Stability of traveling fronts in a population model
with nonlocal delay and advection

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Abstract

In this paper, we are concerned with the stability of traveling fronts in a population model with nonlocal delay and advection under the large initial perturbation (i.e. the initial perturbation around the traveling wave decays exponentially as $x \to -\infty$, but it can be arbitrarily large in other locations). The globally exponential stability of traveling fronts is established by the weighted-energy method combining with comparison principle, including even the slower waves whose wave speed are close to the critical speed.

Keywords: Traveling fronts, stability, weighted-energy method.

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1 Introduction

In this paper, we consider a population model with nonlocal delay and advection

$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} - B\frac{\partial u}{\partial x} + d(u(t,x)) = \varepsilon \int_{-\infty}^{+\infty} b(u(t-r, x-y+Br))g(y)dy, \quad t > 0, x \in R \tag{1.1}$$

with the initial condition

$$u(s,x) = u_0(s,x), \quad s \in [-r,0], \ x \in R, \tag{1.2}$$

which describes the population growth of a single-species population dynamics with two age classes and a fixed maturation period living in a spatial transport field(see [5, 24, 26]). Here, $u(t,x)$ denotes the total mature population in time $t \geq 0$ and at location $x \in R$, $D > 0$ is the diffusion rate for the mature, $\varepsilon > 0$ is the impact of the death rate of the immature. $r > 0$ is the mature age of the species, which is the so-called time delay, $\alpha > 0$ represents the effect of the dispersal rate of the immature and satisfies $\alpha \leq rD$, $B \in R$ is the velocity of the spatial transport field and $g(y)$ is the heat kernel

$$g(y) = \frac{1}{\sqrt{4\pi\alpha}}e^{-\frac{y^2}{4\alpha}} \ with \ \int_{-\infty}^{+\infty} g(y)dy = 1.$$

Lastly, $d(u)$ and $b(u)$ denote the death and birth rates of the mature, respectively, and satisfy the following hypotheses:

(A_1) There exist $u_- = 0$ and $u_+ > 0$ such that $d(0) = b(0) = 0$, $\varepsilon b'(0) > d'(0) \geq 0$, $d(u_+) = \varepsilon b(u_+)$, and $0 \leq \varepsilon b'(u_+) < d'(u_+)$;

(A_2) For $0 \leq u \leq u_+, d''(u) \geq 0, b''(u) \leq 0$, and $d(u), b(u) \in C^2[0, u_+]$.

Particularly, the birth function $b(u)$ can be taken as the following three important types

$$b_1(u) = pue^{-au^q}, \quad b_2(u) = \frac{pu}{1 + au^q}, \quad or \quad b_3(u) = \left\{ \begin{array}{ll} pu(1 - \frac{u^q}{K^q}) & 0 \leq u \leq K \\ 0, & u > K \end{array} \right. \quad a, p, q > 0, \tag{1.3}$$

where $K > 0$ represents the carrying capacity and $p > 0$ is the effect of varying the birth rate. When $q = 1$, $b_1(u)$ is the so-called Nicholson's birth function, $b_2(u)$ is the Monod function and $b_3(u)$ is the nonlinear term of Logistic model.

On the other hand, when $B \neq 0$, by taking the death rate of the mature population as $d(u) = \delta u$, $\delta > 0$, (1.1) reduces to the following reaction-advection-diffusion model with nonlocal delayed effect in [5]

$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} - B\frac{\partial u}{\partial x} + \delta u = \varepsilon \int_{-\infty}^{+\infty} b(u(t - r, x - y + Br))g(y)dy, \quad t > 0, \ x \in R. \tag{1.4}$$

When $B = 0$, (1.1) can be reduced to all kinds of reaction-diffusion models with nonlocal delay, such as So, Wu and Zou's age-structured population model(where $d(u) = \delta u, \delta > 0$, see[19]), Nicholson's blowflies model(where $d(u) = \delta u$, $\delta > 0$, $b(u) = b_1(u)$ with $q = 1$, see [12, 13, 15]), Al-Omari and Gourley's age-structured population model(where $d(u) = \delta u^2$ and $\varepsilon b(u) = \tilde{p}e^{-\gamma r}u$, $\tilde{p} > 0$, $\gamma > 0$, see [1]), and so on.

Recent years, there have been extensively investigations on the stability of traveling waves for various reaction-diffusion equations with nonlocal delay and the issue of the stability of traveling waves become more interesting and important, please refer to [2–5, 8, 9, 11–18, 21, 23, 24] and references therein. One of the most effective methods for the stability of monostable waves is the weighted energy method used and developed by Mei(see [11–16]). For example, Mei[13, 15] established the exponential stability of monostable waves for the Nicholson's blowflies equations with nonlocal delay by the weighted energy method combining with the comparison principle. Motivated by Mei's idea, Wu's[24] established the exponential stability of the traveling wavefronts in monostable reaction-advection-diffusion equations with nonlocal delay, but it is only for the faster waves(i.e. the noncritical speed). More recently, by the weighted energy method combining with the comparison principle and the Green function technique, Mei[15, 16] proved the stability for the noncritical waves including the slower waves whose wave speed are close to the critical speed and even for the critical waves of (1.1) when $B = 0$. As a result, here we are interested in the stability of traveling waves of the reaction-advection-diffusion equation (1.1) for all faster waves including those slow waves by the weighted energy method combining with the comparison principle, which recovers and improves Wu[24] stability results. By constructing a non-piecewise weighted function concerning with the critical speed [20, 27, 28] which is different from Mei[14, 16], the difficulty caused by the nonlocality can be overcome, some energy estimates in the weighted L^2 space is first established, and the energy estimates in the H^1 space is thus built up. Finally, we prove the globally exponential stability of traveling fronts of (1.1). The stability of critical waves is our pursuit in another subsequent paper [10].

The rest of this paper is organized as follows. In Section 2, we introduce some preliminaries and state our main result on the stability of traveling fronts. In Section 3, we prove the main result after establishing the boundedness of solutions , the comparison principle and some energy estimates.

2   Preliminaries and Main Result

Throughout this paper, $C > 0$ denotes a generic constant, $C_i > 0$ ($i = 1, 2, ...$) represents a specific constant. Let I be an interval. $L^2(I)$ is the space of the square integrable functions defined on I, and $H^k(I)(k \geq 0)$ is the Sobolev space of the L^2-functions $h(x)$ defined on the interval I whose derivatives $\frac{d^i}{dx^i}h(x)$ ($i = 1, 2, ..., k$) also belong to $L^2(I)$. $L_w^p(I)$ denotes the weighted L^p-space with a weight function $w(x) > 0$ and its norm is defined by

$$\|h\|_{L_w^p} = \left(\int_I w(x)|h(x)|^p dx \right)^{\frac{1}{p}},$$

$H_w^k(I)$ is the weighted Sobolev space with the norm given by

$$\|h\|_{H_w^k} = \left(\sum_{i=0}^{k} \int_I w(x)|\frac{d^i}{dx^i}h(x)|^2 dx \right)^{\frac{1}{2}}.$$

Let $T > 0$ be a number and $\mathcal{B}$ be a Banach space. We denote by $C([0,T];\mathcal{B})$ the space of the $\mathcal{B}$−valued continuous functions on $[0,T]$. $L^2([0,T];\mathcal{B})$ is the space of the $\mathcal{B}$−valued L^2−functions on $[0,T]$. The corresponding spaces of $\mathcal{B}$−valued functions on $[0,\infty)$ are defined similarly.

From (A_1), it is not difficult to verify that (1.1) has two constant equilibria $u_\pm$ and from (A_2) we know that u_- is unstable and u_+ is stable. A traveling wavefront of (1.1) connecting with u_- and u_+ is a monotone solution in the form of $u(t,x) = \phi(x+ct)$ with a speed c and satisfies the following differential equation

$$\begin{cases} (c-B)\phi'(\xi) - D\phi''(\xi) + d(\phi) = \varepsilon \int_{-\infty}^{+\infty} b(\phi(\xi - y + (B-c)r))g(y)dy, \\ \phi(\pm\infty) = u_\pm, \end{cases} \tag{2.5}$$

where $\xi = x + ct, \phi' = \frac{d\phi}{d\xi}$.

By using the monotone iteration technique as well as the upper and lower solution method as in [5–7, 21, 22], the existence of traveling wavefronts of (1.1) can be obtained in a similar way as follows.

Proposition 2.1. (Existence of traveling wavefronts) (see[21, 22, 24].) *Assume that (A_1)-(A_2) hold. Then there exist a minimum wave speed (is also called the critical wave speed) $\widetilde{c}_* = c_* + B$, where $c_* = c_*(\gamma, \alpha, \varepsilon, D, d'(0), b'(0)) \in (0, 2\sqrt{D(\varepsilon b'(0) - d'(0))})$ is the the critical wave speed of traveling wavefront of (1.1) when $B = 0$, and a corresponding number $\lambda_* = \lambda_*(\widetilde{c}_*) > 0$ satisfying*

$$\Delta(\lambda_*, c_* + B) = 0, \quad \frac{\partial}{\partial\lambda}\Delta(\lambda, c_* + B)|_{\lambda=\lambda_*} = 0, \tag{2.6}$$

where

$$\Delta(\lambda, c) = F_c(\lambda) - G_c(\lambda) = \varepsilon b'(0)e^{\alpha\lambda^2 - \lambda(c-B)r} - [(c-B)\lambda - D\lambda^2 + d'(0)], \tag{2.7}$$

such that for all $c \geq c_* + B$, the traveling wavefront $\phi(x+ct)$ of (1.1) connecting with u_- and u_+ exists. Furthermore, $(\lambda_*, c_* + B)$ is the tangent point of $F_c(\lambda) = \varepsilon b'(0)e^{\alpha\lambda^2 - \lambda(c-B)r}$ and $G_c(\lambda) = (c-B)\lambda - D\lambda^2 + d'(0)$, i.e., for $c = c_* + B$, it holds that

$$\varepsilon b'(0)e^{\alpha\lambda_*^2 - \lambda_* c_* r} = c_*\lambda_* - D\lambda_*^2 + d'(0), \tag{2.8}$$

and for $c > c_* + B$, there exist two numbers depending on c: $\lambda_1 = \lambda_1(c) > 0$ and $\lambda_2 = \lambda_2(c) > 0$ as the solutions to the equation $F_c(\lambda_i) = G_c(\lambda_i)$, i.e.,

$$\varepsilon b'(0)e^{\alpha\lambda_i^2 - \lambda_i cr + \lambda_i Br} = c\lambda_i - B\lambda_i - D\lambda_i^2 + d'(0), \quad i = 1, 2 \tag{2.9}$$

such that

$$F_c(\lambda) < G_c(\lambda) \quad \text{for } \lambda_1 < \lambda < \lambda_2,$$

and particularly

$$F_c(\lambda_*) = G_c(\lambda_*) \quad \text{for } \lambda_1 < \lambda_* < \lambda_2.$$

Now, we define a weight function as

$$w(x) = e^{-2\lambda_* x}, \quad x \in R,$$

where $\lambda_* = \lambda_*(\widetilde{c}_*)$ is the positive constant determined in Proposition2.1 and it satisfies $\frac{c_*}{2D} = \frac{\widetilde{c}_*-B}{2D} < \lambda_* < \frac{\widetilde{c}_*-B}{D} = \frac{c_*}{D}$ as shown in[25]. Obviously, $w(x) \to +\infty$ as $x \to -\infty$ and $w(x) \to 0$ as $x \to +\infty$.

Next, we are going to state our main result about the stability of the traveling wavefront of (1.1) and (1.2).

Theorem 2.1. (Nonlinear Stability). *Let $d(u)$ and $b(u)$ satisfy (A_1)-(A_2). For a given traveling wavefront $\phi(x+ct)$ of (1.1) with $c > c_* + B$ and $\phi(\pm\infty) = u_\pm$, if c satisfies*

$$e^{\lambda_*^2\alpha} < \frac{(c-B)\lambda_* - D\lambda_*^2 + d'(0)}{c_*\lambda_* - D\lambda_*^2 + d'(0)}, \tag{2.10}$$

the initial data holds $u_- \leq u_0(s,x) \leq u_+$ for $(s,x) \in [-r,0] \times R$, and the initial perturbation is $u_0(s,x) - \phi(x+cs) \in C([-r,0]; H_w^1(R))$, then the solution of (1.1) and (1.2) satisfies $u(t,x) - \phi(x+ct) \in C([0,\infty); H_w^1(R))$, and

$$u_- \leq u(t,x) \leq u_+, \quad \text{for } (t,x) \in R_+ \times R,$$

$$\|(u - \phi)(t)\|_{H^1_w(R)} \le Ce^{-\mu t}, \quad t \ge 0,$$

for some positive constant μ.

In particular, $u(t, x)$ also converges asymptotically exponential to the wavefront $\phi(x + ct)$ in the $L^\infty-$norm:

$$\sup_{x \in R} |u(t, x) - \phi(x + ct)| \le Ce^{-\mu t}, \quad t \ge 0.$$

Remark 2.1. (i) When $\alpha \to 0$ in $g(y)$, by the property of the heat kernel, we have $\lim_{\alpha \to 0} \int_{-\infty}^{+\infty} b(u(t - r, x - y))g(y)dy = b(v(t - r, x))$, then the nonlocal equation (1.1) is reduced to a reaction-advection-diffusion equation with local delay. So the result of this paper also includes the stability of traveling fronts of (1.1) with local nonlinearity.

(ii) Obviously, the condition (2.10) is equivalent to $\alpha < \frac{1}{\lambda_*^2} \ln \frac{c\lambda_* - \lambda_* B - D\lambda_*^2 + d'(0)}{c_*\lambda_* - D\lambda_*^2 + d'(0)}$. Here α is independent of c, D and d, but both λ_* and c_* are related to α. For given α, we need the wave speed c to be large shown as in (2.10). Conversely, when c is sufficiently large, one can easily verify that $\frac{1}{\lambda_*^2} \ln \frac{c\lambda_* - \lambda_* B - D\lambda_*^2 + d'(0)}{c_*\lambda_* - D\lambda_*^2 + d'(0)}$ is sufficiently large. This ensures α is sufficiently large as well. When c is sufficiently close to the critical wave speed $\widetilde{c}_* = c_* + B$, then one can recognize that $\frac{1}{\lambda_*^2} \ln \frac{c\lambda_* - \lambda_* B - D\lambda_*^2 + d'(0)}{c_*\lambda_* - D\lambda_*^2 + d'(0)} \ll 1$, which means α needed to be sufficiently small. Thus, when α is small enough, we may obtain the stability for those slower waves with the speed $c \in (c_* + B, 2\sqrt{D(\varepsilon b'(0) - d'(0))} + B)$.

3 Proof of asymptotic stability

As shown in [11], we can similarly prove the global existence and uniqueness of the solution for the initial value problem (1.1) and (1.2). In order to prove our stability result, we also need to prove the following boundedness and establish the comparison principle for (1.1) and (1.2), which can be proved similarly as shown in [9, 11, 12, 14, 15], here we omit them.

Lemma 3.1. (Boundedness). *Let the initial data satisfy*

$$u_- = 0 \le u_0(s, x) \le u_+, \text{ for } (s, x) \in [-r, 0] \times R. \tag{3.11}$$

Then the solution $u(t, x)$ of the Cauchy problem (1.1) and (1.2) satisfies

$$u_- \le u(t, x) \le u_+, \text{ for } (t, x) \in [0, \infty) \times R. \tag{3.12}$$

Lemma 3.2. (Comparison principle). *Let $\bar{u}(t, x)$ and $\underline{u}(t, x)$ be the solutions of (1.1) and (1.2) with the initial data $\bar{u}_0(s, x)$ and $\underline{u}_0(s, x)$, respectively. If*

$$u_- \le \underline{u}_0(s, x) \le \bar{u}_0(s, x) \le u_+, \text{ for } (s, x) \in [-r, 0] \times R, \tag{3.13}$$

then

$$u_- \le \underline{u}(t, x) \le \bar{u}(t, x) \le u_+, \text{ for } (t, x) \in [0, \infty) \times R. \tag{3.14}$$

In what follows, we are going to prove the main result, Theorem 2.1, by means of the weighted-energy method combining with comparison principle.

For given initial data $u_0(s, x)$ satisfying

$$u_- = 0 \le u_0(s, x) \le u_+, \text{ for } (s, x) \in [-r, 0] \times R,$$

let

$$\begin{cases} U_0^+(s, x) = \max\{u_0(s, x), \phi(x + cs)\}, \text{ for } (s, x) \in [-r, 0] \times R. \\ U_0^-(s, x) = \min\{u_0(s, x), \phi(x + cs)\}, \text{ for } (s, x) \in [-r, 0] \times R. \end{cases} \tag{3.15}$$

so

$$u_- \le U_0^-(s, x) \le u_0(s, x) \le U_0^+(s, x) \le u_+, \text{ for } (s, x) \in [-r, 0] \times R, \tag{3.16}$$

$$u_- \le U_0^-(s, x) \le \phi(x + cs) \le U_0^+(s, x) \le u_+, \text{ for } (s, x) \in [-r, 0] \times R. \tag{3.17}$$

Define $U^+(t,x)$ and $U^-(t,x)$ as the corresponding solutions of Eq.(1.1) with respect to the initial data $U_0^+(s,x)$ and $U_0^-(s,x)$, respectively, i.e.,

$$\begin{cases} \frac{\partial U^\pm}{\partial t} - D\frac{\partial^2 U^\pm}{\partial x^2} - B\frac{\partial U^\pm}{\partial x} + d(U^\pm) = \varepsilon \int_{-\infty}^{+\infty} b(U^\pm(t-r,x-y+Br))g(y)dy, \quad t > 0, x \in R, \\ U^\pm(s,x) = U_0^\pm(s,x), \quad (s,x) \in [-r,0] \times R. \end{cases} \tag{3.18}$$

By the comparison principle, it follows that

$$u_- \le U^-(t,x) \le u(t,x) \le U^+(t,x) \le u_+, \text{ for } (t,x) \in R_+ \times R, \tag{3.19}$$

$$u_- \le U^-(t,x) \le \phi(x+ct) \le U^+(t,x) \le u_+, \text{ for } (t,x) \in R_+ \times R. \tag{3.20}$$

In order to prove the stability of the traveling wavefronts presented in Theorem 2.1, we also need the following three steps as shown in [9, 14–16].

Step 1. The convergence of $U^+(t,x)$ to $\phi(x+ct)$.

Let $\xi := x + ct$ and

$$v(t,\xi) := U^+(t,x) - \phi(x+ct), \quad v_0(s,\xi) := U_0^+(s,x) - \phi(x+cs), \tag{3.21}$$

then by (3.17) and (3.20), we have

$$v(t,\xi) \ge 0 \text{ and } v_0(s,\xi) \ge 0. \tag{3.22}$$

From Eq.(1.1) and (A_1), it can be verified that $v(t,\xi)$ defined in (3.21) satisfies (by linearizing it at 0)

$$\begin{cases} v_t + (c-B)v_\xi - Dv_{\xi\xi} + d'(0)v - \varepsilon b'(0) \int_{-\infty}^{+\infty} v(t-r,\xi-y+(B-c)r)g(y)dy \\ = -Q_1(t,\xi) + \varepsilon \int_{-\infty}^{+\infty} Q_2(t-r,\xi-y+(B-c)r)g(y)dy + [d'(0) - d'(\phi(\xi))]v \\ +\varepsilon \int_{-\infty}^{+\infty} [b'(\phi(\xi-y+(B-c)r)) - b'(0)]v(t-r,\xi-y+(B-c)r)g(y)dy \\ =: J_1(t,\xi) + J_2(t,\xi) + J_3(t,\xi) + J_4(t,\xi), \quad (t,\xi) \in R_+ \times R, \end{cases} \tag{3.23}$$

with the initial data

$$v(s,\xi) = v_0(s,\xi), \quad (s,\xi) \in [-r,0] \times R,$$

where

$$Q_1(t,\xi) = d(\phi + v) - d(\phi) - d'(\phi)v. \tag{3.24}$$

with $\phi = \phi(\xi)$ and $v = v(t,\xi)$.

$$Q_2(t-r,\xi-y+(B-c)r) = b(\phi + v) - b(\phi) - b'(\phi)v. \tag{3.25}$$

with $\phi = \phi(\xi-y+(B-c)r)$ and $v = v(t-r,\xi-y+(B-c)r)$.

Multiplying (3.23) by $e^{2\mu t}w(\xi)v(t,\xi)$, we obtain

$$\{\frac{1}{2}e^{2\mu t}wv^2\}_t + e^{2\mu t}\{\frac{c-B}{2}wv^2 - Dwvv_\xi\}_\xi + De^{2\mu t}wv_\xi^2 + De^{2\mu t}w'vv_\xi$$
$$+\{-\frac{c-B}{2}\cdot\frac{w'}{w} + d'(0) - \mu\}e^{2\mu t}wv^2$$
$$-\varepsilon e^{2\mu t}w(\xi)v(t,\xi)b'(0)\int_{-\infty}^{+\infty} v(t-r,\xi-y+(B-c)r)g(y)dy$$
$$= e^{2\mu t}w(\xi)v(t,\xi)[J_1(t,\xi) + J_2(t,\xi) + J_3(t,\xi) + J_4(t,\xi)]. \tag{3.26}$$

By the Cauchy-Schwarz inequality $xy \le x^2 + \frac{1}{4}y^2$, we have

$$|De^{2\mu t}w'vv_\xi| = De^{2\mu t}w|v_\xi \cdot \frac{w'}{w}v| \le De^{2\mu t}wv_\xi^2 + \frac{D}{4}e^{2\mu t}(\frac{w'}{w})^2wv^2,$$

then (3.26) is reduced to

$$\{\frac{1}{2}e^{2\mu t}wv^2\}_t + e^{2\mu t}\{\frac{c-B}{2}wv^2 - Dwvv_\xi\}_\xi$$
$$+\{-\frac{c-B}{2}\cdot\frac{w'}{w} + d'(0) - \mu - \frac{D}{4}(\frac{w'}{w})^2\}e^{2\mu t}wv^2$$
$$-\varepsilon e^{2\mu t}w(\xi)v(t,\xi)b'(0)\int_{-\infty}^{+\infty}v(t-r,\xi-y+(B-c)r)g(y)dy$$
$$\leq\ e^{2\mu t}w(\xi)v(t,\xi)[J_1(t,\xi)+J_2(t,\xi)+J_3(t,\xi)+J_4(t,\xi)]. \tag{3.27}$$

Integrating (3.27) over $R\times[0,t]$ with respect to ξ and t, and noting the vanishing term at far fields,

$$\{\frac{c-B}{2}wv^2 - Dwvv_\xi\}|_{\xi=-\infty}^{\xi=+\infty} = 0,$$

because $\sqrt{w}v\in H^1(R)$, this implies, by the property of Sobolev space $H^1(R)$, that $(\sqrt{w}v)|_{\xi=\pm\infty}=0$ and $(\sqrt{w}v_\xi)|_{\xi=\pm\infty}=0$. Thus we further have

$$e^{2\mu t}\|v(t)\|_{L^2_w}^2 + \int_0^t\int_R\{(B-c)\cdot\frac{w'}{w}+2d'(0)-2\mu-\frac{D}{2}(\frac{w'}{w})^2\}e^{2\mu s}w(\xi)v^2(s,\xi)d\xi ds$$
$$-2\varepsilon b'(0)\int_0^t\int_R\int_R e^{2\mu s}w(\xi)v(s,\xi)v(s-r,\xi+(B-c)r-y)g(y)dyd\xi ds$$
$$\leq\ \|v_0(0)\|_{L^2_w}^2 + 2\int_0^t\int_R e^{2\mu s}w(\xi)v(s,\xi)[J_1(t,\xi)+J_2(t,\xi)+J_3(t,\xi)+J_4(t,\xi)]d\xi ds. \tag{3.28}$$

We now turn to estimate the third term on the left-hand-side of (3.28). First of all, by changing variables : $\xi-y+(B-c)r\to\xi, s-r\to s, y\to y$, we have

$$b'(0)\int_0^t\int_R\int_R e^{2\mu s}w(\xi)v^2(s-r,\xi+(B-c)r-y)g(y)dyd\xi ds$$
$$=\ b'(0)\int_{-r}^{t-r}\int_R\int_R e^{2\mu(s+r)}w(\xi+y+(c-B)r)v^2(s,\xi)g(y)dyd\xi ds$$
$$=\ e^{2\mu r}b'(0)\int_0^{t-r}\int_R e^{2\mu s}[\int_R\frac{w(\xi+y+(c-B)r)}{w(\xi)}g(y)dy]w(\xi)v^2(s,\xi)d\xi ds$$
$$+e^{2\mu r}b'(0)\int_{-r}^0\int_R e^{2\mu s}[\int_R\frac{w(\xi+y+(c-B)r)}{w(\xi)}g(y)dy]w(\xi)v_0^2(s,\xi)d\xi ds$$
$$\leq\ e^{2\mu r}b'(0)\int_0^t\int_R e^{2\mu s}[\int_R\frac{w(\xi+y+(c-B)r)}{w(\xi)}g(y)dy]w(\xi)v^2(s,\xi)d\xi ds$$
$$+e^{2\mu r}b'(0)\int_{-r}^0\int_R e^{2\mu s}[\int_R\frac{w(\xi+y+(c-B)r)}{w(\xi)}g(y)dy]w(\xi)v_0^2(s,\xi)d\xi ds. \tag{3.29}$$

Again, using the Cauchy-Schwarz inequality we obtain

$$|v(s,\xi)v(s-r,\xi+(B-c)r-y)| \leq \frac{\eta}{2}v^2(s,\xi) + \frac{1}{2\eta}v^2(s-r,\xi+(B-c)r-y) \tag{3.30}$$

for any positive constant η, which will be specified later, and using (3.29), we have

$$2\varepsilon b'(0)|\int_0^t \int_R \int_R e^{2\mu s}w(\xi)v(s,\xi)v(s-r,\xi+(B-c)r-y)g(y)dyd\xi ds|$$

$$\leq \quad \varepsilon b'(0)|\int_0^t \int_R \int_R e^{2\mu s}w(\xi)[\eta v^2(s,\xi)+\frac{1}{\eta}v^2(s-r,\xi+(B-c)r-y)]g(y)dyd\xi ds|$$

$$= \quad \varepsilon\eta b'(0)\int_0^t \int_R \int_R e^{2\mu s}w(\xi)v^2(s,\xi)g(y)dyd\xi ds$$

$$+\frac{\varepsilon}{\eta}b'(0)\int_0^t \int_R \int_R e^{2\mu s}w(\xi)v^2(s-r,\xi+(B-c)r-y)g(y)dyd\xi ds$$

$$\leq \quad \varepsilon\eta b'(0)\int_0^t \int_R e^{2\mu s}w(\xi)[\int_R g(y)dy]v^2(s,\xi)d\xi ds$$

$$+\frac{\varepsilon e^{2\mu r}}{\eta}b'(0)\int_0^t \int_R e^{2\mu s}[\int_R \frac{w(\xi+y+(c-B)r)}{w(\xi)}g(y)dy]w(\xi)v^2(s,\xi)d\xi ds$$

$$+\frac{\varepsilon e^{2\mu r}}{\eta}b'(0)\int_{-r}^0 \int_R e^{2\mu s}[\int_R \frac{w(\xi+y+(c-B)r)}{w(\xi)}g(y)dy]w(\xi)v_0^2(s,\xi)d\xi ds. \tag{3.31}$$

Substituting (3.31) into (3.28) leads to

$$e^{2\mu t}\|v(t)\|_{L_w^2}^2 + \int_0^t \int_R e^{2\mu s}B_{\eta,\mu,w}(\xi)w(\xi)v^2(s,\xi)d\xi ds$$

$$\leq \quad \|v_0(0)\|_{L_w^2}^2 + \frac{\varepsilon e^{2\mu r}b'(0)}{\eta}\int_{-r}^0 \int_R e^{2\mu s}[\int_R \frac{w(\xi+y+(c-B)r)}{w(\xi)}g(y)dy]w(\xi)v_0^2(s,\xi)d\xi ds$$

$$+2\int_0^t \int_R e^{2\mu s}w(\xi)v(s,\xi)[J_1(t,\xi)+J_2(t,\xi)+J_3(t,\xi)+J_4(t,\xi)]d\xi ds. \tag{3.32}$$

where

$$B_{\eta,\mu,w}(\xi) \quad := \quad A_{\eta,w}(\xi) - 2\mu - \frac{\varepsilon}{\eta}(e^{2\mu r}-1)b'(0)\int_R \frac{w(\xi+y+(c-B)r)}{w(\xi)}g(y)dy. \tag{3.33}$$

where

$$A_{\eta,w}(\xi) \quad := \quad -(c-B)\cdot\frac{w'}{w} + 2d'(0) - \frac{D}{2}(\frac{w'}{w})^2$$

$$-\varepsilon\eta b'(0)\int_R g(y)dy$$

$$-\frac{\varepsilon b'(0)}{\eta}\int_R \frac{w(\xi+y+(c-B)r)}{w(\xi)}g(y)dy. \tag{3.34}$$

Using Taylor's formula for the nonlinearity $Q_1(t,\xi)$, $Q_2(t-r,\xi-y+(B-c)r)$, and noting $(A2)$, we have

$$Q_1(t,\xi) = d(\phi+v) - d(\phi) - d'(\phi)v = d''(\bar{\phi}_1)v^2 \geq 0.$$

$$Q_2(t-r,\xi-y+(B-c)r) = b(\phi+v) - b(\phi) - b'(\phi)v = b''(\bar{\phi}_2)v^2 \leq 0$$

for some $\bar{\phi}_1$, $\bar{\phi}_2 \in [\phi,\phi+v]$, namely,

$$J_1(t,\xi) \leq 0 \quad \text{and} \quad J_2(t,\xi) \leq 0.$$

Notice that $d'(u)$ is increasing and $b'(u)$ is decreasing from (A_2) and the fact $v(t,\xi) \geq 0$(see (3.22)), which implies

$$d'(0) - d'(\phi) \leq 0 \ and \ b'(\phi) - b'(0) \leq 0 \ for \ \phi \geq 0,$$

namely,

$$J_3(t,\xi) \leq 0 \quad \text{and} \quad J_4(t,\xi) \leq 0,$$

which leads that

$$2 \int_0^t \int_R e^{2\mu s} w(\xi) v(s,\xi) [J_1(t,\xi) + J_2(t,\xi) + J_3(t,\xi) + J_4(t,\xi)] d\xi ds \leq 0. \tag{3.35}$$

On the other hand, since $\frac{w(\xi+(c-B)r+y)}{w(\xi)} = e^{-2\lambda_*((c-B)r+y)}$, and using the fact

$$\int_R \frac{w(\xi+y+(c-B)r)}{w(\xi)} g(y)dy = e^{4\alpha\lambda_*^2 - 2\lambda_*(c-B)r},$$

it follows that

$$\frac{\varepsilon e^{2\mu r}}{\eta} b'(0) \int_{-r}^0 \int_R e^{2\mu s} [\int_R \frac{w(\xi+y+(c-B)r)}{w(\xi)} g(y)dy] w(\xi) v_0^2(s,\xi) d\xi ds$$

$$= \frac{\varepsilon b'(0) e^{2\mu r}}{\eta} \int_{-r}^0 e^{2\mu s} \int_R e^{4\alpha\lambda_*^2 - 2\lambda_*(c-B)r} w(\xi) v_0^2(s,\xi) d\xi ds$$

$$= \frac{\varepsilon b'(0)}{\eta} e^{2\mu r + 4\alpha\lambda_*^2 - 2\lambda_*(c-B)r} \int_{-r}^0 \int_R e^{2\mu s} w(\xi) v_0^2(s,\xi) d\xi ds$$

$$\leq C \int_{-r}^0 \|v_0(s)\|_{L_w^2}^2 ds. \tag{3.36}$$

Applying (3.35) and (3.36) in (3.32), we then obtain

$$e^{2\mu t} \|v(t)\|_{L_w^2}^2 + \int_0^t \int_R e^{2\mu s} B_{\eta,\mu,w}(\xi) w(\xi) v^2(s,\xi) d\xi ds$$

$$\leq \|v_0(0)\|_{L_w^2}^2 + C \int_{-r}^0 \|v_0(s)\|_{L_w^2}^2 ds. \tag{3.37}$$

Next, we will prove $B_{\eta,\mu,w}(\xi) > 0$ by selecting the numbers η, μ. For that purpose, we need the following lemma.

Lemma 3.3. *Let $\eta = e^{2\alpha\lambda_*^2 - \lambda_*(c-B)r}$. Then*

$$A_{\eta,w}(\xi) \geq C_1 > 0, \quad \xi \in R \tag{3.38}$$

for some positive constant C_1.

Proof. Notice that $\eta = e^{2\alpha\lambda_*^2 - \lambda_*(c-B)r}, w(\xi) = e^{-2\lambda_*\xi}, \frac{w'(\xi)}{w(\xi)} = -2\lambda_*$ and $\int_R g(y)dy = 1$. We may obtain

$$
\begin{aligned}
A_{\eta,w}(\xi) &= -(c-B) \cdot \frac{w'}{w} + 2d'(0) - \frac{D}{2} (\frac{w'}{w})^2 - \varepsilon\eta b'(0) \int_R g(y)dy \\
&\quad - \frac{\varepsilon b'(0)}{\eta} \int_R \frac{w(\xi+y+(c-B)r)}{w(\xi)} g(y)dy \\
&= 2(c-B)\lambda_* + 2d'(0) - 2D\lambda_*^2 - \varepsilon\eta b'(0) - \frac{\varepsilon b'(0)}{\eta} e^{4\alpha\lambda_*^2 - 2\lambda_*(c-B)r} \\
&= 2[(c-B)\lambda_* + d'(0) - D\lambda_*^2 - \varepsilon\eta b'(0)] \\
&= 2[(c-B)\lambda_* + d'(0) - D\lambda_*^2 - \varepsilon b'(0)e^{2\alpha\lambda_*^2 - \lambda_*(c-B)r}] \\
&= 2[(c-B)\lambda_* + d'(0) - D\lambda_*^2 - \varepsilon b'(0)e^{\alpha\lambda_*^2 - \lambda_*(c-B)r} e^{\alpha\lambda_*^2}] \\
&\geq 2[(c-B)\lambda_* + d'(0) - D\lambda_*^2 - \varepsilon b'(0)e^{\alpha\lambda_*^2 - \lambda_* c_* r} e^{\alpha\lambda_*^2}] \\
&= 2[((c-B)\lambda_* + d'(0) - D\lambda_*^2) - (c_*\lambda_* + d'(0) - D\lambda_*^2)e^{\alpha\lambda_*^2}] [by(2.8)] \\
&= 2(c_*\lambda_* + d'(0) - D\lambda_*^2)[\frac{(c-B)\lambda_* + d'(0) - D\lambda_*^2}{c_*\lambda_* + d'(0) - D\lambda_*^2} - e^{\alpha\lambda_*^2}] \\
&=: C_1 > 0.[by(2.10), \text{ see Remark } 2.1 \text{ } (ii)).]
\end{aligned}
\tag{3.39}
$$

This completes the proof. $\square$

Lemma 3.4. *Let $\mu_1 > 0$ be the unique solution of the equation*

$$C_1 = 2\mu + \varepsilon\eta b'(0)(e^{2\mu r} - 1) \tag{3.40}$$

If $0 < \mu < \mu_1$, then

$$B_{\eta,\mu,w}(\xi) \geq C_2 > 0, \quad \xi \in R. \tag{3.41}$$

Proof. Applying (3.39) to (3.33), it can be examined that

$$
\begin{aligned}
B_{\eta,\mu,w}(\xi) &\geq& C_1 - 2\mu - \frac{\varepsilon b'(0)}{\eta}(e^{2\mu r} - 1)\int_R \frac{w(\xi + y + (c - B)r)}{w(\xi)}g(y)dy \\
&=& C_1 - 2\mu - \frac{\varepsilon b'(0)}{\eta}(e^{2\mu r} - 1)e^{4\alpha\lambda_*^2 - 2\lambda_*(c-B)r} \\
&=& C_1 - 2\mu - \varepsilon\eta b'(0)(e^{2\mu r} - 1) \\
&=:& C_2 > 0, \ for \ 0 < \mu < \mu_1.
\end{aligned} \tag{3.42}
$$

This completes the proof. □

Applying (3.41) to (3.37), and dropping $\int_0^t \int_R e^{2\mu s} B_{\eta,\mu,w}(\xi)w(\xi)v^2(s,\xi)d\xi ds$, we then immediately establishes the first basic energy estimate as follows which is crucial for our main stability result.

Lemma 3.5. *It holds that*

$$e^{2\mu t}\|v(t)\|_{L_w^2}^2 \leq C\left(\|v_0(0)\|_{L_w^2}^2 + \int_{-r}^0 \|v_0(s)\|_{L_w^2}^2 ds\right), \quad t \geq 0. \tag{3.43}$$

Similarly, differentiating Eq. (3.23) with respect to ξ, and multiplying it by $e^{2\mu t}w(\xi)v_\xi(t,\xi)$, and integrating the resultant equation over $R \times [0,t]$ with respect to ξ and t, then by using (3.43) in Lemma 3.5, we can obtain the second energy estimate as follows.

Lemma 3.6. *It holds that*

$$e^{2\mu t}\|v_\xi(t)\|_{L_w^2}^2 \leq C\left(\|v_0(0)\|_{H_w^1}^2 + \int_{-r}^0 \|v_0(s)\|_{H_w^1}^2 ds\right), \quad t \geq 0. \tag{3.44}$$

Therefore, (3.43) and (3.44) imply the following fact.

Lemma 3.7. *It holds that*

$$\|v(t)\|_{H_w^1}^2 \leq Ce^{-2\mu t}\left(\|v_0(0)\|_{H_w^1}^2 + \int_{-r}^0 \|v_0(s)\|_{H_w^1}^2 ds\right), \quad t \geq 0. \tag{3.45}$$

Notice that $w(\xi) \to 0$ as $\xi \to \infty$, we can not conclude $H_w^1(R) \hookrightarrow C(R)$. However, for any interval $I = (-\infty, \xi_*]$ for some large $\xi_* \gg 1$, there is the Sobolev's embedding result $H_w^1(I) \hookrightarrow C(I)$, which can be combined with (3.45) and be given the following $L^\infty-$ estimate.

Lemma 3.8. *It holds that*

$$\sup_{\xi \in I} |v(t,\xi)| \leq Ce^{-\mu t}\left(\|v_0(0)\|_{H_w^1}^2 + \int_{-r}^0 \|v_0(s)\|_{H_w^1}^2 ds\right)^{\frac{1}{2}}, \quad t \geq 0, \tag{3.46}$$

for any interval $I = (-\infty, \xi_*]$ with some large $\xi_* \gg 1$.

However, we need the $L^\infty-$ convergence in (3.46) in the whole space $(-\infty, +\infty)$. Thus, we are going to prove the convergence at $\xi = +\infty$.

Lemma 3.9. *It holds that*

$$\lim_{\xi \to +\infty} |v(t,\xi)| \leq Ce^{-\mu_2 t}, \quad t \geq 0, \tag{3.47}$$

where $\mu_2 > 0$ and μ_2 satisfies $d'(u_+) - \mu_2 - \varepsilon b'(u_+)e^{\mu_2 r} > 0$.

Proof. From Eq.(1.1), it can be verified that $v(t, \xi)$ defined in (3.21) satisfies

$$
\begin{aligned}
& v_t + cv_\xi - Dv_{\xi\xi} - Bv_\xi + d'(\phi(\xi))v \\
& \quad -\varepsilon \int_{-\infty}^{+\infty} b'(\phi(\xi - y + (B - c)r))v(t - r, \xi - y + (B - c)r)g(y)dy \\
& = \quad -Q_1(t, \xi) + \varepsilon \int_{-\infty}^{+\infty} Q_2(t - r, \xi - y + (B - c)r))g(y)dy \\
& = \quad J_1(t, \xi) + J_2(t, \xi), \quad (t, \xi) \in R_+ \times R.
\end{aligned}
\tag{3.48}
$$

As shown in proving (3.35), $J_1(t, \xi) \leq 0$ *and* $J_2(t, \xi) \leq 0$, above equation is reduced to

$$
\begin{aligned}
& v_t + cv_\xi - Dv_{\xi\xi} - Bv_\xi + d'(\phi(\xi))v \\
& \quad -\varepsilon \int_{-\infty}^{+\infty} b'(\phi(\xi - y + (B - c)r))v(t - r, \xi - y + (B - c)r)g(y)dy \leq 0.
\end{aligned}
\tag{3.49}
$$

Taking limits as $\xi \to +\infty$, and noting that $v_\xi(t, +\infty) = 0$, $v_{\xi\xi}(t, +\infty) = 0$ by the fact that the boundedness of $v(t, \xi)$ for all $\xi \in R$, and $\int_{-\infty}^{+\infty} g(y)dy = 1$, we obtain

$$
\frac{d}{dt}v(t, \infty) + d'(u_+)v(t, \infty) - \varepsilon b'(u_+)v(t - r, \infty) \leq 0.
\tag{3.50}
$$

Multiplying (3.50) by $e^{\mu_2 t}$(μ_2 is a positive constant to be specified later) and integrating it over $[0, t]$, we then have

$$
\int_0^t e^{\mu_2 s}\frac{d}{ds}v(s, \infty)ds + d'(u_+)\int_0^t e^{\mu_2 s}v(s, \infty)ds - \varepsilon b'(u_+)\int_0^t e^{\mu_2 s}v(s - r, \infty)ds \leq 0.
\tag{3.51}
$$

For the first term in (3.51), we get

$$
\int_0^t e^{\mu_2 s}\frac{d}{ds}v(s, \infty)ds = e^{\mu_2 t}v(t, \infty) - v_0(0, \infty) - \mu_2\int_0^t e^{\mu_2 s}v(s, \infty)ds.
\tag{3.52}
$$

By the change of variable $s - r \to s$ for the third term in (3.51), we obtain

$$
\begin{aligned}
& \varepsilon b'(u_+)\int_0^t e^{\mu_2 s}v(s - r, \infty)ds \\
& = \quad \varepsilon b'(u_+)\int_{-r}^{t-r} e^{\mu_2(s+r)}v(s, \infty)ds \\
& \leq \quad \varepsilon b'(u_+)e^{\mu_2 r}\int_{-r}^t e^{\mu_2 s}v(s, \infty)ds \\
& = \quad \varepsilon b'(u_+)e^{\mu_2 r}[\int_{-r}^0 e^{\mu_2 s}v_0(s, \infty)ds + e^{\mu_2 r}\int_0^t e^{\mu_2 s}v(s, \infty)ds] \\
& = \quad \varepsilon b'(u_+)e^{\mu_2 r}\int_{-r}^0 e^{\mu_2 s}v_0(s, \infty)ds + e^{\mu_2 r}\varepsilon b'(u_+)\int_0^t e^{\mu_2 s}v(s, \infty)ds.
\end{aligned}
\tag{3.53}
$$

Substituting (3.52) and (3.53) into (3.51), we have

$$
e^{\mu_2 t}v(t, \infty) + [d'(u_+) - \mu_2 - \varepsilon b'(u_+)e^{\mu_2 r}]\int_0^t v(s, \infty)e^{\mu_2 s}ds \leq C_3,
\tag{3.54}
$$

where $C_3 = v_0(0, \infty) + \varepsilon b'(u_+)e^{\mu_2 r}\int_{-r}^0 v_0(s, \infty)e^{\mu_2 s}ds$. Moreover, by (A_1), we can obtain $d'(u_+) - \varepsilon b'(u_+) > 0$. Then, there exist $\mu_2 > 0$ such that $d'(u_+) - \mu_2 - \varepsilon b'(u_+)e^{\mu_2 r} > 0$, and therefore (3.54) yields

$$
v(t, \infty) \leq Ce^{-\mu_2 t}.
$$

This completes the proof. $\qquad\square$

Combining Lemma 3.8 and Lemma 3.9, and taking $0 < \mu < \min\{\mu_1, \mu_2\}$, we prove the $L^\infty-$ convergence in Theorem 2.1 for all $\xi \in R$, i.e.,

Lemma 3.10. *It holds that*

$$
\sup_{x \in R}|U^+(t, x) - \phi(x + ct)| \leq Ce^{-\mu t}, \quad t \geq 0,
\tag{3.55}
$$

where $0 < \mu < \min\{\mu_1, \mu_2\}$.

Step 2. The convergence of $U^-(t, x)$ to $\phi(x + ct)$.

Let $\xi := x + ct$ and

$$v(t, \xi) = \phi(x + ct) - U^-(t, x), \quad v_0(s, \xi) = \phi(x + cs) - U_0^-(s, x). \tag{3.56}$$

As shown in the process of Step 1, we can similarly prove the convergence of $U^-(t, x)$ to $\phi(x + ct)$, i.e.,

Lemma 3.11. *It holds that*

$$\sup_{x \in R} |U^-(t, x) - \phi(x + ct)| \leq Ce^{-\mu t}, \quad t \geq 0. \tag{3.57}$$

for $0 < \mu < \min\{\mu_1, \mu_2\}$.

   Step 3. The convergence of $u(t, x)$ to $\phi(x + ct)$.

   In this step, we are going to prove Theorem 2.1, namely,

Lemma 3.12. *It holds that*

$$\sup_{x \in R} |u(t, x) - \phi(x + ct)| \leq Ce^{-\mu t}, \quad t \geq 0,$$

for $0 < \mu < \min\{\mu_1, \mu_2\}$.

Proof. Since the initial data satisfy $U_0^-(s, x) \leq u_0(s, x) \leq U_0^+(s, x)$, by Lemma 3.2, it can be proved that the corresponding solutions of (1.1) and (1.2) satisfy $U^-(t, x) \leq u(t, x) \leq U^+(t, x)$, $(t, x) \in R_+ \times R$. Thanks to Lemma 3.10, Lemma 3.11, we have the following convergence results:

$$\sup_{x \in R} |U^\pm(t, x) - \phi(x + ct)| \leq Ce^{-\mu t}, \quad t \geq 0.$$

Using the squeeze Theorem, we finally prove

$$\sup_{x \in R} |u(t, x) - \phi(x + ct)| \leq Ce^{-\mu t}, \quad t \geq 0.$$

This completes the proof.   □

   As a final remark, we consider a reaction-advection-diffusion equation with nonlocal delay (see [21, 24])

$$\begin{cases} \frac{\partial u}{\partial t} = D\Delta u + B\frac{\partial u}{\partial x} + f(u(t, x), \tilde{g} * S(u)), & t > 0, x \in R, \\ u(s, x) = u_0(s, x), & s \in [-r, 0], \ x \in R, \end{cases} \tag{3.58}$$

which is a generalized version of the model (1.1) in the case where $S(u) = b(u)$, $f(u, v) = -d(u) + \varepsilon v$, $\tilde{g}(t, x) = J_\alpha(x + Bt)\delta(t - r)$, $r > 0$ is the time delay and $J_\alpha(x) = \frac{1}{\sqrt{4\pi\alpha}}e^{-\frac{x^2}{4\alpha}}$. Under appropriate assumptions, the exponential stability of noncritical traveling fronts of (3.58) can be obtained similarly as in this paper, including even the slower waves whose wave speed are close to the critical speed, which recovers and improves Wang and Wu's stability results [21, 24] for the noncritical waves. Particularly, the spreading speed and its coincidence with the minimal wave speed(i.e. the critical speed) can be established for (3.58) in the same way as Zhao's [20, 27, 28].

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