(T) + \int_0^T \phi(s)G(s,x(s))ds, \end{cases} \tag{5.1}$$

where $t_0 = 0 < t_1 < t_2 = T$, $b_1 \in R$ with $b_1 \neq 0$, $\phi : (0,T) \mapsto \mathbb{R}$ with $\phi \in L^1(0,T)$, $p : (0,T) \mapsto \mathbb{R}$ is continuous and there exist numbers $k > -1, l \in (-\alpha, -\alpha-k, 0]$ such that $|p(t) \leq t^k(T-t)^l$ for all $t \in (0,T)$, $f : (0,T] \times \mathbb{R} \mapsto \mathbb{R}$ is a II-Carathéodory function, $I : \{t_1,\}, \times\mathbb{R} \mapsto \mathbb{R}$ is a **discrete II-Carathéodory function**.

If x is a solution of (5.1), then by Theorem 3.1, we see that there exist constants c_0, c_1 such that

$$x(t) = \sum_{j=0}^{i} c_i + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f(s,x(s))ds, t \in (t_i, t_{i+1}], i = 0, 1.$$

By $x(0) = x(T) + \int_0^T \phi(s)G(s,x(s))ds$, we get $c_0 = c_0 + c_1 + \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f(s,x(s))ds + \int_0^T \phi(s)G(s,x(s))ds$. So $c_1 = -\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f(s,x(s))ds - \int_0^T \phi(s)G(s,x(s))ds$. By $\Delta x(t_1) = b_1x(t_1) + I_1(t_1,x(t_1))$, we get $c_1 = b_1\left(c_0 + \int_0^{t_1} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f(s,x(s))ds\right) + I(t_1,x(t_1))$. The $c_0 = -\int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f(s,x(s))ds - \frac{1}{b_1}I(t_1,x(t_1)) -$

$\frac{1}{b_1}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f(s,x(s))ds - \frac{1}{b_1}\int_0^T \phi(s)G(s,x(s))ds$. So

$$x(t)=\begin{cases} -\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f(s,x(s))ds-\frac{1}{b_1}I(t_1,x(t_1))-\frac{1}{b_1}\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f(s,x(s))ds\\ -\frac{1}{b_1}\int_0^T\phi(s)G(s,x(s))ds+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f(s,x(s))ds, t\in(0,t_1],\\ -\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f(s,x(s))ds-\frac{1}{b_1}I(t_1,x(t_1))-\left(1+\frac{1}{b_1}\right)\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f(s,x(s))ds\\ -\left(1+\frac{1}{b_1}\right)\int_0^T\phi(s)G(s,x(s))ds+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f(s,x(s))ds, t\in(t_1,T].\end{cases}$$

It is easy to show that if x satisfies above integral equation, then x is a solution of (5.1).

Define the operator $T: P_1C(0,T]\mapsto P_1C(0,T]$ by

$$(Tx)(t)=\begin{cases} -\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f(s,x(s))ds-\frac{1}{b_1}I(t_1,x(t_1))-\frac{1}{b_1}\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f(s,x(s))ds\\ -\frac{1}{b_1}\int_0^T\phi(s)G(s,x(s))ds+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f(s,x(s))ds, t\in(0,t_1],\\ -\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f(s,x(s))ds-\frac{1}{b_1}I(t_1,x(t_1))-\left(1+\frac{1}{b_1}\right)\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f(s,x(s))ds\\ -\left(1+\frac{1}{b_1}\right)\int_0^T\phi(s)G(s,x(s))ds+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)f(s,x(s))ds, t\in(t_1,T].\end{cases}$$

Theorem 5.19. *$T: P_1C(0,T]\mapsto P_1C(0,T]$ is well defined and is completely continuous. x is a solution of mentioned problem (5.1) if and only if x is a fixed point of T in $P_1C(0,T]$.*

Proof. It follows from Theorem 3.10 and the details are omitted. □

Theorem 5.20. *Suppose that there exist nondecreasing functions $b, B, \overline{B}: [0,+\infty)\mapsto \mathbb{R}$ such that*

$$|f(t,x)|\le b(|x|), t\in(0,1), x\in R,$$

$$|G(t,x)|\le B(|x|), t\in(0,1), x\in R,$$

$$|I(t_1,x)|\le \overline{B}(|x|), x\in R.$$

Then problem (5.1) has at least one solution if there exists $r_0>0$ such that

$$\begin{aligned}&\left(1+\frac{||\phi||_1}{|b_1|}\right)B(r_0)+\frac{1}{|b_1|}\overline{B}(r_0)\\ &+\left(\frac{t_1^{\alpha+k+l}}{\Gamma(\alpha)}+\left(1+\frac{1}{|b_1|}\right)\frac{T^{\alpha+k+l}}{\Gamma(\alpha)}+\frac{T^{\alpha+k+l}}{\Gamma(\alpha)}\right)\mathbf{B}(\alpha+l,k+1)b(r_0)<r_0.\end{aligned} \tag{5.2}$$

Proof. In fact, for $x\in P_1C(0,T]$, we have $|f(t,x(t))|\le b(||x||), |G(t,x(t))|\le B(|x||)$ and $|I(t_1,x(t_1))|\le \overline{B}(||x||)$ for all $t\in(0,T]$. Then

$$\begin{aligned}||Tx||\le&\left(1+\frac{||\phi||_1}{|b_1|}\right)B(||x||)+\frac{1}{|b_1|}\overline{B}(||x||)\\ &+\left(\frac{t_1^{\alpha+k+l}}{\Gamma(\alpha)}+\left(1+\frac{1}{|b_1|}\right)\frac{T^{\alpha+k+l}}{\Gamma(\alpha)}+\frac{T^{\alpha+k+l}}{\Gamma(\alpha)}\right)\mathbf{B}(\alpha+l,k+1)b(||x||).\end{aligned}$$

The remainder of the proof is similar to the proof of Theorem 4.3 and is omitted. □

Example 5.1. *Consider the following problem*

$$\begin{cases} {}^{RL}D_{0^+}^{\alpha}x(t)-\lambda x(t)=t^{-\frac{1}{4}}(1-t)^{-\frac{1}{8}}(t^2+\arctan x(t)), t\in(t_i,t_{i+1}], i=0,1,\\ x(1)=\lim_{t\to0^+}t^{1-\alpha}x(t),\\ \lim_{t\to t_1^+}(t-t_1)^{1-\alpha}x(t)-x(t_1)=0,\end{cases} \tag{5.3}$$

where $\alpha \in (0,1)$, $0 = t_0 < \frac{1}{2} = t_1 < t_2 = 1$, $\lambda \in R$ *are fixed constant. Corresponding to BVP(1.7), we have* $\phi(t) \equiv 0$, $G(t,x) \equiv 0$ *and* $I(t_1,x) \equiv 0$.

It is easy to see that (a), (b) and (c) hold. $p(t) = t^{-\frac{1}{4}}(1-t)^{-\frac{1}{8}}$ satisfies (d) with $k = -\frac{1}{4}, l = -\frac{1}{8}$. $f(t,x) = t^2 + \arctan x$. One sees that f, G, I satisfy (e). Choose $\phi(t) = t^2$, $\psi(t) = 0$ and $I_0 = 0$. Then (B1) holds with $B(x) = \overline{B}(x) = 0$ and $b(x) = \frac{\pi}{2}$. Thus by Theorem 4.1, we know BVP(5.3) has at least one solution since there exists a constant $r_0 > 0$ such that (4.1) holds obviously.

According to the results in [69], BVP(5.3) can not be solved since the nonlinearity $p(t)f(t,x) = t^{-\frac{1}{4}}(1-t)^{-\frac{1}{8}}(t^2 + \arctan x)$ is unbounded and the impulse function $I(x) = x$ is also unbounded.

Example 5.2. *Consider the following BVP*

$$\begin{cases} {}^{RL}D_{0^+}^{\alpha}x(t) - \lambda x(t) = t^{-\frac{1}{4}}(1-t)^{-\frac{1}{8}}(t^2 + \sqrt[3]{(t-t_i)^{1-\alpha}x(t)}), t \in (t_i, t_{i+1}], i = 0, 1, \\ x(1) - \lim\limits_{t \to 0^+} t^{1-\alpha}x(t) = \int_0^1 s^{-\frac{1}{2}}(1-s)^{-\frac{1}{6}}[s + \sqrt{(t-t_i)^{1-\alpha}|x(s)|}]ds, \\ \lim\limits_{t \to t_1^+}(t-t_1)^{1-\alpha}x(t) - x(t_1) = 8 + \sqrt[5]{t_1^{1-\alpha}x(t_1)}, \end{cases} \tag{5.4}$$

where $\alpha \in (0,1)$, $0 = t_0 < \frac{1}{2} = t_1 < t_2 = 1$, $\lambda \in R$ *are fixed constant.*

Corresponding to BVP(1.7), we have $\phi(t) = t^{-\frac{1}{2}}(1-t)^{-\frac{1}{6}}$, $G(t,x) = t + \sqrt{(t-t_i)^{1-\alpha}|x|}$ and $I(t_1,x) = 8 + \sqrt[5]{x}$. It is easy to see that (a), (b) and (c) hold.$p(t) = t^{-\frac{1}{4}}(1-t)^{-\frac{1}{8}}$ satisfies (d) with $k = -\frac{1}{4}, l = -\frac{1}{8}$. $f(t,x) = t^2 + \sqrt[3]{(t-t_i)^{1-\alpha}x}$. One sees that f, G, I satisfy (e). Choose $\phi(t) = t^2$, $\psi(t) = t$ and $I_0 = 8$. Then (B1) holds with $B(x) = \sqrt{|x|}, \overline{B}(x) = \sqrt[5]{x}$ and $b(x) = \sqrt[3]{x}$. Thus by Theorem 4.1, we know BVP(5.4) has at least one solution since there exists a constant $r_0 > 0$ such that (4.1) holds obviously.

Example 5.3. *Consider the following BVP*

$$\begin{cases} {}^{C}D_{0^+}^{\alpha}x(t) = t^{-\frac{1}{4}}(1-t)^{-\frac{1}{8}}\sqrt[3]{x(t)}, t \in (t_i, t_{i+1}], i = 0, 1, \\ x(0) - x(1) = \int_0^1 s^{-\frac{1}{2}}(1-s)^{-\frac{1}{6}}\sqrt[3]{x(s)}ds, \\ \Delta x(t_1) = x(t_1^+) - x(t_1) = x(t_1) + 8, \end{cases} \tag{5.5}$$

where $\alpha \in (0,1)$, $0 = t_0 < \frac{1}{2} = t_1 < t_2 = 1$ *are fixed constant.*

Corresponding to BVP(5.1), we have $\phi(t) = t^{-\frac{1}{2}}(1-t)^{-\frac{1}{6}}$, $G(t,x) = \sqrt[3]{x}$ and $I(t_1,x) = 8$. It is easy to see that (a), (b) and (c) hold.$p(t) = t^{-\frac{1}{4}}(1-t)^{-\frac{1}{8}}$ satisfies (d) with $k = -\frac{1}{4}, l = -\frac{1}{8}$. $f(t,x) = \sqrt[3]{x}$. One sees that f, G, I satisfy (e). Then (B1) holds with $B(x) = \sqrt[3]{|x|}, \overline{B}(x) = 8$ and $b(x) = \sqrt[3]{x}$. Thus by Theorem 5.0.2, we know BVP(5.5) has at least one solution since there exists a constant $r_0 > 0$ such that

$$\left(1 + \frac{\mathbf{B}(1/2,5/6)}{|b_1|}\right)\sqrt[3]{r_0} + \frac{8}{|b_1|} + \left(\frac{t_1^{\alpha+k+l}}{\Gamma(\alpha)} + \left(2 + \frac{1}{|b_1|}\right)\frac{1}{\Gamma(\alpha)}\right)\mathbf{B}(\alpha+l, k+1)\sqrt[3]{r_0} < r_0.$$

holds obviously.

Example 5.4. *Consider the following periodic boundary value problem*

$$\begin{cases} {}^{C}D_{0^+}^{\alpha}x(t) = r(t), t \in (t_i, t_{i+1}], i \in N_0, \\ x(0) - x(T) = 0, \\ \Delta x(t_i) = x(t_i^+) - x(t_1) = b_i x(t_i), i \in N, \end{cases} \tag{5.6}$$

where $0 = t_0, < t_1 < \cdots < t_m < t_{m+1} = T$, $\alpha \in (0,1)$, $r : (0,T) \mapsto \mathbb{R}$ *satisfies that there exist constants* $k > -1$ *and* $l \in (-\alpha, -\alpha - k, 0]$ *such that* $|r(t)| \le t^k(T-t)^l$ *for all* $t \in (0,T)$, $b_i \in \mathbb{R}(i \in \mathbb{N})$. *Then BVP(5.6) has a unique solution if and only if* $\sum\limits_{i=1}^{m} b_i \prod\limits_{j=1}^{i-1}(1+b_j) \neq 0$.

By Theorem 3.10, we see that there exist constants $c_i (i \in \mathbb{N}_0)$ such that

$$x(t) = \sum_{j=0}^{i} c_i + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} r(s)ds, t \in (t_i, t_{i+1}], i \in N_0.$$

By $x(0) = x(T)$, we get

$$c_0 = \sum_{j=0}^{m} c_i + \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} r(s)ds.$$

So

$$\sum_{j=1}^{m} c_j = -\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} r(s)ds. \tag{5.7}$$

By $\Delta x(t_i) = b_i x(t_1)$, we get

$$c_i = b_i \left(\sum_{j=0}^{i-1} c_j + \int_0^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} r(s)ds \right).$$

Then

$$\frac{c_1}{b_1} = c_0 + \int_{t_0}^{t_1} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} r(s)ds,$$

$$\frac{c_i}{b_i} = \sum_{j=1}^{i-1} b_j \frac{c_j}{b_j} + c_0 + \int_0^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} r(s)ds, i \in N.$$

Thus

$$\frac{c_i}{b_i} = \prod_{j=1}^{i-1} (1+b_j) c_0 + \sum_{j=0}^{i} \prod_{v=j}^{i-1} (1+b_v) \int_{t_{j-1}}^{t_j} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} r(s)ds.$$

Substituting c_i into (5.7), we get

$$\sum_{i=1}^{m} b_i \left(\prod_{j=1}^{i-1} (1+b_j) c_0 + \sum_{j=0}^{i} \prod_{v=j}^{i-1} (1+b_v) \int_{t_{j-1}}^{t_j} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} r(s)ds \right) = -\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} r(s)ds.$$

It is easy to see that if (5.6) has a unique solution if and only if $\sum_{i=1}^{m} b_i \prod_{j=1}^{i-1} (1+b_j) \neq 0$.

6 Acknowledgment

The research was supported by the National Natural Science Foundation of China (No: 11401111), the Natural Science Foundation of Guangdong province (No:S2011010001900), the Natural Science Foundation of institution of higher education of Guangdong province (No:2014KTSCX126) and the Foundation for High-level talents in Guangdong Higher Education Project.

The author would like to thank the referees and the editors for their careful reading and some useful comments on improving the presentation of this paper.

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Received: February 5, 2015; *Accepted*: August 23, 2015

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