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Spectrum of fuzzy prime filters of a 0 - distributive lattice

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Abstract

Stone's topology on the set of fuzzy prime filters of a bounded 0 - distributive lattice is introduced and many properties of this space of fuzzy prime filters are furnished.

Keywords: fuzzy sublattice, fuzzy filter, fuzzy prime filter, 0-distributive lattice.

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1 Introduction

For topological concepts which have now become commonplace the reader is referred to [5] and for lattice theoretic concepts the reader is referred to [4]. Venkatanarasimhan [15] has studied the Stone's space of prime filters for a pseudocomplemented lattice. The concept of a 0-distributive lattice introduced by Varlet [13] is a generalization of a distributive lattice and a pseudo-complemented lattice. A 0-distributive lattice is a lattice L with 0 in which for all $a, b, c \in L$, $a \wedge b = 0 = a \wedge c$ implies $a \wedge (b \vee c) = 0$. In [9], [2] authors have studied Stone's topology on set of prime filters of a 0 - distributive lattice. Such a study of prime spectrum plays an important role in the field of lattice theory.

Fuzzy set theory introduced by Zadeh [16] is generalization of classical set theory. After the inception of the notion fuzzy sets, Rosenfield started the pioneering work in the domain of fuzzification of algebraic objects viz fuzzy groups [11]. Many researchers have applied this concept to mathematical branches such as semi-group, ring, semi-ring, field, near ring, lattice etc. In particular while fuzzifying the notions in lattice theory, Bo et al [3] and Swami et al [12] have laid down the foundation for fuzzy ideals, fuzzy filters of a lattice. In [10], we have introduced and studied spectrum of L - fuzzy prime ideals of a bounded distributive lattice. In this paper our aim is to introduce Stone's topology τ on the set of fuzzy prime filters Σ of a bounded 0 - distributive lattice L and study many properties of the space $Fspec\,(L) = (\Sigma, \tau)$. Mainly we prove $Fspec\,(L)$ is compact and it contains a subspace homeomorphic with the spectrum of L which is dense in it. If L and L' are isomorphic bounded 0 - distributive lattices, $Fspec\,(L)$ and $Fspec\,(L')$ are homeomorphic.

2 Preliminaries

In this article we collect basic definitions and results which are used in subsequent sections.

Let $L = \langle L, \wedge, \vee \rangle$ be a bounded lattice.

Definition 2.1. *A fuzzy subset of L is a map of L into $\langle [0,1], \wedge, \vee \rangle$, where $\alpha \wedge \beta = min(\alpha, \beta)$ and $\alpha \vee \beta = max(\alpha, \beta)$ for all $\alpha, \beta \in [0,1]$. Let μ be a fuzzy subset of L. For $\alpha \in [0,1]$, the set $\mu_\alpha = \{x \in L : \mu(x) \geq \alpha\}$ is called α - cut (or α - level set) of μ.*

Definition 2.2. *A fuzzy subset μ of L is said to be a fuzzy sublattice of L if for all $x, y \in L$, $\mu\,(x \wedge y) \wedge \mu\,(x \vee y) \geq \mu\,(x) \wedge \mu(y)$.*

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Definition 2.3. *A monotonic fuzzy sublattice is a fuzzy filter of L. Here μ is monotonic means $\mu(x) \leq \mu(y)$ whenever $x \leq y$ in L.*

Definition 2.4. *The smallest fuzzy filter containing fuzzy subset μ of L is called fuzzy filter generated by μ and is denoted by $\langle\mu\rangle$. Here by a fuzzy subset σ contains a fuzzy subset μ we mean $\mu(a) \leq \sigma(a), \forall a \in L$ and will be denoted by $\mu \subseteq \sigma$.*

Definition 2.5. *A proper fuzzy filter of L is a non-constant fuzzy filter of L.*

Definition 2.6. *A proper fuzzy filter of L is said to be a fuzzy prime filter of L if for any $x, y \in L$, $\mu(x \vee y) \leq \mu(x) \vee \mu(y)$*

Result 2.1. *A fuzzy subset μ of L is a fuzzy filter of L if and only if $\mu(x \wedge y) = \mu(x) \wedge \mu(y)$.*

Result 2.2. *A fuzzy subset μ of L is a fuzzy prime filter of L if and only if μ is a homomorphism from $\langle L, \wedge, \vee\rangle$ into $\langle[0,1], \wedge, \vee\rangle$.*

Result 2.3. *A fuzzy subset μ of L is a fuzzy filter of L if and only if each level set μ_α is a filter of L, $\forall\, \alpha \in [0,1]$ such that $\mu_\alpha \neq \emptyset$.*

Result 2.4. *If μ is a fuzzy subset of L, then*

$$\langle\mu\rangle = \bigcap \{\sigma \mid \sigma \text{ is a fuzzy filter of } L,\ \mu \subseteq \sigma\}.$$

Also $\chi_{[S)} = \langle\chi_S\rangle$, where $S \subseteq L$.

Result 2.5. *A non-constant fuzzy filter μ of L is a fuzzy prime filter of L if and only if each level set μ_α is a prime filter of L; $\forall\, \alpha \in [0,1]$ such that μ_α is a proper filter of L.*

Result 2.6. *A non-empty subset P of L is a prime filter of L if and only if χ_P is a fuzzy prime filter of L.*

Result 2.7. *([3]) Let L and L' be two lattices and $f : L \to L'$ an onto homomorphism. Then*
(i) If μ is a fuzzy sublattice (ideal, filter) of L then $f(\mu)$ is a fuzzy sublattice (ideal, filter) of L' where $f(\mu)$ is defined as

$$f(\mu)(y) = sup\{\mu(x) : f(x) = y, x \in L\} \text{ for all } y \in L';$$

(ii) If ν is a fuzzy sublattice (ideal, prime ideal, filter, prime filter) of L' then $f^{-1}(\nu)$ is a fuzzy sublattice (ideal, prime ideal, filter, prime filter) of L where $f^{-1}(\nu)$ is defined as

$$f^{-1}(\nu)(x) = \nu(f(x)) \text{ for all } x \in L.$$

Result 2.8. *Let $L,\ L'$ be two bounded lattices. Let $f : L \to L'$ be a lattice isomorphism. If μ is a fuzzy prime filter of L, then $f(\mu)$ is a fuzzy prime filter of L' and $f^{-1}(f(\mu)) = \mu$.*

3 Spectrum of fuzzy prime filters

Now onwards L will denote a bounded 0 - distributive lattice. Let Σ denote the set of fuzzy prime filters of L. For each $\mu \in \Sigma$, we assume that $\mu(1) = 1$. For a fuzzy subset θ of L define $V(\theta) = \{\mu \in \Sigma \mid \theta \subseteq \mu\}$. If $\theta = \chi_{\{a\}}$ then we denote $V(\theta)$ by $V(a)$.

At the outset we prove some properties of $V\ (.)$.

Theorem 3.1. *Let θ and σ be fuzzy subsets of L.*

1. *If $\theta \subseteq \sigma$, then $V(\sigma) \subseteq V(\theta)$.*
2. *$V(\sigma) \cup V(\theta) \subseteq V(\sigma \cap \theta)$.*
3. *$V(\theta) = V(\langle\theta\rangle)$*
4. *$V(0) = \emptyset$ and $V(1) = \Sigma$.*

Proof. Proof of (1) follows by definition of $V(.)$.

(2) We have $\sigma \cap \theta \subseteq \sigma$ and $\sigma \cap \theta \subseteq \theta$. By using (1) we get $V(\sigma) \subseteq V(\sigma \cap \theta)$ and $V(\theta) \subseteq V(\sigma \cap \theta)$. Hence $V(\sigma) \cup V(\theta) \subseteq V(\sigma \cap \theta)$.

(3) As $\theta \subseteq \langle\theta\rangle$, again by (1) we get $V(\langle\theta\rangle) \subseteq V(\theta)$. Let $\mu \in V(\theta)$ then $\theta \subseteq \mu$. Therefore $\bigcap\{\sigma \in \Sigma : \theta \subseteq \sigma\} \subseteq \mu$ that is $\langle\theta\rangle \subseteq \mu$. Thus $\mu \in V(\langle\theta\rangle)$ proving that $V(\theta) \subseteq V(\langle\theta\rangle)$. Combining both inclusions we get $V(\theta) = V(\langle\theta\rangle)$.

(4) $V(0) = \{\mu \in \Sigma : \chi_{\{0\}} \subseteq \mu\} = \{\mu \in \Sigma : \mu(0) = 1\}$. As $\mu \in \Sigma$ and our assumption that $\mu(1) = 1$ it follows $V(0) = \emptyset$. Again by assumption that $\mu(1) = 1\ \forall \mu \in \Sigma$ we get $V(1) = \{\mu \in \Sigma : \chi_{\{1\}} \subseteq \mu\} = \{\mu \in \Sigma : \mu(1) = 1\} = \Sigma$. □

Remark 3.1. *Let $F(L)$ be the set of all fuzzy subsets of L and $\wp(\Sigma)$ be the power set of Σ. Then $V(.)$ defines a function $V : F(L) \longrightarrow \wp(\Sigma)$ such that $V(\theta) = \{\mu \in \Sigma \mid \theta \subseteq \mu\}$. Clearly it is not an injective map as $V(\theta) = V(\langle\theta\rangle)$.*

Theorem 3.2. *Let I and J be filters of L. Then $V(\chi_I) \cup V(\chi_J) = V(\chi_{I\cap J})$.*

Proof. By Theorem 3.1 (2) we have $V(\chi_I) \cup V(\chi_J) \subseteq V(\chi_I \cap \chi_J) = V(\chi_{I\cap J})$. Let $\mu \in V(\chi_{I\cap J})$. Then $\chi_{I\cap J} \subseteq \mu$ implies $\mu(x) = 1$ for all $x \in I \cap J$. If $\chi_I \nsubseteq \mu$ and $\chi_J \nsubseteq \mu$, then there exist $x \in I$ and $y \in J$ such that $\mu(x) \neq 1$ and $\mu(y) \neq 1$. But I and J being filters we have $x \vee y \in I \cap J$ so that $\mu(x \vee y) = 1$. As μ is a fuzzy prime filter of L, by Result 2.2, $\mu(x \vee y) = \mu(x) \vee \mu(y) = 1$ that is $\mu(x) = 1$ or $\mu(y) = 1$; which contradicts to the choice of x and y. Hence either $\chi_I \subseteq \mu$ or $\chi_J \subseteq \mu$. Therefore $\mu \in V(\chi_I)$ or $\mu \in V(\chi_J)$ and consequently $\mu \in V(\chi_I) \cup V(\chi_J)$. Thus $V(\chi_I \cap \chi_J) \subseteq V(\chi_I) \cup V(\chi_J)$ and the result follows. □

Theorem 3.3. *If $\{\theta_i \mid i \in \Lambda\}$ (Λ is any indexing set) is a family of fuzzy subsets of L, then $V(\bigcup\{\theta_i \mid i \in \Lambda\}) = \bigcap\{V(\theta_i) \mid i \in \Lambda\}$.*

Proof. We have

$$\begin{aligned} \mu \in V\left(\bigcup\{\theta_i \mid i \in \Lambda\}\right) &\iff \bigcup\{\theta_i \mid i \in \Lambda\} \subseteq \mu \\ &\iff \theta_i \subseteq \mu \quad \forall\, i \in \Lambda \\ &\iff \mu \in V(\theta_i) \quad \forall\, i \in \Lambda \\ &\iff \mu \in \bigcap\{V(\theta_i) \mid i \in \Lambda\}. \end{aligned}$$

This sows that $V(\bigcup\{\theta_i \mid i \in \Lambda\}) = \bigcap\{V(\theta_i) \mid i \in \Lambda\}$. □

Remark 3.2. *Unlike in a crisp case $\{V(\theta) \mid \theta$ is a fuzzy subset of $L\}$ does not offer a system of closed sets for a topology on the set Σ though $V(\bigcup\{\theta_i | i \in \Lambda\}) = \bigcap\{V(\theta_i) | i \in \Lambda\}$ (where Λ is any indexing set) holds. This happens as $V(\sigma) \cup V(\theta) \neq V(\sigma \cap \theta)$ for some fuzzy subsets θ and σ of L.*

Theorem 3.4.

1. *$V(a) \cup V(b) = V(a \vee b)$; $\forall\, a, b \in L$*

2. *$V(\chi_E) = \bigcap\{V(a) \mid a \in E\}$; $\forall\, E \subseteq L$.*

Proof. (1) Let $\mu \in V(a) \cup V(b)$. Then $\mu \in V(a)$ or $\mu \in V(b)$. If $\mu \in V(a)$, then $\mu(a) = 1$. As μ is a fuzzy filter, $\mu(a \vee b) \geq \mu(a) = 1$ that is $\mu(a \vee b) = 1$. But then $\mu \in V(a \vee b)$ so that $V(a) \subseteq V(a \vee b)$. Similarly if $\mu \in V(b)$, then $\mu \in V(a \vee b)$. Therefore $V(b) \subseteq V(a \vee b)$. Hence $V(a) \cup V(b) \subseteq V(a \vee b)$. Let $\mu \in V(a \vee b)$ then $\mu(a \vee b) = 1$. μ being a fuzzy prime filter, by Result 2.2 , $\mu(a \vee b) = \mu(a) \vee \mu(b)$. Thus either $\mu(a) = 1$ or $\mu(b) = 1$ that is $\mu \in V(a)$ or $\mu \in V(b)$ so that $\mu \in V(a) \cup V(b)$. Thus $V(a \vee b) \subseteq V(a) \cup V(b)$. Combining both the inclusions, (1) follows

(2) As $E = \bigcup\{a \mid a \in E\}$, we have $\chi_E = \bigcup\left\{\chi_{\{a\}} \mid a \in E\right\}$. Therefore $V(\chi_E) = V\left(\bigcup\left\{\chi_{\{a\}} \mid a \in E\right\}\right) = \bigcap\{V(a) \mid a \in E\}$ (By Theorem 3.3). □

Theorem 3.5. *Let $\mathfrak{B} = \{X(a) \mid a \in L\}$ where $X(a) = X\left(\chi_{\{a\}}\right) = \Sigma \setminus V(a)$. Then $\mathfrak{B}$ constitutes a base for the open sets of some topology on Σ.*

Proof. By Theorem 3.1, (4), we have $X(0) = \Sigma \setminus V(0) = \Sigma \setminus \emptyset = \Sigma$. Therefore

$$\Sigma = \bigcup \{X(a) \mid a \in L\} \tag{1}$$

Let $a, b \in L$. Then

$$\begin{aligned} \mu \in X(a) \cap X(b) &\iff \mu \in (\Sigma \setminus V(a)) \cap (\Sigma \setminus (V(b))) \\ &\iff \mu \in \Sigma \setminus (V(a) \cup V(b)) \\ &\iff \mu \in \Sigma \setminus V(a \vee b) \text{ (by Theorem3.4 (1))} \\ &\iff \mu \in X(a \vee b) \end{aligned}$$

Thus

$$X(a) \cap X(b) = X(a \vee b) \tag{2}$$

From (1) and (2) it follows that $\mathfrak{B}$ forms a base for a topology on Σ. □

Let τ denote the topology with base $\mathfrak{B}$ on Σ. The topological space $\langle \Sigma, \tau \rangle$ is called fuzzy prime spectrum of L and is denoted by $Fspec(L)$.

Theorem 3.6. *If L is a chain or a finite lattice, then* $\mathfrak{B} = \tau$.

Proof. Any open set O in $Fspec(L)$ is expressed as $O = \bigcup \{X(a) \mid a \in A \subseteq L\}$. By assumption, $[A) = [t)$ for some $t \in A$. Hence

$$\begin{aligned} O = \bigcup \{X(a) \mid a \in A \subseteq L\} &= \Sigma \setminus \bigcap \{V(a) \mid a \in A\} \\ &= \Sigma \setminus V(\chi_A) \text{ (by Theorem 3.4 (2))} \\ &= \Sigma \setminus V\left(\chi_{[A)}\right) \text{ (by Theorem 3.1 (3))} \\ &= \Sigma \setminus V\left(\chi_{[t)}\right) \\ &= \Sigma \setminus V(t) \\ &= X(t) \end{aligned}$$

Thus any open set $O = X(t)$ for some $t \in L$ imply $\tau \subseteq \mathfrak{B}$. But always we have $\mathfrak{B} \subseteq \tau$. Hence $\mathfrak{B} = \tau$. □

Theorem 3.7. *The space* $Fspec(L)$ *is a compact space.*

Proof. Consider an open cover $\{X(a) \mid a \in A \subseteq L\}$ of Σ consisting of basic open sets. Therefore

$$\begin{aligned} \Sigma &= \bigcup \{X(a) \mid a \in A\} \\ &= \bigcup \{\Sigma \setminus V(a) \mid a \in A\} \\ &= \Sigma \setminus \bigcap \{V(a) \mid a \in A\} \\ &= \Sigma \setminus V(\chi_A) \quad \text{(by Theorem 3.4, (2))} \\ &= \Sigma \setminus V\left(\chi_{[A)}\right) \text{ (by Theorem 3.1, (3))} \end{aligned}$$

Therefore $V\left(\chi_{[A)}\right) = \emptyset$ (1)
If $[A) \subset L$, then as every proper filter of a 0 - distributive lattice is contained in a prime filter (see [14]) there exists a prime filter P of L containing $[A)$. Hence $\chi_{[A)} \subseteq \chi_P$ and χ_P is a fuzzy prime filter of L (by Result 2.6). Thus $\chi_P \in V\left(\chi_{[A)}\right) = \emptyset$ (by (1)); a contradiction. This proves that $[A) = L$. This results into $0 \in [A)$ and consequently $0 = a_1 \wedge a_2 \wedge \ldots \wedge a_n$; n is finite, $a_i \in A \quad \forall\, i = 1, 2, \ldots, n$. Let $S = \{a_1, a_2, \ldots, a_n\} \subseteq A$. Then by using (1) and the fact that $[A) = L = [S)$ we have $V(\chi_S) = V\left(\chi_{[S)}\right) = V(\chi_L) = V\left(\chi_{[A)}\right) = \emptyset$
Therefore

$$\begin{aligned} \bigcup \{X(a_i) \mid i = 1, 2, \ldots, n \text{ and } a_i \in S\} &= \bigcup \{\Sigma \setminus V(a_i) \mid i = 1, 2, \ldots, n \text{ and } a_i \in S\} \\ &= \Sigma \setminus \bigcap \{V(a_i) \mid i = 1, 2, \ldots, n \text{ and } a_i \in S\} \\ &= \Sigma \setminus V(\chi_S) \\ &= \Sigma \setminus \emptyset = \Sigma \end{aligned}$$

This shows that $\{X(a_i) \mid a_i \in S,\ i = 1, 2, \dots, n\}$ is a finite subcover of the basic open cover $\{X(a) \mid a \in A\}$ of Σ. Hence $Fspec(L)$ is compact. □

Theorem 3.8. *A subset $\mathcal{F}$ of Σ is closed in $Fspec(L)$ if and only if there exists $F \subseteq L$ such that $\mathcal{F} = V(\chi_F)$*

Proof. Let $\mathcal{F} = V(\chi_F)$ for some subset F of L. Then $\mathcal{F} = V(\chi_F) = \bigcap\{V(a) \mid a \in F\}$ (by Theorem 3.4, (2)). As $V(a)$ is a closed set in $Fspec(L)$, we get $\mathcal{F}$ is closed in $Fspec(L)$.
Conversely suppose $\mathcal{F}$ is a closed set in $Fspec(L)$. Then $\Sigma \setminus \mathcal{F}$ is open in $Fspec(L)$ and therefore we have

$$\begin{aligned}
\Sigma \setminus \mathcal{F} &= \bigcup\{X(a) \mid a \in F\} \text{ for some } F \subseteq L \\
&= \bigcup\{\Sigma \setminus V(a) \mid a \in F\} \\
&= \Sigma \setminus \bigcap\{V(a) \mid a \in F\} \\
&= \Sigma \setminus V(\chi_F) \quad \text{(by Theorem 3.4, (2))}
\end{aligned}$$

Thus $\mathcal{F} = V(\chi_F)$ for some $F \subseteq L$. This completes the proof.

□

Theorem 3.9. *Let $\wp$ denote the set of all prime filters of L. Then the set $\mathcal{H} = \{\chi_P \mid P \in \wp\}$ is dense in $Fspec(L)$.*

Proof. By Result 2.6, $\mathcal{H} \subseteq \Sigma$. Let $\mu \in \Sigma \setminus \mathcal{H}$. Let $X(a)$ be a basic open subset of $Fspec(L)$ containing μ. Then the 1 - level subset $\mu_1 = \{x \in L \mid \mu(x) = 1\}$ is proper and hence a prime filter of L (By Result 2.5). $\mu \in X(a) = \Sigma \setminus V(a)$ imply $\mu(a) \neq 1$ so that $a \notin \mu_1$ that is $\chi_{\mu_1}(a) = 0 \neq 1$. This gives $\chi_{\mu_1} \in X(a)$. As μ_1 is a prime filter of L, $\chi_{\mu_1} \in \mathcal{H}$. Thus any $X(a)$ containing μ contains a point of $\mathcal{H}$. Thus every member of $\Sigma \setminus \mathcal{H}$ is a limit point of $\mathcal{H}$. Hence $\overline{\mathcal{H}} = \Sigma$ which proves the result. □

Remark 3.3. *The subspace $\mathcal{H}$ of $Fspec(L)$ is homeomorphic with the prime spectrum $\wp$ (the set equipped with Zariski topology) under the homeomorphism $f : \wp \to \mathcal{H}$ defined by $f(P) = \chi_P$; $\forall P \in \wp$.*

Theorem 3.10. *For any subset $\mathcal{U}$ of Σ, the closure of $\mathcal{U}$ i.e. $\overline{\mathcal{U}} = V(\chi_F)$, where $F = \bigcap\{\mu_1 \mid \mu \in \mathcal{U}\}$.*

Proof. Let $\sigma \in \mathcal{U}$. Also if $\chi_F(x) = 1$ then $x \in F$ and by definition of F we have $x \in \sigma_1$ (since $\sigma \in \mathcal{U}$). But then $\sigma(x) = 1$ which gives $\chi_F \subseteq \sigma$. Thus $\sigma \in V(\chi_F)$. This proves that $\mathcal{U} \subseteq V(\chi_F)$. As $V(\chi_F)$ is a closed set we get $\overline{\mathcal{U}} \subseteq V(\chi_F)$
Now let $\mu \in V(\chi_F)$. If $\mu \in \mathcal{U}$, then $\mu \in \overline{\mathcal{U}}$ and we get $V(\chi_F) \subseteq \overline{\mathcal{U}}$. Otherwise suppose $X(a)$ be a basic open set containing μ. Then $\mu \notin V(a)$ so that $\mu(a) \neq 1$. As $\mu \in V(\chi_F)$ we get $\chi_F(a) \neq 1$. Thus $a \notin F = \bigcap\{\sigma_1 \mid \sigma \in \mathcal{U}\}$ imply $\sigma(a) \neq 1$ for some $\sigma \in \mathcal{U}$. Therefore $\chi_{\{a\}} \nsubseteq \sigma$ and consequently $\sigma \notin V\left(\chi_{\{a\}}\right) = V(a)$ that is $\sigma \in X(a)$. Thus any basic open set $X(a)$ containing μ contains a point σ of $\mathcal{U}$. Therefore μ is a limit point of $\mathcal{U}$. Thus $\mu \in \overline{\mathcal{U}}$. Hence $V(\chi_F) \subseteq \overline{\mathcal{U}}$. Combining both the inclusions the result follows. □

Corollary 3.1. *For any $\mu \in \Sigma$, $\overline{\{\mu\}} = V(\chi_{\mu_1})$ and for $\mu, \theta \in \Sigma$, $\overline{\{\mu\}} = \overline{\{\theta\}}$ if and only if $\mu_1 = \theta_1$.*

Proof. $\overline{\{\mu\}} = V(\chi_{\mu_1})$ follows by Theorem 3.10.
Suppose $\overline{\{\mu\}} = \overline{\{\theta\}}$. Then $V(\chi_{\mu_1}) = V(\chi_{\theta_1})$ (by Theorem 3.10).
Now

$$\begin{aligned}
a \in \mu_1 &\Rightarrow \chi_{\mu_1}(a) = 1 \\
&\Rightarrow \sigma(a) = 1\,;\ \forall\, \sigma \in V(\chi_{\mu_1}) \\
&\Rightarrow \sigma(a) = 1\ \ \forall\, \sigma \in V(\chi_{\theta_1})\ \dots\ (\text{since } V(\chi_{\mu_1}) = V(\chi_{\theta_1})) \\
&\Rightarrow \chi_{\theta_1}(a) = 1\ \dots\ (\text{as } \chi_{\theta_1} \in V(\chi_{\theta_1})) \\
&\Rightarrow a \in \theta_1
\end{aligned}$$

This shows that $\mu_1 \subseteq \theta_1$. Similarly we can prove $\theta_1 \subseteq \mu_1$. Therefore $\mu_1 = \theta_1$.
Conversely suppose $\mu_1 = \theta_1$. Then $\chi_{\mu_1} = \chi_{\theta_1} \Longrightarrow V(\chi_{\mu_1}) = V(\chi_{\theta_1}) \Longrightarrow \overline{\{\mu\}} = \overline{\{\theta\}}$ (by Theorem 3.10). □

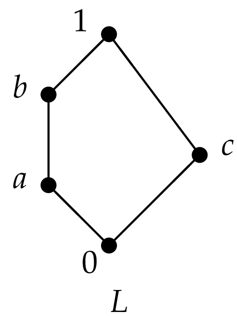

Figure 1:

This result suggests that for a 0 - distributive lattice L the space $Fspec(L)$ need not be T_0 - space. To verify this consider the 0 - distributive lattice L as depicted by the Hasse diagram in Figure 1.

Define $\mu = \{(0,0.2),(a,0.4),(b,0.8),(c,1),(1,1)\}$ and $\theta = \{(0,0.3),(a,0.5),(b,0.7),(c,1),(1,1)\}$. Then μ and θ are fuzzy prime filters of L i.e. $\mu,\theta \in Fspec(L)$. Clearly, $\mu \neq \theta$. But $\mu_1 = \theta_1 = \{c,1\}$. Hence by Corollary 3.1 we have $\overline{\{\mu\}} = \overline{\{\theta\}}$. This shows that $Fspec(L)$ is not a T_0 - space.

Theorem 3.11. *Let L and L' be bounded 0 - distributive lattices and let $f : L \to L'$ be a lattice homomorphism. For each $\mu' \in Fspec(L')$ define $f^* : Fspec(L') \to Fspec(L)$ by $f^*(\mu') = f^{-1}(\mu')$. Then*
(i) f^* is a continuous mapping.
(ii) If f is surjective, then f^* is injective.

Proof. By Result 2.7 , f^* is well defined map.
(i) For any basic closed set $V(a)$ in $Fspec(L)$ where $a \in L$ we have

$$\begin{aligned} f^{*-1}(V(a)) &= \{\mu' \in Fspec(L') \mid f^*(\mu') \in V(a)\} \\ &= \{\mu' \in Fspec(L') \mid [f^*(\mu')](a) = 1\} \\ &= \left\{\mu' \in Fspec(L') \mid \left[f^{-1}(\mu')\right](a) = 1\right\} \\ &= \{\mu' \in Fspec(L') \mid \mu'(f(a)) = 1\} \\ &= \left\{\mu' \in Fspec(L') \mid \chi_{\{f(a)\}} \subseteq \mu'\right\} \\ &= V\left(\chi_{\{f(a)\}}\right) = V(f(a)) \end{aligned}$$

which is a closed set in $Fspec(L')$. Thus inverse image under f^* of a basic closed set in $Fspec(L)$ is a closed set in $Fspec(L')$. Hence f^* is continuous.
(ii) Let f be surjective and μ', $\theta' \in Fspec(L')$ such that $f^*(\mu') = f^*(\theta')$. Then

$$\begin{aligned} (f^*(\mu'))(x) = (f^*(\theta'))(x)\ ;\ \forall\, x \in L &\Rightarrow \left(f^{-1}(\mu')\right)(x) = \left(f^{-1}(\theta')\right)(x)\ ;\ \forall\, x \in L \\ &\Rightarrow \mu'(f(x)) = \theta'(f(x))\ \ \forall\, x \in L \\ &\Rightarrow \mu' = \theta' \end{aligned}$$

This proves f^* is an injective map. □

Theorem 3.12. *If $f : L \to L'$ is an isomorphism, then $Fspec(L)$ is homeomorphic to $Fspec(L')$.*

Proof. By hypothesis, the functions $f^* : Fspec(L') \longrightarrow Fspec(L)$ defined by $f^*(\mu') = f^{-1}(\mu')$ and $g^* : Fspec(L) \longrightarrow Fspec(L')$ defined by $g^*(\mu) = f(\mu)$, are well defined and inverses of each other (see Result 2.7 and Result 2.8). Further, analogous to the proof in theorem 3.17, it can be proved that they are continuous. □

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