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On certain subclass of p - valent analytic functions associated with differintegral operator

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Abstract

In this paper, by making use of the fractional differintegral operator, we introduce a certain subclass of multivalent analytic functions. We study some properties such as inclusion relationship, integral preserving, convolution and some interesting results for multivalent starlikeness are proved.

Keywords: Multivalent function, subordination, superordination, hadamard product, differintegral operator, starlike function.

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1 Introduction

Let $\mathcal{H}$ be the class of functions analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{H}(a,m)$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_{m+1}z^{m+1} + a_{m+2}z^{m+2} + \cdots$.

Let $\mathcal{A}_p$ be the class of functions analytic in the open unit disc $\mathbb{U} = \{z : | z |< 1\}$ of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (p \geq 1). \tag{1.1}$$

and let $\mathcal{A} = \mathcal{A}_1$.

For the functions $f(z)$ of the form (1.1) and $g(z) = z^p + \sum_{n=1}^{\infty} b_{n+p} z^{n+p}$, the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z^p + \sum_{n=1}^{\infty} a_{n+p}\, b_{n+p}\, z^{n+p}.$$

Let $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$. Then we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, if there exists an analytic function $w(z)$ in $\mathbb{U}$ such that $|w(z)| < |z|$ and $f(z) = g(w(z))$, denoted by $f(z) \prec g(z)$. If $g(z)$ is univalent in $\mathbb{U}$, then the subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

In our present investigation, we shall also make use of the Guassian hypergeometric function

$$\begin{aligned} {}_2F_1(a,b;\,c;\,z) &= 1 + \frac{ab}{c}.\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}.\frac{z^2}{2!} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n}\frac{z^n}{n!}, \qquad (a,\,b,\,c \in \mathbb{C}, \quad with \quad c \notin \mathbb{Z}_0^- = \{0,\,-1,\,-2,\,\ldots\}) \end{aligned} \tag{1.2}$$

where the Pochhammer symbol $(x)_k$ is defined, in terms of the Gamma function Γ, by

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} = \begin{cases} 1 & if\, k = 0 \\ x(x+1)(x+2)\ \ldots\ (x+k-1) & if\, k \in \mathbb{N} = \{1,2,\ldots\}. \end{cases}$$

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Definition 1.1. *Let $\alpha > 0$ and $\beta, \gamma \in \mathbb{R}$, then the generalized fractional integral operator $I_{0,z}^{\alpha,\beta,\gamma}$ of order α of a function $f(z)$ is defined by*

$$I_{0,z}^{\alpha,\beta,\gamma} f(z) = \frac{z^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^z (z-t)^{\alpha-1} \quad {}_2F_1\left(\alpha+\beta, \gamma; \alpha; 1-\frac{t}{z}\right) f(t)dt, \tag{1.3}$$

where the function $f(z)$ is analytic in a simply - connected region of the z - plane containing the origin and the multiplicity of $(z-t)^{\alpha-1}$ is removed by requiring $log(z-t)$ to be real when $(z-t) > 0$ provided further that

$$f(z) = O\left(|z|\right)^{\epsilon}, z \to 0 \quad for \quad \epsilon > max\,(0, \beta-\gamma) - 1. \tag{1.4}$$

Definition 1.2. *Let $0 \leq \alpha < 1$ and $\beta, \gamma \in \mathbb{R}$, then the generalized fractional derivative operator $J_{0,z}^{\alpha,\beta,\gamma}$ of order α of a function $f(z)$ is defined by*

$$\begin{aligned} J_{0,z}^{\alpha,\beta,\gamma} f(z) &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz}\left[z^{\alpha-\beta} \int_0^z (z-t)^{-\alpha} \quad {}_2F_1\left(\beta-\alpha, 1-\gamma; 1-\alpha; 1-\frac{t}{z}\right) f(t)dt\right] \\ &= \frac{d^n}{dz^n} J_{0,z}^{\alpha-n,\beta,\gamma} f(z) \end{aligned} \tag{1.5}$$

where the function $f(z)$ is analytic in a simply - connected region of the z - plane containing the origin, with the order as given in (1.4)and multiplicity of $(z-t)^{\alpha}$ is removed by requiring $log(z-t)$ to be real when $(z-t) > 0$.

Definition 1.3. *For real number $\alpha\,(-\infty < \alpha < 1)$, $\beta\,(-\infty < \beta < 1)$ and a positive real number γ, the fractional operator $U_{0,z}^{\alpha,\beta,\gamma} : A_p \to A_p$ is defined in terms of $J_{0,z}^{\alpha,\beta,\gamma}$ and $I_{0,z}^{\alpha,\beta,\gamma}$ by*

$$U_{0,z}^{\alpha,\beta,\gamma} = z^p + \sum_{n=1}^{\infty} \frac{(1+p)_n (1+p+\gamma-\beta)_n}{(1+p-\beta)_n (1+p+\gamma-\alpha)_n} a_{n+p} z^{n+p} \tag{1.6}$$

which for $f(z) \neq 0$ may be written as

$$U_{0,z}^{\alpha,\beta,\gamma} = \begin{cases} \frac{\Gamma(1+p-\beta)\Gamma(1+p+\gamma-\alpha)}{\Gamma(1+p)\Gamma(1+p+\gamma-\beta)} z^{\beta} J_{0,z}^{\alpha,\beta,\gamma} f(z); & 0 \leq \alpha \leq 1 \\ \frac{\Gamma(1+p-\beta)\Gamma(1+p+\gamma-\alpha)}{\Gamma(1+p)\Gamma(1+p+\gamma-\beta)} z^{\beta} I_{0,z}^{-\alpha,\beta,\gamma} f(z); & if\ -\infty \leq \alpha < 0. \end{cases}$$

where $J_{0,z}^{\alpha,\beta,\gamma} f(z)$ and $I_{0,z}^{-\alpha,\beta,\gamma} f(z)$ are, respectively the fractional derivative of f of order α if $0 \leq \alpha < 1$ and the fractional integral of f of order $-\alpha$ if $\infty \leq \alpha < 0$.

Recently, using the operator $U_{0,z}^{\alpha,\beta,\gamma}$, Ahmed S. Galiz [1], introduce the linear operator $\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f : A_p \to A_p$ by

$$\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z) = z^p + \sum_{n=1}^{\infty} \left[\frac{p+l+\lambda n}{p+l}\right]^m \frac{(1+p)_n (1+p+\gamma-\beta)_n}{(1+p-\beta)_n (1+p+\gamma-\alpha)_n} a_{n+p} z^{n+p} \tag{1.7}$$

where $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $l \geq 0$, $\lambda \geq 0$ and $p \in \mathbb{N}$.

The above operator generates several operators studied by many authors such as El - Ashwah and Aouf [4], Selvaraj and Karthikeyan [21], Dziok - Srivastava operator [6], Salagean [19], Goyal and Prajapat [7] and others.

From (1.7), we can easily verified that

$$\lambda z \left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z)\right)' = (p+l)\, \phi_{\alpha,\beta,\gamma}^{m+1,l,\lambda} f(z) - [p(1-\lambda)+l]\, \phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z). \tag{1.8}$$

On differentiating (1.8), we get

$$\lambda z \left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z)\right)'' = (p+l) \left(\phi_{\alpha,\beta,\gamma}^{m+1,l,\lambda} f(z)\right)' - [p+l+(1-p)\lambda] \left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z)\right)'. \tag{1.9}$$

We note that the operator $\phi_{\alpha,\beta,\gamma}^{m,l,\lambda}$ is a generalization of several familiar operators and we will show some of the interesting special cases:

(1). If $m = 0$, $\alpha = \lambda$, $\beta = \mu$ and $\gamma = \eta$ then the operator is reduced into the well - known fractional differintegral operator $I_p^{\lambda}(\mu, \eta)$ which was introduced and investigated by Goyal and Prajapat [7].

(2). If we take $m = 0$, $\alpha = \lambda$, $\beta = \mu$ and $\gamma = \eta = 0$ then the operator is reduced into the known fractional differintegral operator Ω_p^{λ}. It was studied by Patel and Mishra [17] and also in [18].

(3). $\phi_{-\alpha,0,\beta-1}^{0,l,\lambda} = \mathcal{Q}_{\beta,p}^{\alpha}$ $(\beta > -p)$, where $\mathcal{Q}_{\beta,p}^{\alpha}$ is the Liu - Owa operator (see in [11] and [3]). Also put $p = 1$, it is well known Jung - Kim - Srivastava operator [8].

(4). $\phi_{-1,0,\beta-1}^{0,l,\lambda} = \mathcal{J}_{\beta,p}$ $(\beta > -1)$, where $\mathcal{J}_{\beta,p}$ is the Bernardi integral operator (see [5]).

2 Definitions and Preliminaries

We denote by $\mathcal{P}$ the class of functions $\chi(z)$ given by

$$\chi(z) = 1 + c_1 z + c_2 z^2 + \cdots , \tag{2.10}$$

which are analytic in $\mathbb{U}$ and satisfy the following inequality $Re\{\chi(z)\} > 0$ for $z \in \mathbb{U}$.

Definition 2.4. *A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{S}^{m,l,\lambda}_{p,b}(\alpha, \beta, \gamma; \psi)$ if it satisfies the following subordination condition;*

$$1 + \frac{1}{b}\left(\frac{z\left(\phi^{m,l,\lambda}_{\alpha,\beta,\gamma} f(z)\right)'}{p\phi^{m,l,\lambda}_{\alpha,\beta,\gamma} f(z)} - 1\right) \prec \psi(z) \qquad (z \in \mathbb{U}; \psi \in \mathcal{P}) \tag{2.11}$$

where (and throughout this paper unless otherwise mentioned) the parameters p, γ, λ, b and β are constrained as follows:
$p \in \mathbb{N}$, $b \in \mathbb{C}^* = \mathbb{C}\setminus\{0\}$, $\gamma \in \mathbb{R}$, $\beta < p+1$, $-\infty < \alpha < \gamma + p + 1$ *and* $\lambda \geq 0$.

For the sake of conveniance, we set

$$\mathcal{S}^{m,l,\lambda}_{p,b}\left(\alpha, \beta, \gamma; \frac{1+Az}{1+Bz}\right) = \mathcal{S}^{m,l,\lambda}_{p,b}(\alpha, \beta, \gamma; A, B) \qquad (-1 \leq B < A \leq 1).$$

For $A = 1 - \frac{2\eta}{p}$, $B = -1$, we have

$$\mathcal{S}^{m,l,\lambda}_{p,b}\left(\alpha, \beta, \gamma; 1 - \frac{2\eta}{p}, B = -1\right) = \mathcal{S}^{m,l,\lambda}_{p,b}(\alpha, \beta, \gamma; \eta) \qquad (0 \leq \eta < 1).$$

In order to estabilish our main results, we shall require the following known lemmas:

Lemma 2.1. *[10] Let the function $\psi(z)$ be analytic and convex(univalent) in $\mathbb{U}$ with $\psi(0) = 1$. Suppose also that the function $\phi(z)$ given by*

$$\phi(z) = 1 + c_k z^k + c_{k+1} z^{k+1} + \cdots \tag{2.12}$$

is analytic in $\mathbb{U}$. If

$$\phi(z) + \frac{z\phi'(z)}{\nu} \prec \psi(z) \quad (\mathcal{R}(\nu) > 0\,; \nu \neq 0\,; z \in \mathbb{U}), \tag{2.13}$$

then

$$\phi(z) \prec q(z) = \frac{\nu}{k} z^{-\frac{\nu}{k}} \int_0^z \psi(t) t^{\frac{\nu}{k}-1} dt \prec \psi(z),$$

and $q(z)$ is the best dominant of (2.12).

Lemma 2.2. *[26] Let μ be a positive measure on the unit interval $[0, 1]$. Let $g(z, t)$ be a complex valued function defined on $\mathbb{U} \times [0, 1]$ such that $g(0, t)$ is analytic in $\mathbb{U}$ for each $t \in [0, 1]$ and such that $g(z, 0)$ is μ integrable on $[0, 1]$ for all $z \in \mathbb{U}$. In addition, suppose that $Re\{g(z, t)\} > 0$, $g(-r, t)$ is real and*

$$Re\left\{\frac{1}{g(z, t)}\right\} \geq \frac{1}{g(-r, t)} \qquad (|z| \leq r < 1; t \in [0, 1]).$$

If G is defined by $G(z) = \int_0^1 g(z, t) d\mu(t)$, then

$$Re\left\{\frac{1}{G(z)}\right\} \geq \frac{1}{G(-r)} \qquad (|z| \leq r < 1).$$

Lemma 2.3. *[15] Let φ be analytic in $\mathbb{U}$ with $\varphi(0) = 1$ and $\varphi(z) = 0$ for $0 < |z| < 1$ and let $A, B \in \mathbb{C}$ with $A \neq B$, $|B| \leq 1$.*

(i). Let $B \neq 0$ and $v \in \mathbb{C}^* = \mathbb{C}\setminus\{0\}$ satisfy either $\left|\frac{v(A-B)}{B} - 1\right| \leq 1$ or $\left|\frac{v(A-B)}{B} + 1\right| \leq 1$. If φ satisfies

$$1 + \frac{z\varphi'(z)}{v\varphi(z)} \prec \frac{1+Az}{1+Bz}, \tag{2.14}$$

then

$$\varphi(z) \prec (1 + Bz)^{v\left(\frac{A-B}{B}\right)},$$

and this is best dominant.

(ii). Let $B = 0$ and $\upsilon \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ be such that $|\upsilon A| < \pi$. If φ satisfies (2.14), then

$$\varphi(z) \prec e^{\upsilon A z}$$

and this is the best dominant.

Lemma 2.4. *[12] Let κ, $\tau \in \mathbb{C}$. Suppose that ϕ is convex and univalent in $\mathbb{U}$ with $\phi(0) = 1$ and $Re(\kappa\phi + \tau) > 0$. If the function g is analytic in $\mathbb{U}$ with $g(0) = 1$, then the subordination*

$$g(z) + \frac{zg'(z)}{\kappa g(z) + \tau} \prec \phi(z) \qquad (z \in \mathbb{U})$$

implies that

$$g(z) \prec \phi(z) \qquad (z \in \mathbb{U}).$$

Lemma 2.5. *[20] Let the function g be analytic in $\mathbb{U}$ with $g(0) = 1$ and $Re\{g(z)\} > \frac{1}{2}$. Then, for any function F analytic in $\mathbb{U}$, $(g * F)(\mathbb{U})$ is contained in the convex hull of $F(\mathbb{U})$.*

Lemma 2.6. *[25] For real and complex numbers a, n and c $(c \notin \mathbb{Z}_0^-)$*

$$\int_0^1 t^{n-1}(1-t)^{c-n-1}(1-tz)^{-a}dt = \frac{\Gamma(n)\Gamma(c-n)}{\Gamma(c)}\, {}_2F_1(a, n; c; z) \quad (Re\{n\}, Re\{c\} > 0), \tag{2.15}$$

$$ {}_2F_1(a, n; c; z) = (1-z)^{-a}\, {}_2F_1\left(a, c-n; c; \frac{z}{z-1}\right). \tag{2.16}$$

Motivated by the concept of Aouf et. al. [2], Huo Tang, Guan Tie Deng and Shu Hai Li [24] and Selvaraj et. al. [22], in this paper, we investigate some inclusion relations and other interesting properties for certain classes of p - valent functions involving an integral operator.

3 Inclusion Relationship

Theorem 3.1. *Let $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $l \geq 0$, $\lambda \geq 0$, $b = b_1 + ib_2 \neq 0$, $\tan\sigma = \frac{b_1}{b_2}$ and $\psi \in \mathcal{P}$ with $Im(\psi) < (Re(\psi) - 1)\cot\sigma$. Then*

$$\mathcal{S}_{p,b}^{m+1,l,\lambda}(\alpha, \beta, \gamma; \psi) \subset \mathcal{S}_{p,b}^{m,l,\lambda}(\alpha, \beta, \gamma; \psi) \tag{3.17}$$

Proof. Let $\mathcal{S}_{p,b}^{m+1,l,\lambda}(\alpha, \beta, \gamma; \psi)$ and suppose that

$$g(z) = 1 + \frac{1}{b}\left[\frac{z\left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z)\right)'}{p\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z)} - 1\right], \qquad (z \in \mathbb{U}) \tag{3.18}$$

where g is analytic in $\mathbb{U}$ with $g(0) = 1$. In view of (1.8) and (3.18), we obtain

$$(p+l)\frac{\phi_{\alpha,\beta,\gamma}^{m+1,l,\lambda} f(z)}{\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z)} = \lambda bp\,(g(z) - 1) + (p + l). \tag{3.19}$$

Differentiating (3.19) both sides with respect to z, and using (3.18), we get

$$1 + \frac{1}{b}\left[\frac{z\left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z)\right)'}{p\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z)} - 1\right] = g(z) + \frac{\lambda z g'(z)}{\lambda bp(g(z) - 1) + p + l}. \tag{3.20}$$

Since $Re(\lambda bp(\psi(z) - 1) + p + l) > 0$ for $Im(\psi) < (Re(\psi) - 1)\cot\sigma$ and where $\tan\sigma = \frac{b_1}{b_2}$, so applying Lemma 2.4 to (3.20), it follows that $g(z) \prec \psi(z)$, that is $f \in \mathcal{S}_{p,b}^{m,l,\lambda}(\alpha, \beta, \gamma; \psi)$. □

Taking $\psi(z) = \dfrac{1 + Az}{1 + Bz}$ in Theorem 3.1, we have the following corollary.

Corollary 3.1. *Let $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $l \geq 0$, $\lambda \geq 0$, $b = b_1 + ib_2 \neq 0$ and $-1 \leq B < A \leq 1$, then*

$$\mathcal{S}_{p,b}^{m+1,l,\lambda}(\alpha, \beta, \gamma; A, B) \subset \mathcal{S}_{p,b}^{m,l,\lambda}(\alpha, \beta, \gamma; A, B).$$

Remark 3.1. *If we put $m = 0$, $\alpha = \lambda$, $\beta = \mu$ and $\gamma = \eta$, then this result is reduced into the class of functions $M_p^\lambda(\mu, \eta; \gamma; \phi)$ which is studied by [24].*

4 Convolution properties

Now, we derive certain convolution properties for the function class $\mathcal{S}_{p,b}^{m,l,\lambda}(\alpha, \beta, \gamma; \psi)$.

Theorem 4.1. *Let $f \in \mathcal{S}_{p,b}^{m,l,\lambda}(\alpha, \beta, \gamma; \psi)$. Then*

$$f(z) = \left[z^p \cdot \exp\left(bp \int_0^z \frac{\psi(\omega(\xi)) - 1}{\xi} d\xi\right)\right] * \left(z^p + \sum_{n=1}^{\infty} \left(\frac{p+l+\lambda n}{p+l}\right)^{-m} \frac{(1+p-\beta)_n (1+p+\gamma-\alpha)_n}{(1+p)_n (1+p+\gamma-\beta)_n} z^{n+p}\right), \tag{4.21}$$

where ω is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in \mathbb{U})$.

Proof. Let $f \in \mathcal{S}_{p,b}^{m,l,\lambda}(\alpha, \beta, \gamma; \psi)$. From (2.11)

$$\frac{z\left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z)\right)'}{\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z)} = [\psi(\omega(z)) - 1]\, bp + p \tag{4.22}$$

where ω is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in \mathbb{U})$. By virtue of (4.22), we can easily find that

$$\frac{\left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z)\right)'}{\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z)} - \frac{p}{z} = \frac{[\psi(\omega(z)) - 1]\, bp}{z} \tag{4.23}$$

Integrating (4.23), we get

$$\begin{aligned} \log\left(\frac{\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z)}{z^p}\right) &= bp \int_0^z \frac{[\psi(\omega(\xi)) - 1]\, bp}{\xi} d\xi \\ \Rightarrow \phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z) &= z^p \cdot \exp\left[bp \int_0^z \frac{[\psi(\omega(\xi)) - 1]}{\xi} d\xi\right] \end{aligned} \tag{4.24}$$

Then, from (1.7) and (4.24), we deduce that the required assertion of the Theorem 4.1. □

Corollary 4.2. *Let $f \in \mathcal{S}_{p,b}^{m,l,\lambda}(\alpha, \beta, \gamma; A, B)$ with $-1 \le B < A \le 1$. Then*

$$f(z) = \left[z^p \cdot \exp\left(bp \int_0^z \frac{(A-B)(\omega(\xi)) - 1}{\xi} d\xi\right)\right] * \left(z^p + \sum_{n=1}^{\infty} \left(\frac{p+l+\lambda n}{p+l}\right)^{-m} \frac{(1+p-\beta)_n (1+p+\gamma-\alpha)_n}{(1+p)_n (1+p+\gamma-\beta)_n} z^{n+p}\right),$$

where ω is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in \mathbb{U})$.

Theorem 4.2. *Let $f \in \mathcal{A}_p$ and $\psi \in \mathcal{P}$. Then $f \in \mathcal{S}_{p,b}^{m,l,\lambda}(\alpha, \beta, \gamma; \psi)$ if and only if*

$$\begin{aligned} &\frac{1}{z^p}\left\{f * \left(pz^p + \sum_{n=1}^{\infty} \left[\frac{p+l+\lambda n}{p+l}\right]^m \frac{(n+p)(1+p)_n (1+p+\gamma-\beta)_n}{(1+p-\beta)_n (1+p+\gamma-\alpha)_n} z^{n+p}\right.\right. \\ &\left.\left. -p\left[(b\psi(e^{i\theta}) - 1) + 1\right] \times \left(z^p + \sum_{n=1}^{\infty} \left(\frac{p+l+\lambda n}{p+l}\right)^{-m} \frac{(1+p-\beta)_n (1+p+\gamma-\alpha)_n}{(1+p)_n (1+p+\gamma-\beta)_n} z^{n+p}\right)\right)\right\} \neq 0 \end{aligned} \tag{4.25}$$

$(z \in \mathbb{U}; 0 < \theta < 2\pi)$.

Proof. Suppose that $f \in \mathcal{S}_{p,b}^{m,l,\lambda}(\alpha, \beta, \gamma; \psi)$. We know that (2.11) holds true, which implies that

$$1 + \frac{1}{b}\left(\frac{z\left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z)\right)'}{p\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z)} - 1\right) \neq \psi(e^{i\theta}) \qquad (z \in \mathbb{U}; 0 < \theta < 2\pi). \tag{4.26}$$

One can easily verify that, from (4.26)

$$\frac{1}{z^p}\left\{z\left(\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}f(z)\right)' - p\left[(b\psi(e^{i\theta})-1)+1\right]\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}f(z)\right\} \neq 0 \qquad (z\in\mathbb{U}; 0<\theta<2\pi). \tag{4.27}$$

On the otherhand, we find from (1.7) that

$$z\left(\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}f(z)\right)' = pz^p + \sum_{n=1}^{\infty}\left[\frac{p+l+\lambda n}{p+l}\right]^m \frac{(n+p)(1+p)_n(1+p+\gamma-\beta)_n}{(1+p-\beta)_n(1+p+\gamma-\alpha)_n} a_{n+p}z^{n+p} \tag{4.28}$$

Combining (1.7), (4.27) and (4.28), we can easily get the convolution property (4.25) asserted by Theorem 4.2. □

5 Some properties of the operator $\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}$

Now we discuss some properties of the operator $\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}$.

Theorem 5.1. *Let $\sigma>0$, $\gamma\in\mathbb{R}$, $p\in\mathbb{N}\setminus\{1\}$, $-1\leq B<A\leq 1$ and the function $f\in\mathcal{A}_p$ satisfies the following subordination:*

$$(1-\sigma)\frac{\left(\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}f(z)\right)'}{pz^{p-1}} + \sigma\frac{\left(\phi^{m+1,l,\lambda}_{\alpha,\beta,\gamma}f(z)\right)'}{pz^{p-1}} \prec \frac{1+Az}{1+Bz}, \qquad (z\in\mathbb{U}). \tag{5.29}$$

Then

$$\frac{\left(\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}f(z)\right)'}{pz^{p-1}} \prec \psi(z) \prec \frac{1+Az}{1+Bz}, \tag{5.30}$$

where

$$\psi(z) = \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1+Bz)^{-1} \quad {}_2F_1\left(1,1;\frac{p+l}{k\sigma\lambda}+1;\frac{Bz}{1+Bz}\right) & for B\neq 0,\\ 1+\frac{p+l}{k\sigma\lambda+p+l}Az & for\ B=0. \end{cases} \tag{5.31}$$

is the best dominant of (5.30). Furthermore,

$$f\in\mathcal{S}^{m,l,\lambda}_{p,b}(\alpha,\beta,\gamma;\delta) \tag{5.32}$$

where

$$\delta = \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1-B)^{-1} \quad {}_2F_1\left(1,1;\frac{p+l}{k\sigma\lambda}+1;\frac{B}{B-1}\right) & for B\neq 0,\\ 1+\frac{p+l}{k\sigma\lambda+p+l}A & for\ B=0. \end{cases}$$

The result is best possible.

Proof. Let

$$g(z) = \frac{\left(\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}f(z)\right)'}{pz^{p-1}}, \tag{5.33}$$

where g is of the form (2.12) and is analytic in $\mathbb{U}$. Differentiating (5.33) with respect to z and making use of (1.9), we get

$$\begin{aligned}(1-\sigma)\frac{\left(\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}f(z)\right)'}{pz^{p-1}} + \sigma\frac{\left(\phi^{m+1,l,\lambda}_{\alpha,\beta,\gamma}f(z)\right)'}{pz^{p-1}} &= g(z) + \frac{\lambda\sigma z g'(z)}{p+l}\\ &\prec \frac{1+Az}{1+Bz}. \qquad (z\in\mathbb{U})\end{aligned}$$

Applying Lemma 2.1 and Lemma 2.6, we have

$$\begin{aligned}\frac{\left(\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}f(z)\right)'}{pz^{p-1}} &\prec \psi(z)\\ &= \frac{p+l}{k\sigma\lambda}z^{-\frac{p+l}{k\sigma\lambda}}\int_0^z t^{\frac{p+l}{k\sigma\lambda}-1}\left(\frac{1+At}{1+Bt}\right)dt\\ &= \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1+Bz)^{-1} \quad {}_2F_1\left(1,1;\frac{p+l}{k\sigma\lambda}+1;\frac{Bz}{1+Bz}\right) & for B\neq 0,\\ 1+\frac{p+l}{k\sigma\lambda+p+l}Az & for\ B=0. \end{cases}\end{aligned}$$

This proves the assertion (5.30) of Theorem 5.1. Next, inorder to prove the assertion (5.32), it suffices to prove that

$$\inf_{|z|<1}\{Re(\psi(z))\} = \psi(-1).$$

Indeed, we have

$$Re\frac{1+Az}{1+Bz} \geq \frac{1-Ar}{1-Br} \qquad (|z| = r < 1)$$

Setting

$$G(z,\zeta) = \frac{1+A\zeta z}{1+B\zeta z} \qquad \text{and} \qquad dv(\zeta) = \frac{p+l}{k\sigma\lambda}\zeta^{\frac{p+l}{k\sigma\lambda}-1}d\zeta \qquad (0 \leq \zeta \leq 1),$$

which is a positive measure on the closed interval [0, 1], we get

$$\psi(z) = \int_0^z G(z,\zeta)dv(\zeta).$$

Then

$$Re\{\psi(z)\} \geq \int_0^1 \frac{1-A\zeta r}{1-B\zeta r}dv(\zeta) = \psi(-r) \qquad (|z| = r < 1).$$

Letting $r \to 1^-$ in the above inequality, we obtain the assertion (5.32). Finally, the estimate (5.32) is best possible as ψ is the best dominant of (5.30). This completes the proof of the theorem. □

Theorem 5.2. *Let* $f \in \mathcal{S}_{p,b}^{m,l,\lambda}(\alpha, \beta, \gamma; \eta) \quad (0 \leq \eta < 1)$, *then*

$$Re\left\{(1-\sigma)\frac{\left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda}f(z)\right)'}{pz^{p-1}} + \sigma\frac{\left(\phi_{\alpha,\beta,\gamma}^{m+1,l,\lambda}f(z)\right)'}{pz^{p-1}}\right\} > \eta \qquad (|z| < R),$$

where

$$R = \left\{\frac{\sqrt{(p+l)^2+(\sigma\lambda k)^2}-\sigma\lambda k}{(p+l)}\right\}^{\frac{1}{k}}. \tag{5.34}$$

The result is best possible.

Proof. Let $f \in \mathcal{S}_{p,b}^{m,l,\lambda}(\alpha, \beta, \gamma; \eta)$, then we write

$$\frac{\left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda}f(z)\right)'}{pz^{p-1}} = \eta + (1-\eta)u(z) \qquad (z \in \mathbb{U}) \tag{5.35}$$

where u is of the form (2.12) and is analytic in $\mathbb{U}$. Differentiating (5.35) with respect to z, we have

$$\frac{1}{1-\eta}\left\{(1-\sigma)\frac{\left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda}f(z)\right)'}{pz^{p-1}} + \sigma\frac{\left(\phi_{\alpha,\beta,\gamma}^{m+1,l,\lambda}f(z)\right)'}{pz^{p-1}} - \eta\right\} = u(z) + \frac{\sigma\lambda z u'(z)}{(p+l)} \tag{5.36}$$

Applying the following well-knowing estimate [9]:

$$\frac{|zu'(z)|}{Re\{u(z)\}} \leq \frac{2kr^k}{1-r^{2k}} \qquad (|z| = r < 1),$$

in (5.36), we have

$$\begin{aligned}\frac{1}{1-\eta}Re&\left\{(1-\sigma)\frac{\left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda}f(z)\right)'}{pz^{p-1}} + \sigma\frac{\left(\phi_{\alpha,\beta,\gamma}^{m+1,l,\lambda}f(z)\right)'}{pz^{p-1}} - \eta\right\}\\ &\geq Re\{u(z)\}\left(1-\frac{2\sigma\lambda kr^k}{(p+l)(1-r^{2k})}\right),\end{aligned} \tag{5.37}$$

such that the right hand side of (5.37) is positive, if $r < R$, where R is given by (5.34). In order to show that the bound R is best possible, we consider the function $f \in \mathcal{A}_p$ defined by

$$\frac{\left(\phi^{m,l,\lambda}_{\alpha,\beta,\gamma} f(z)\right)'}{pz^{p-1}} = \eta + (1-\eta)\frac{1+z^k}{1-z^k} \qquad (0 \leq \eta < 1; z \in \mathbb{U}).$$

Note that

$$\frac{1}{1-\eta}\left\{(1-\sigma)\frac{\left(\phi^{m,l,\lambda}_{\alpha,\beta,\gamma} f(z)\right)'}{pz^{p-1}} + \sigma\frac{\left(\phi^{m+1,l,\lambda}_{\alpha,\beta,\gamma} f(z)\right)'}{pz^{p-1}} - \eta\right\} = \frac{(p+l)(1-z^{2k}) - 2\sigma\lambda k z^k}{(p+l)(1-z^{2k})} = 0, \tag{5.38}$$

for $z = Re^{\frac{i\pi}{k}}$. □

For a function $f \in \mathcal{A}_p$, the generalized Bernardi - Libera - Livingston integral operator $F_{c,p}$ is defined by

$$\begin{aligned} F_{c,p} f(z) &= \frac{c+p}{z^c}\int_0^z t^{c-1} f(t)dt = \left(z^p + \sum_{n=1}^{\infty}\frac{c+p}{c+p+n}z^{n+p}\right) * f(z) \\ &= z^p\, {}_2F_1(1, c+p; c+p+1; z) * f(z) \qquad (c > -p; z \in \mathbb{U}) \end{aligned} \tag{5.39}$$

From (1.7) and (5.39), we have

$$z\left(\phi^{m,l,\lambda}_{\alpha,\beta,\gamma} F_{c,p}(f(z))\right)' = (c+p)\phi^{m,l,\lambda}_{\alpha,\beta,\gamma} f(z) - c\phi^{m,l,\lambda}_{\alpha,\beta,\gamma} F_{c,p}(f(z)) \tag{5.40}$$

Theorem 5.3. *Let $f \in \mathcal{S}^{m,l,\lambda}_{p,b}(\alpha, \beta, \gamma; A, B)$ and $F_{c,p}$ be defined by (5.39). Then*

$$\frac{\left(\phi^{m,l,\lambda}_{\alpha,\beta,\gamma} F_{c,p}(f(z))\right)'}{pz^{p-1}} \prec \theta(z) \prec \frac{1+Az}{1+Bz}, \tag{5.41}$$

where

$$\theta(z) = \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1+Bz)^{-1} \quad {}_2F_1\left(1, 1; \frac{p+c}{k}+1; \frac{Bz}{1+Bz}\right) & for B \neq 0, \\ 1 + \frac{p+c}{k+p+c}Az & for\ B = 0. \end{cases} \tag{5.42}$$

is the best dominant of (5.41). Furthermore,

$$Re\frac{\left(\phi^{m,l,\lambda}_{\alpha,\beta,\gamma} F_{c,p}(f(z))\right)'}{pz^{p-1}} > \mu$$

where

$$\mu = \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1-B)^{-1} \quad {}_2F_1\left(1, 1; \frac{p+c}{k}+1; \frac{B}{B-1}\right) & for B \neq 0, \\ 1 + \frac{p+c}{k+p+c}Az & for\ B = 0. \end{cases}$$

The result is best possible.

Proof. Let

$$K(z) = \frac{\left(\phi^{m,l,\lambda}_{\alpha,\beta,\gamma} F_{c,p}(f(z))\right)'}{pz^{p-1}} \qquad (z \in \mathbb{U}). \tag{5.43}$$

where K is of the form (2.12) and is analytic in $\mathbb{U}$. Using (5.40) and (5.43) and differentiating the resulting equation with respect to z we have

$$\frac{\left(\phi^{m,l,\lambda}_{\alpha,\beta,\gamma} F_{c,p}(f(z))\right)'}{pz^{p-1}} = K(z) + \frac{zK'(z)}{p+c} \prec \frac{1+Az}{1+Bz}.$$

The remaining part of the proof is similar to that of Theorem 5.1 and so we omit it. □

Theorem 5.4. *Let $f, g \in \mathcal{A}_p$ satisfy the following inequality:*

$$Re\left(\frac{\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}g(z)}{z^p}\right) > 0 \qquad (z \in \mathbb{U}).$$

If

$$\left|\frac{\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}f(z)}{\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}g(z)} - 1\right| < 1 \qquad (z \in \mathbb{U}).$$

then

$$Re\left(\frac{z\left(\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}f(z)\right)'}{\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}f(z)}\right) > 0 \qquad (|z| < R_1; z \in \mathbb{U}),$$

where

$$R_1 = \left(\frac{-3k + \sqrt{9k^2 + 4p(p+k)}}{2(p+k)}\right)^{\frac{1}{k}}. \tag{5.44}$$

Proof. Let

$$\begin{aligned} q(z) &= \frac{\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}f(z)}{\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}g(z)} - 1 \\ &= c_k z^k + c_{k+1}z^{k+1} + \cdots, \end{aligned} \tag{5.45}$$

where $q(z)$ is analytic in $\mathbb{U}$ with $q(0) = 0$ and $|q(z)| \leq |z|^k$. Then, by applying the familiar Schwarz Lemma [14], we have $q(z) = z^k\chi(z)$, where χ is analytic in $\mathbb{U}$ and $\chi(z) \leq 1$.
From (5.45),

$$\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}f(z) = \phi^{m,l,\lambda}_{\alpha,\beta,\gamma}g(z)\left[1 + z^k\chi(z)\right] \tag{5.46}$$

Differentiating (5.46) logarithmically w.r.t z, we have

$$\frac{z\left(\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}f(z)\right)'}{\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}f(z)} = \frac{z\left(\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}g(z)\right)'}{\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}g(z)} + \frac{z^k\left[k\chi(z) + z\chi'(z)\right]}{1 + z^k\chi(z)}. \tag{5.47}$$

Letting

$$\omega(z) = \frac{\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}g(z)}{z^p} \qquad (z \in \mathbb{U}),$$

where ω is in the form (2.12) is analytic in $\mathbb{U}$, $Re\{\omega(z)\} > 0$ and

$$\frac{z\left(\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}g(z)\right)'}{\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}g(z)} = \frac{z\omega'(z)}{\omega(z)} + p$$

then we have

$$Re\left\{\frac{z\left(\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}f(z)\right)'}{\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}f(z)}\right\} \geq p - \left|\frac{z\omega'(z)}{\omega(z)}\right| - \left|\frac{z^k\left[k\chi(z) + z\chi'(z)\right]}{1 + z^k\chi(z)}\right| \tag{5.48}$$

Using the following known estimates [9],

$$\left|\frac{\omega'(z)}{\omega(z)}\right| \leq \frac{2kr^{k-1}}{1 - r^{2k}} \quad \textit{and} \quad \left|\frac{k\chi(z) + z\chi'(z)}{1 + z^k\chi(z)}\right| \leq \frac{k}{1 - r^k} \quad (|z| = r < 1),$$

in (5.48), we have

$$Re\left\{\frac{z\left(\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}f(z)\right)'}{\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}f(z)}\right\} \geq \frac{p - 3kr^k - (p+k)r^{2k}}{1 - r^{2k}} \tag{5.49}$$

which is certainly positive, provided that $r < R_1$, where R_1 is given by (5.44). □

Theorem 5.5. *Let $f \in \mathcal{S}^{m,l,\lambda}_{p,b}(\alpha, \beta, \gamma; A, B)$ and $g \in \mathcal{A}_p$ satisfy the following inequality:*

$$Re\left\{\frac{g(z)}{z^p}\right\} > \frac{1}{2} \qquad (z \in \mathbb{U}) \tag{5.50}$$

then $(f * g)(z) \in \mathcal{S}^{m,l,\lambda}_{p,b}(\alpha, \beta, \gamma; A, B)$.

Proof. We have

$$\frac{(\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}(f*g)(z))'}{pz^{p-1}} = \frac{(\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}(f)(z))'}{pz^{p-1}} * \frac{g(z)}{z^p} \qquad (z \in \mathbb{U}),$$

where g satisfies (5.50) and $\frac{1+Az}{1+Bz}$ is convex in $\mathbb{U}$. By using (5.30) and applying Lemma 2.5, we get the required assertion of this theorem. □

Theorem 5.6. *Let $\vartheta \in \mathbb{C}^*$ and $A, B \in \mathbb{C}$ with $A \neq B$ and $|B| \leq 1$. Suppose that*

$$\left|\frac{\vartheta(p+l)(A-B)}{\lambda B} - 1\right| \leq 1 \quad or \quad \left|\frac{\vartheta(p+l)(A-B)}{\lambda B} + 1\right| \leq 1, if B \neq 0,$$

$$\left|\frac{\vartheta(p+l)}{\lambda}A\right| \leq \pi, if B = 0.$$

If $f \in \mathcal{A}_p$ with $\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}f(z) \neq 0$ for all $z \in \mathbb{U}^* = \mathbb{U} \setminus \{0\}$, then

$$\frac{\phi^{m+1,l,\lambda}_{\alpha,\beta,\gamma}f(z)}{\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}f(z)} \prec \frac{1+Az}{1+Bz},$$

implies $\left(\frac{\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}f(z)}{z^p}\right)^{\vartheta} \prec g_1(z)$. where

$$g_1(z) = \begin{cases} (1+Bz)^{\frac{\vartheta(p+l)(A-B)}{\lambda B}} & for B \neq 0, \\ e^{\frac{\vartheta(p+l)}{\lambda}Az} & for\ B = 0, \end{cases}$$

is the best dominant.

Proof. Let

$$\varphi(z) = \left(\frac{\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}f(z)}{z^p}\right)^{\vartheta} \qquad (z \in \mathbb{U}). \tag{5.51}$$

Then φ is analytic in $\mathbb{U}$, $\varphi(0) = 1$ and $\varphi(z) \neq 0$ for all $\mathbb{U}$. Taking the logrithamic differentiation on both sides of (5.51) and using the identity (1.8), we obtain

$$1 + \frac{\lambda z\varphi'(z)}{\vartheta(p+l)\varphi(z)} = \frac{\phi^{m+1,l,\lambda}_{\alpha,\beta,\gamma}f(z)}{\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}f(z)} \prec \frac{1+Az}{1+Bz}.$$

Now the assertions of Theorem 5.6 follows from Lemma 2.3. □

Taking $B = -1$ and $A = 1 - 2\eta$, $0 \leq \eta < 1$ in Theorem 5.6, we get the following corollary:

Corollary 5.3. *Let $\vartheta \in \mathbb{C}^*$ satisfies either*

$$\left|\frac{2\vartheta(p+l)(1-\eta)}{\lambda} - 1\right| \leq 1 \quad or \quad \left|\frac{2\vartheta(p+l)(1-\eta)}{\lambda} + 1\right| \leq 1.$$

If $f \in \mathcal{A}_p$ with $\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}f(z) \neq 0$ for all $z \in \mathbb{U}^*$, then

$$Re\left\{\frac{\phi^{m+1,l,\lambda}_{\alpha,\beta,\gamma}f(z)}{\phi^{m,l,\lambda}_{\alpha,\beta,\gamma}f(z)}\right\} \prec \frac{1+Az}{1+Bz}$$

implies

$$\left(\frac{\phi^{m,l,\lambda}_{\alpha,\beta,\gamma} f(z)}{z^p}\right)^{\vartheta} \prec g_1(z)$$

where

$$g_1(z) = (1-z)^{\frac{-2\vartheta(p+l)(1-\eta)}{\lambda}}$$

is the best dominant.

Theorem 5.7. *Let $\sigma > 0$, $\epsilon > 0$, $-1 \leq B < A \leq 1$ and the function $f \in \mathcal{A}_p$ satisfies the following subordination:*

$$(1-\sigma)\frac{\phi^{m,l,\lambda}_{\alpha,\beta,\gamma} f(z)}{z^p} + \sigma\frac{\left(\phi^{m,l,\lambda}_{\alpha,\beta,\gamma} f(z)\right)'}{pz^{p-1}} \prec \frac{1+Az}{1+Bz}, \qquad (z \in \mathbb{U}). \tag{5.52}$$

Then

$$Re\left\{\frac{\phi^{m,l,\lambda}_{\alpha,\beta,\gamma} f(z)}{z^p}\right\}^{\frac{1}{\epsilon}} > \delta^{\frac{1}{\epsilon}}, \tag{5.53}$$

where

$$\delta = \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1-B)^{-1} \quad {}_2F_1\left(1, 1; \frac{p}{k\sigma}+1; \frac{B}{B-1}\right) & for B \neq 0, \\ 1 + \frac{p}{k\sigma+p}A & for\ B = 0. \end{cases}$$

The result is best possible.

Proof. Let

$$G(z) = \frac{\phi^{m,l,\lambda}_{\alpha,\beta,\gamma} f(z)}{z^p}, \tag{5.54}$$

where G is of the form (2.12) and is anlytic in $\mathbb{U}$. Differentiating (5.54) with respect to z, we get

$$\begin{aligned}(1-\sigma)\frac{\phi^{m,l,\lambda}_{\alpha,\beta,\gamma} f(z)}{z^p} + \sigma\frac{\left(\phi^{m,l,\lambda}_{\alpha,\beta,\gamma} f(z)\right)'}{pz^{p-1}} &= G(z) + \frac{\sigma z G'(z)}{p} \\ &\prec \frac{1+Az}{1+Bz}. \qquad (z \in \mathbb{U})\end{aligned}$$

Now, applying similar steps invoved in Theorem 5.1 and using the elementary inequality

$$Re\{\Omega^{\kappa}\} \geq Re\{\Omega\}^{\kappa} \qquad (Re\{\Omega\} > 0; \kappa \in \mathbb{N}),$$

we obtain the required result. □

Remark 5.2. *Taking $m = 0$ and the choices of α, β and γ , this subclass is reduced into the class $S^{\lambda}_{p,n}(A, B)$ which is studied by A. O. Mostafa and M.K.Aouf [13].*

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