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Qualitative behavior of rational difference equations of higher order

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Abstract

In this paper we study the behavior of the solution of the following rational difference equation

$$x_{n+1}=\frac{ax_{n-r}^2+bx_{n-l}x_{n-k}^2}{cx_{n-r}^2+dx_{n-l}x_{n-k}^2}\quad n=0,1,...,$$

where the parameters a,b,c and d are positive real numbers and the initial conditions $x_{-t},x_{-t+1},...,x_{-1}$ and x_0 are posistive real numbers where $t=\max\{r,k,l\}$.

Keywords: stability, rational difference equation, global attractor, periodic solution.

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1 Introduction

In the past two decades, the study of Difference Equations has been growing continuously. This is largely due to the fact that difference equations appear as mathematical models describing real life situations in probability theory, queuing theory, statistical problems, stochastic time series, combinatorial analysis, number theory, geometry, electrical network, quanta in radiation, genetics in biology, economics, psychology, sociology, etc. Moreover, difference equations also appear in the study of discretization schemes for nonlinear differential equations. The need for a discretization of nonlinear differential equations arises from the fundamental realization that nonlinear systems generally do not have analytic solutions expressible in terms of a finite representation of elementary functions. In fact, now it occupies a central position in applicable analysis and will no doubt continue to play an important role in mathematics as a whole. Our objective in this paper is to investigate the global stability character, boundedness and the periodicity of solutions of the rational difference equation

$$x_{n+1}=\frac{ax_{n-r}^2+bx_{n-l}x_{n-k}^2}{cx_{n-r}^2+dx_{n-l}x_{n-k}^2}\quad n=0,1,..., \tag{1.1}$$

where the parameters a,b,c and d are positive real numbers and the initial conditions $x_{-t},x_{-t+1},...,x_{-1}$ and x_0 are positive real numbers where $t=\max\{r,k,l\}$.

Recently there has a lot of interest in studying the global attractivity, the boundedness character and the periodicity nature of nonlinear difference equations for example ([1], [2], [3], [4], [5], [6], [7], [8], [9]).

Many researchers studied qualitative behaviors of the solution of difference equations for example; in [5] Elabbasy et al studied the global stability character, boundedeness and the periodicity of solutions of the difference equation

$$x_{n+1}=\frac{\alpha x_n+\beta x_{n-1}+\gamma x_{n-2}}{Ax_n+Bx_{n-1}+Cx_{n-2}}.$$

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Elabbasy et al. [4] analyzed the global stability, periodicity character and gave the solution of special cases of the difference equation

$$x_{n+1} = \frac{dx_{n-l}x_{n-k}}{cx_{n-s} - b} + a.$$

Wang et al. [24] studied the global attractivity of equilibrium points and the asymptotic behavior of the solutions of the solutions of the difference equation

$$x_{n+1} = \frac{ax_{n-l}x_{n-k}}{\alpha + bx_{n-s} + cx_{n-t}}.$$

Saleh and Baha [17] investigated the behavior of nonlinear rational difference equation

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-k}}{Bx_n + Cx_{n-k}}.$$

Yan, Li and Zhao [26] studied boundedeness, periodic character, invariant intervals and the global asymptotic stability of the all nonnegative solutions of the difference equation

$$x_{n+1} = \frac{ax_n + bx_{n-k}}{A + Bx_n}.$$

See also ([10],[11],[12],[13],[14],[15],[16],[17],[18]). Other related results can be found in ([19],[20],[21],[22],[23],[24],[25],[27],[28],[29],[30],[31],[32]).Let us introduce some basic definitions and some theorems that we need sequel.

Let I be some interval of real numbers and let

$$f : I^{k+1} \to I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ..., \tag{1.2}$$

Definition 1.1. *([13], [16])(Equilibrium Point) A point $\bar{x} \in I$ is called an equilibrium point of Eq.*(1.2) *if*

$$\bar{x} = F(\bar{x}, \bar{x}, ..., \bar{x}).$$

Definition 1.2. *([13], [16])The difference equation* (1.2) *is said to be persistence if there exist numbers m and M with $0 < m \leq M < \infty$ such that for any initial conditions $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in (0, \infty)$ there exists a positive integer N which depends on the initial conditions such that*

$$m \leq x_0 \leq M \quad \textit{for all } n \geq N.$$

Definition 1.3. *([13], [16]) Stability*

(a) The equilibrium point $\bar{x}$ of Eq.(1.2) is called stable (or locally stable) if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\| x_0 - \bar{x} \| < \delta$ implies $\| x_n - \bar{x} \| < \epsilon$ for $n \geq 0$. Otherwise the equilibrium $\bar{x}$ is called unstable.

(b) The equilibrium point $\bar{x}$ of Eq.(1.2) is called asymptotically stable (or locally asymptotically stable) if it stable and there exists $\gamma > 0$ such that $\| x_0 - \bar{x} \| < \gamma$ implies

$$\lim_{n\to\infty} \| x_n - \bar{x} \| = 0.$$

(c) The equilibrium point $\bar{x}$ of Eq.(1.2) is called globally asymptotically stable if it is asymptotically stable, and if every x_0,

$$\lim_{n\to\infty} \| x_n - \bar{x} \| = 0.$$

(d) The equilibrium point $\bar{x}$ of Eq.(1.2) is called globally asymptotically stable relative to a set $s \subset \mathbb{R}^{k+1}$ if it is asymptotically stable, and if for every $x_0 \in s$,

$$\lim_{n\to\infty} \| x_n - \bar{x} \| = 0.$$

(e) The equilibrium point $\bar{x}$ of Eq.(1.2) is said to be a global attractor with basin of attraction a set $s \subset \mathbb{R}^{k+1}$ if

$$\lim_{n\to\infty} x_n = \bar{x}$$

for every solution with $x_0 \in s$.

Theorem 1.1. *([13], [16])Assume $p, q \in \mathbb{R}$. Then a necessary and sufficient for the asymptotic stability of the difference equation*

$$x_{n+2} + px_{n+1} + qx_n = 0, \ \ n = 0, 1, ... \tag{1.3}$$

is that

$$| \ p \ |< 1 + q < 2.$$

Theorem 1.2. *([13], [16])Assume that $p, q \in \mathbb{R}$ and $k \in \{0, 1, ...\}$. Then*

$$| \ p \ | + | \ q \ |< 1$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} + px_n + qx_{n-k} = 0, \quad n = 0, 1, ... \tag{1.4}$$

Theorem 1.3. *([13], [16])Assume $p_1, ..., p_k \in \mathbb{R}$ and $k \in \{1, 2, ...\}$. Then the difference equation*

$$x_{n+k} + p_1 x_{n+k-1} + ... + p_k x_n = 0$$

is asymptotically stable provided that

$$\sum_{i=1}^{k} | \ p_i \ |< 1$$

Remark 1.1. *([12], [13])The Linear equation*

$$x_{n+1} - x_n + \sum_{i=1}^{m} p_i x_{n-k_i} = 0, \quad n = 0, 1, 2, ... \tag{1.5}$$

where $p_1, ..., p_m \in (0, \infty)$ and $k_1, ...k_m$ are positive integers, is asymptotically stable provided that

$$\sum_{i=1}^{m} k_i p_i < 1.$$

([12], [13])Periodicity (a) A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if

$$x_{n+p} = x_n \quad \text{for } n \geq -k. \tag{1.6}$$

The theory of Full Limiting Sequences was indicated in [15]. The following theorem was given in [5].

Theorem 1.4. *([12], [13])Let $F \in [I^{k+1}, I]$ for some interval I of real numbers and for some non-negative integer k, and consider the difference equation*

$$x_{n+1} = F(x_n, x_{n-1}..., x_{n-k}), \tag{1.7}$$

Let l_0 be a limit point of the sequence $\{x_n\}_{n=-k}^{\infty}$. Then the following statements are true.

(i) There exists a solution $\{L_n\}_{n=-\infty}^{\infty}$ of Eq.(1.7), called a full limiting sequence of $\{x_n\}_{n=-k}^{\infty}$, such that $L_0 = l_0$, and such that for every $N \in \{..., -1, 0, 1, ...\}$ L_N is a limit point of $\{x_n\}_{n=-k}^{\infty}$.

(ii) For every $i_0 \leq -k$, there exists a subsequence $\{x_i\}_{i=-0}^{\infty}$ of $\{x_n\}_{n=-k}^{\infty}$ such that

$$L_N = \lim_{i\to\infty} x_{r_i+N} \ \ \textit{for all} \ \ N \geq i_0.$$

2 Local Stability of the Equilibrium Point

In this section we investigate the local stability character of the solutions of Eq.(1.1). Eq.(1.1) has an equilibrium points are given by

$$\begin{aligned}\bar{x} &= f(\bar{x},\bar{x}) \\ &= \frac{a\bar{x}^2 + b\bar{x}^3}{c\bar{x}^2 + d\bar{x}^3} \\ &= \frac{a+b\bar{x}}{c+d\bar{x}}\end{aligned}$$

Then Eq.(1.1) has an equilibrium points $\bar{x} = \frac{b-c\pm\sqrt{(b-c)^2+4ad}}{2d}$.

Let $f : (0,\infty)^2 \to (0,\infty)$ be a function defined by

$$f(u,v,w) = \frac{au^2 + bvw^2}{cu^2 + dvw^2}. \tag{2.8}$$

Therefore it follows that

$$\begin{aligned}f_u(u,v,w) &= \frac{2uvw^2(ad-bc)}{(cu^2+dvw^2)^2}, \\ f_v(u,v,w) &= -\frac{u^2w^2(ad-bc)}{(cu^2+dvw^2)^2}, \\ f_w(u,v,w) &= -\frac{2u^2vw(ad-bc)}{(cu^2+dvw^2)^2}\end{aligned}$$

we see that

$$\begin{aligned}f_u(\bar{x},\bar{x},\bar{x}) &= \frac{2(ad-bc)}{(c+d\bar{x})^2} = -c_0, \\ f_v(\bar{x},\bar{x},\bar{x}) &= -\frac{(ad-bc)}{(c+d\bar{x})^2} = -c_1, \\ f_w(\bar{x},\bar{x},\bar{x}) &= -\frac{2(ad-bc)}{(c+d\bar{x})^2} = -c_2.\end{aligned}$$

At $\bar{x} = \frac{b-c+\sqrt{(b-c)^2+4ad}}{2d}$, one has $(c+d\bar{x})^2 = \frac{1}{4}(b+c+\sqrt{(b-c)^2+4ad})$.

Thus

$$\begin{aligned}f_u(\bar{x},\bar{x},\bar{x}) &= \frac{8(ad-bc)}{(b+c+\sqrt{(b-c)^2+4ad})} = -c_0, \\ f_v(\bar{x},\bar{x},\bar{x}) &= -\frac{4(ad-bc)}{(b+c+\sqrt{(b-c)^2+4ad})} = -c_1, \\ f_w(\bar{x},\bar{x},\bar{x}) &= -\frac{8(ad-bc)}{(b+c+\sqrt{(b-c)^2+4ad})} = -c_2.\end{aligned}$$

Then the linearized equation of Eq.(1.1) about is $\bar{x}$ is

$$y_{n+1} + c_0y_{n-r} + c_1y_{n-l} + c_2y_{n-k} = 0 \tag{2.9}$$

Theorem 2.5. *Assume that*

$$20 \mid (ad-bc) \mid < b+c+\sqrt{(b-c)^2+4ad}.$$

Then the positive equilibrium point $\bar{x} = \frac{b-c+\sqrt{(b-c)^2+4ad}}{2d}$ *of Eq.(1.1) is locally asymptotically stable.*

Proof. It follows by Theorem 1.3 that, Eq.(1.1) is asymptotically stable if

$$\mid c_0 \mid + \mid c_1 \mid + \mid c_2 \mid < 1$$

$$\mid \frac{8(ad-bc)}{(b+c+\sqrt{(b-c)^2+4ad})} \mid + \mid \frac{4(ad-bc)}{(b+c+\sqrt{(b-c)^2+4ad})} \mid + \mid \frac{8(ad-bc)}{(b+c+\sqrt{(b-c)^2+4ad})} \mid < 1,$$

or

$$20 \mid (ad-bc) \mid < b+c+\sqrt{(b-c)^2+4ad}.$$

The proof is complete. □

3 Existence of Periodic Solutions

In this section we study the existence of prime period two solutions of Eq.(1.1).

Theorem 3.6. (i) *Let* r, l, k *odd, then Eq.*(1.1) *has a prime period two solution for all* $a, b, c, d \in \mathbb{R}^+$.
(ii) *Let* r, k *even,* l *odd, then Eq.*(1.1) *has a prime period two solution for all* $a, b, c, d \in \mathbb{R}^+$.

Proof. We will prove the theorem when Case (i) is true. The proof of Case (ii) is similar.
First suppose that there exists a prime period two solution

$$..., p, q, p, q, ...,$$

of Eq.(1.1).
We see from Eq.(1.1) that

$$p = \frac{ap^2+bp^3}{cp^2+dp^3} = \frac{a+bp}{c+dp},$$

and

$$q = \frac{aq^2+bq^3}{cq^2+dq^3} = \frac{a+bq}{c+dq}.$$

Then

$$cp + dp^2 = a + bp \tag{3.10}$$

and

$$cq + dq^2 = a + bq \tag{3.11}$$

Subtracting (3.10) from (3.11) gives

$$c(p-q) + d(p^2-q^2) = b(p-q).$$

Since $p \neq q$, it follows that

$$p + q = \frac{b-c}{d}. \tag{3.12}$$

Also, since p and q are positive, $(b-c)$ should be positive.
Again, adding (3.10) and (3.11) yields

$$c(p+q) + d(p^2+q^2) = 2a + b(p+q). \tag{3.13}$$

It follows by (3.12), (3.13) and the relation

$$p^2 + q^2 = (p+q)^2 - 2pq \quad \text{for all} \ \ p, q \in \mathbb{R},$$

that

$$pq = -\frac{a}{d}. \tag{3.14}$$

It is clear that p and q are two real distinct roots of quadratic equation given by:

$$dt^2 - (b-c)t - a = 0,$$

for all $a, b, c, d \in \mathbb{R}^+$.

Second suppose that $a, b, c, d \in \mathbb{R}^+$.We will show that Eq.(1.1) has prime period two solutions.
Assume that

$$p = \frac{(b-c) + \sqrt{(b-c)^2 + 4ad}}{2d},$$

and

$$q = \frac{(b-c) - \sqrt{(b-c)^2 + 4ad}}{2d}$$

Therefore p and q are distinct real numbers.
Set

$$x_{-t} = p, x_{-t+1} = q, \cdots, x_{-1} = p, x_0 = q.$$

We wish to show that

$$x_1 = x_{-1} = p \text{ and } x_2 = x_0 = q.$$

It follows from Eq.(1.1) that

$$x_1 = \frac{a + bp}{c + dp} = p$$

Similarly we see that

$$x_2 = q.$$

Then Eq.(1.1) has the prime period two solution

$$..., p, q, p, q, ...,$$

where p and q are distinct roots of a quadratic equation and the proof is complete. □

4 Global Attractor of the Equilibrium Point of Eq.(1.1)

In this section we investigate the global attractivity character of solutions of Eq.(1.1).

Lemma 4.1. *For any values of the quotient $\frac{a}{c}$ and $\frac{b}{d}$, the function $f(u, v, w)$ defined by Eq.(2.8) is monotone in each of its three arguments.*

Theorem 4.7. *The equilibrium point $\bar{x}$ of Eq.(1.1) is global attractor if one of the following statments hold:*
(i) $ad \geq bc$ and $4c(\frac{b}{d})^4 - 4a(\frac{b}{d})^3 > -(b+c)(\frac{a}{c})^4$.
(ii) $ad \leq bc$ and $5d(\frac{a}{c})^4 - 4b(\frac{a}{c})^3 > -a(\frac{b}{d})^2$.

Proof. Let $\{x_n\}_{n=-t}^{\infty}$ be solution of Eq.(1.1) and again let f be function defined by Eq.(2.8).
We will prove the theorem when Case (i) is true. The proof of Case (ii) is similar. In case of (i), when $ad \geq bc$, the function $f(u, v, w)$ is non-decreasing in u and non-increasing in v, w. Thus from Eq.(1.1), we see that

$$x_{n+1} = \frac{ax_{n-r}^2 + bx_{n-l}x_{n-k}^2}{cx_{n-r}^2 + dx_{n-l}x_{n-k}^2} \leq x_{n+1} = \frac{ax_{n-r}^2 + b(0)}{cx_{n-r}^2 + d(0)} = \frac{a}{c}.$$

Then

$$x_n \leq \frac{a}{c} = H \text{ for all } n \geq 1. \tag{4.15}$$

$$\begin{aligned} x_{n+1} &= \frac{ax_{n-r}^2 + bx_{n-l}x_{n-k}^2}{cx_{n-r}^2 + dx_{n-l}x_{n-k}^2} \geq x_{n+1} = \frac{a(0) + bx_{n-l}x_{n-k}^2}{c(0) + dx_{n-l}x_{n-k}^2} \\ &\geq \frac{b}{d} = h \text{ for all } n \geq 1. \end{aligned} \tag{4.16}$$

Then from Eq.(4.15) and Eq.(4.16), we see that

$$0 < h = \frac{b}{d} \leq x_n \leq \frac{a}{c} = H \quad \text{for all } n \geq 1.$$

let $\{x_n\}_{n=0}^{\infty}$ be solution of Eq.(1.1) with

$$I = \liminf_{n\to\infty} x_n \text{ and } S = \limsup_{n\to\infty}.$$

We want to show that $I = S$.

Now it follows from Eq.(1.1) that

$$I \geq f(I, S, S),$$

or

$$I \geq \frac{aI^2 + bS^3}{cI^2 + dS^3}.$$

and so

$$aI^2 + bS^3 - cI^3 \leq dIS^3. \tag{4.17}$$

Similarly, we see from Eq.(1.1) that

$$S \leq f(S, I, I),$$

or

$$S \leq \frac{aS^2 + bI^3}{cS^2 + dI^3},$$

and so

$$aS^2 + bI^3 - cS^3 \geq dSI^3. \tag{4.18}$$

Therefore it follows from Eq.(4.17) and Eq.(4.18) that

$$aI^4 + bI^2S^3 - cI^5 \leq dI^3S^3 \leq aS^4 + bI^3S^2 - cS^5$$

$$c(I^5 - S^5) + bI^2S^2(I - S) - a(I^4 - S^4) \geq 0,$$

if and only if

$$(I - S)[c(I^4 + I^3S + I^2S^2 + IS^3 + S^4) + bI^2S^2 - a(I + S)(I^2 + S^2)] \geq 0,$$

and so $I \geq S$ if

$$c(I^4 + I^3S + I^2S^2 + IS^3 + S^4) + bI^2S^2 - a(I + S)(I^2 + S^2) \geq 0. \tag{4.19}$$

Inequality (4.19) can be written as:

$$c(I^4 + I^3S + IS^3 + S^4) + (b + c)I^2S^2 - a(I + S)(I^2 + S^2) \geq 0.$$

To prove Inequality (4.19), let us consider

$$\tau = c(I^4 + I^3S + IS^3 + S^4) - a(I + S)(I^2 + S^2)$$

Then, one has

$$\begin{aligned} \tau &\geq 4c(\frac{b}{d})^4 - 4a(\frac{b}{d})^3 \\ &\geq -(b + c)(\frac{a}{c})^4 \\ &\geq -(b + c)I^2S^2, \end{aligned}$$

and so it follows that

$$I \geq S.$$

Therefore

$$I = S.$$

This complete the proof. $\square$

5 Boundedness of Solutions of Eq.(1.1)

In this section we study the boundedness of solutions of Eq.(1.1)

Theorem 5.8. *Every solution of Eq.*(1.1) *is bounded and persists.*

Proof. Let $\{x_n\}_{n=-t}^{\infty}$ be a solution of Eq.(1.1). Then

$$\begin{aligned} x_{n+1} &= \frac{ax_{n-r}^2 + bx_{n-l}x_{n-k}^2}{cx_{n-r}^2 + dx_{n-l}x_{n-k}^2} \\ & \frac{ax_{n-r}^2}{cx_{n-r}^2 + dx_{n-l}x_{n-k}^2} + \frac{bx_{n-l}x_{n-k}^2}{cx_{n-r}^2 + dx_{n-l}x_{n-k}^2} \\ &\leq \frac{ax_{n-r}^2}{cx_{n-r}^2} + \frac{bx_{n-l}x_{n-k}^2}{dx_{n-l}x_{n-k}^2} \\ &= \frac{a}{c} + \frac{b}{d}. \end{aligned}$$

Thus $x_N \leq \frac{a}{c} + \frac{b}{d} = M$ for all $N \geq 1$.

Let there exists $m > 0$ such that $x_N \geq m$ for all $N \geq 1$. Taking $x_N = \frac{1}{y_N}$, then one has

$$\begin{aligned} y_{n+1} &= \frac{cy_{n-r}^2 + dy_{n-l}y_{n-k}^2}{ay_{n-r}^2 + by_{n-l}y_{n-k}^2} \\ & \frac{cy_{n-r}^2}{ay_{n-r}^2 + by_{n-l}y_{n-k}^2} + \frac{dy_{n-l}y_{n-k}^2}{ay_{n-r}^2 + by_{n-l}y_{n-k}^2} \\ &\leq \frac{cy_{n-r}^2}{ay_{n-r}^2} + \frac{dy_{n-l}y_{n-k}^2}{by_{n-l}y_{n-k}^2} \\ &= \frac{c}{a} + \frac{d}{b}. \end{aligned}$$

Thus $x_N = \frac{1}{y_N} \geq \frac{1}{h} = \frac{ab}{ad+bc} = m$ for all $N \geq 1$. Hence, $m \leq x_N \leq M$ for all $N \geq 1$. □

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