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Periodic boundary value problem for the graph differential equation and the matrix differential equation

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Abstract

A network can be represented by graph which is isomorphic to its adjacency matrix. Thus the analysis of networks involving rate of change with respect to time reduces to the study of graph differential equations and its associated matrix differential equations. In this paper we develop monotone iterative technique for graph differential equations and its associated matrix differential equations using Periodic boundary value problem.

Keywords: Graph differential equation, Matrix differential equation, coupled lower and upper solutions, monotone iterative technique.

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1 Introduction

It is a well established fact that a graph represents interconnections in a physical or a biological system and a graph that varies with time models physical phenomena that are time dependent. Such systems naturally arise in all disciplines of knowledge including social sciences. In [1,2] the concept of the derivative of a graph was considered and graph differential equations(GDEs) were introduced. In [2] a solution of the graph differential equations was obtained by developing monotone iterative technique [3] for initial value problems(IVPs) of the corresponding matrix differential equations(MDEs). In [4] the concept of a pseudo graph was introduced and a graph was used to study prey predator problem. The idea of using MDEs to study the corresponding GDEs is justified by the existence of an isomorphism between graphs with n-vertices and $n \times n$ matrices.

In this paper, we obtain a solution for the periodic boundary value problem(PBVP) of a GDE by studying the corresponding PBVP of the MDE. Using the approach used in [5] we develop a sequence of monotone iterates which are solutions of a sequence of IVPs of linear matrix differential equations. We develop different types of iterative sequences as in [6,7] using the iterative scheme given in [8] to obtain the monotone sequences. Next, we show that the sequence converges to a solution of the PBVP and that the corresponding graph function is a solution for the considered PBVP for a GDE.

2 Preliminaries

In this section, we give certain definitions,notations,results and preliminary facts related to GDEs that are required to study the main results in the problem.

Definition 2.1. Pseudo simple graph: A simple graph having loops is called as a pseudo simple graph.

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Let $v_1, v_2, ...v_N$ be N vertices,where N is any positive integer. Let D_N be the set of all weighted directed pseudo simple graphs D=(V, E). Then $(D_N, +, .)$ is a linear space w.r.t the operations + and . defined in [1,2].

Let the set of all corresponding adjacency matrices be E_N. Then $(E_N, +, .)$ is a matrix linear space where '+' denotes matrix addition and '.' denotes scalar multiplication. With this basic structure defined, the comparison theorems, existence and uniqueness results of a solution of a MDE and the corresponding GDE follow as in [2].

Definition 2.2. Continuous and differentiable matrix function:
(1) A matrix function $E : I \to \mathbb{R}^{n\times n}$ defined by $E(t) = (e_{ij}(t))_{N\times N}$ is said to be continuous if and only if each entry $e_{ij}(t)$ is continuous for all $i, j = 1, 2, \ldots, N$ where $e_{ij} : I \to \mathbb{R}$.
(2) A continuous matrix function E(t) is said to be differentiable if and only if each entry $e_{ij}(t)$ is differentiable for all $i, j = 1, 2, \ldots, N$. The derivative of E(t) (if exists) is denoted by E' and is given by $E'(t) = (e'_{ij})_{N\times N}$.

Definition 2.3. Continuous and differentiable graph function: Let $D : I \to D_N$ be a graph function and $E : I \to \mathbb{R}^{n\times n}$ be its associated adjacency matrix function. Then
(1) D(t) is said to be continuous if and only if E(t) is continuous.
(2) D(t) is said to be differentiable if and only if E(t) is differentiable.

If for any graph function D the corresponding adjacency matrix function is differentiable then we say that D is differentiable and the derivative of D (if exists) is denote by D'.

Consider the initial value problem

$$D' = G(t, D), \quad D(t_0) = D_0 \tag{2.1}$$

Let E, E_0 be adjacency matrices corresponding to any graph D and the initial graph D_0.

Then the MDE is given by

$$E' = F(t, E), \quad E(t_0) = E_0 \tag{2.2}$$

where F(t,E) is the adjacency matrix function corresponding to G(t,D).

Definition 2.4. Any continuous differentiable matrix function E(t) is said to be a solution of (2.2), if and only if it satisfies (2.2).

Definition 2.5. By a solution of GDE (2.1), we mean the graph function D(t) corresponding to the matrix solution E(t) of the MDE (2.2).

In order to obtain a solution of (2.1), we use the corresponding adjacency MDE. As there exists an isomorphism between graphs and matrices, the graph function corresponding to the solution obtained for the MDE will be a solution of the corresponding GDE.

Definition 2.6. Let $\{E_n\}$ be a sequence of matrices and E be a matrix. Then $\mathrm{E_n}$ converges to E if and only if given $\epsilon > 0$ there exist n $\geq$ N such that $||\mathrm{E_n} - \mathrm{E}|| \leq$ ffl for all n $\geq$ N. This means $\mathrm{e_{n_{ij}}} \to e_{ij}$ for all $1 \leq i, j \leq N$.

Definition 2.7. Consider two matrices A and B of order N. We say that $A \leq B$ if and only if $a_{ij} \leq b_{ij}$ for all $i, j = 1, 2, \ldots, N$.

Definition 2.8.

Let D_1 and D_2 be two graphs. Let $e^1_{ij}(t)$ be the weight of the edge joining the vertex v_j to v_i, $i, j = 1, 2, ..., n$ in D_1 and $e^2_{ij}(t)$ be the weight of the edge joining the vertex v_j to v_i, $i, j = 1, 2, ..., n$ in D_2 then we say that $D_1 \leq D_2$ if and only if $e^1_{ij} \leq e^2_{ij}$.

Theorem 2.1.

If $\{U_n(t)\} \in C^1[I, \mathbb{R}^{n\times n}]$ is a sequence of equicontinuous and equibounded multimappings defined on an interval I, then we can extract a subsequence that converges uniformly to a continuous multimapping $U(t)$ on I.

With the necessary preliminaries in place, we proceed to the next section to develop the main results.

3 Main results

In this section, we obtain a solution for the PBVP for GDE through the solution of the PBVP for MDE.
Consider the PBVP for GDE given by

$$D' = G_1(t,D) + G_2(t,D), \quad D(0) = D(T), \tag{3.3}$$

where $G_1, G_2 \in C[I \times D_N, D_N]$ and I=[0,T].

The graphs G_1 and G_2 generate two matrix mappings $F_1, F_2 \in C^1[I \times \mathbb{R}^{n\times n}, \mathbb{R}^{n\times n}]$ such that the weight of the edge joining the vertex v_k to the vertex v_l, $k,l = 1,2,...,n$ is given by $e^i_{lk}(t)$, for $i = 1,2$ that is, $(e^i_{lk}(t))_{n\times n}$ is the weight matrix corresponding to the graph G_i and is denoted by $F_i = (e^i_{lk}(t))_{n\times n}$. Further, let $E(t) = (e_{lk}(t))$ be any arbitrary matrix function corresponding to any arbitrary graph function D(t) and then consider the following MDE corresponding to GDE (3.3) as

$$E' = F_1(t,E) + F_2(t,E), \quad E(0) = E(T), \tag{3.4}$$

where $F_1, F_2 \in C[I \times \mathbb{R}^{n\times n}, \mathbb{R}^{n\times n}]$.

We next proceed to gives several possible notions of lower and upper solutions relative to MDE (3.4).

Definition 3.9. *Let $V_0, W_0 \in C^1[I, \mathbb{R}^{n\times n}]$. Then V_0, W_0 are said to be*
(a) natural lower and upper solutions of (3.4) *if*

$$\left.\begin{array}{r} V_0' \le F_1(t,V_0) + F_2(t,V_0), \;\; V_0(0) \le V_0(T). \\ W_0' \ge F_1(t,W_0) + F_2(t,W_0), \;\; W_0(0) \ge W_0(T), \;\; t \in I; \end{array}\right\} \tag{3.5}$$

(b) coupled lower and upper solutions of Type I of (3.4) *if*

$$\left.\begin{array}{r} V_0' \le F_1(t,V_0) + F_2(t,W_0), \;\; V_0(0) \le V_0(T), \\ W_0' \ge F_1(t,W_0) + F_2(t,V_0), \;\; W_0(0) \ge W_0(T), \;\; t \in I; \end{array}\right\} \tag{3.6}$$

(c) coupled lower and upper solutions of Type II of (3.4) *if*

$$\left.\begin{array}{r} V_0' \le F_1(t,W_0) + F_2(t,V_0), \;\; V_0(0) \le V_0(T), \\ W_0' \ge F_1(t,V_0) + F_2(t,W_0), \;\; W_0(0) \ge W_0(T), \;\; t \in I; \end{array}\right\} \tag{3.7}$$

(d) coupled lower and upper solutions of Type III of (3.4) *if*

$$\left.\begin{array}{r} V_0' \le F_1(t,W_0) + F_2(t,W_0), \;\; V_0(0) \le V_0(T), \\ W_0' \ge F_1(t,V_0) + F_2(t,V_0), \;\; W_0(0) \ge W_0(T), \;\; t \in I. \end{array}\right\} \tag{3.8}$$

We observe that whenever $V_0(t) \le W_0(t)$, $t \in I$, if $F_1(t,E)$ is nondecreasing in E for each $t \in I$ and $F_2(t,E)$ is nonincreasing in E for each $t \in I$, the lower and upper solutions defined by (3.5) and (3.8) reduce to (3.7) and consequently, it is sufficient to investigate the cases (3.6) and (3.7).

We prove the following lemma that deals with developing the MIT for the equation (3.4).

Lemma 3.1. *Let $P \in C^1[I, \mathbb{R}^{n\times n}]$ such that $P'(t) \le 0$ and $P(0) \le 0$ then $P(t) \le 0$*

Proof. Consider the linear matrix differential equation

$$P'(t) = M(t)P + H(t), \quad P(0) = 0$$

Whose unique solution is given by

$$P(t) = e^{M(t-t_0)}P(0) + \int_{t_0}^{t} e^{M(t-s)}H(s)ds$$

Then by hypothesis, we get, $P(t) \le 0$

Theorem 3.2. *Assume that*
(A_1) $V_0, W_0 \in C^1[I, \mathbb{R}^{n\times n}]$ *are coupled lower and upper solutions of Type I relative to* (3.4) *with* $V_0(t) \leq W_0(t),\ t \in I$;
(A_2) $F_1, F_2 \in C[I \times \mathbb{R}^{n\times n}, \mathbb{R}^{n\times n}]$, $F_1(t,E)$ *is nondecreasing in E for each* $t \in I$ *and* $F_2(t,E)$ *is nonincreasing in E for each* $t \in I$;
(A_3) F_1 *and* F_2 *map bounded sets into bounded sets in* $\mathbb{R}^{n\times n}$.

Then there exist monotone sequences $\{V_n\}$,$\{W_n\}$ *in* $\mathbb{R}^{n\times n}$ *such that* $V_n \to \rho, W_n \to R$ *where* (ρ, R) *are the coupled minimal and maximal solutions of* (3.4)*, that is, they satisfy*

$$\rho' = F_1(t,\rho) + F_2(t,R), \quad \rho(0) = \rho(T),$$
$$R' = F_1(t,R) + F_2(t,\rho), \quad R(0) = R(T).$$

Proof. For each $n \geq 0$, consider the initial value problems

$$V'_{n+1} = F_1(t,V_n) + F_2(t,W_n), \quad V_{n+1}(0) = V_n(T), \tag{3.9}$$

$$W'_{n+1} = F_1(t,W_n) + F_2(t,V_n), \quad W_{n+1}(0) = W_n(T), \tag{3.10}$$

where $V(0) \leq W(0)$.

Our aim is to prove that the solutions of (3.9) and (3.10) satisfy the relation,

$$V_0 \leq V_1 \leq ... \leq V_n \leq W_n \leq ... \leq W_1 \leq W_0, \quad t \in I. \tag{3.11}$$

Since V_o is the coupled lower solution of Type I of (3.4), we have, using the fact $V_0 \leq W_0$ and the nondecreasing character of F,

$$V'_0 \leq F_1(t,V_0) + F_2(t,W_0).$$

Also from (3.9) we get for $n = 0$,

$$V'_1 = F_1(t,V_0) + F_2(t,W_0), \tag{3.12}$$

$$V_1(0) = V_0(T). \tag{3.13}$$

Clearly equations (3.12), (3.13) have a unique solution denoted by $V_1(t), t \in I$. Now we show that $V_0 \leq V_1$ on I. Set $P(t) = V_0(t) - V_1(t)$,

$$P'(t) \leq F_1(t,V_0) + F_2(t,W_0) - F_1(t,V_0) - F_2(t,V_0) \leq 0$$

and $P(0) = V_0(0) - V_1(0) \leq V_0(T) - V_0(T) \leq 0$. Then by Lemma 3.2, we get $P(t) \leq 0$. Thus $V_0 \leq V_1$ on I. A similar argument shows that $W_1 \leq W_0$ on I. For the purpose of showing $V_1 \leq W_1$, consider (3.12), (3.13) along with

$$W'_1 = F_1(t,W_0) + F_2(t,V_0), \tag{3.14}$$

$$W_1(0) = W_0(T). \tag{3.15}$$

Put $P(t) = V_1(t) - W_1(t)$, *then* $P'(t) = F_1(t,V_0) + F_2(t,W_0) - F_1(t,W_0) - F_2(t,V_0)$. Since $V_0 \leq W_0$ on I, using the monotone nature of F_1 and F_2 gives
$P'(t) \leq 0$ and $P(0) = V_1(0) - W_1(0) = V_0(T) - W_0(T) \leq 0$. Therefore by Lemma 3.2, we get $P(t) \leq 0$. Thus $V_1 \leq W_1$ on I. Hence

$$V_0 \leq V_1 \leq W_1 \leq W_0, \text{ on } I. \tag{3.16}$$

Assume that for $j \geq 1$,

$$V_{j-1} \leq V_j \leq W_j \leq W_{j-1}, \text{ on } I. \tag{3.17}$$

Then we will show that

$$V_j \leq V_{j+1} \leq W_{j+1} \leq W_j, \text{ on } I. \tag{3.18}$$

To do this consider,

$$V'_j = F_1(t,V_{j-1}) + F_2(t,W_{j-1}), \tag{3.19}$$

$$V_j(0) = V_{j-1}(T), \tag{3.20}$$

$$V'_{j+1} = F_1(t, V_j) + F_2(t, W_j), \tag{3.21}$$

$$V_{j+1}(0) = V_j(T). \tag{3.22}$$

Now we show that $V_j(t) \leq V_{j+1}(t)$ on I. Consider $P(t) = V_j(t) - V_{j+1}(t)$. Then $P'(t) = F_1(t, V_{j-1}) + F_2(t, W_{j-1}) - F_1(t, V_j) - F_2(t, W_j)$, Now using the fact that $V_{j-1} \leq V_j$, $W_j \leq W_{j-1}$, and the monotone nature of F_1 and F_2, we obtain $P'(t) \leq 0$ and $P(0) = V_j(0) - V_{j+1}(0) = V_{j-1}(T) - V_j(T) \leq 0$. Again by using Lemma 3.2, we get $P(t) \leq 0$. Thus $V_j \leq V_{j+1}$ on I. Similarly we get $W_{j+1} \leq W_j$ on I.

Next we show that $V_{j+1} \leq W_{j+1}$ $t \in I$. We have from (3.9), (3.10)

$$V'_{j+1} = F_1(t, V_j) + F_2(t, W_j), \quad V_{j+1}(0) = V_j(T), \tag{3.23}$$

$$W'_{j+1} = F_1(t, W_j) + F_2(t, V_j), \quad W_{j+1}(0) = W_j(T). \tag{3.24}$$

Set $P(t) = V_{j+1}(t) - W_{j+1}(t)$, then $P'(t) = F_1(t, V_j) + F_2(t, W_j) - F_1(t, W_j) - F_2(t, V_j)$, Arguing as earlier we conclude $P'(t) \leq 0$ and $P(0) = V_{j+1}(0) - W_{j+1}(0) = V_j(T) - W_j(T) \leq 0$. By Lemma 3.2, we get $P(t) \leq 0$. Thus $V_{j+1} \leq W_{j+1}$ on I.
Hence (3.18) follows and consequently, by induction (3.18) is valid for all n. Clearly the sequences $\{V_n\}$,$\{W_n\}$ are uniformly bounded on I.
To show that these sequences are equicontinuous, consider for any $s \geq t$, where $t, s \in I$,

$$\begin{aligned}
|V_n(t) - V_n(s)| = |V_n(0) + \int_0^t (F_1(\xi, V_{n-1}(\xi)) + F_2(\xi, W_{n-1}(\xi)))d\xi \\
- V_n(0) - \int_0^s (F(\xi, V_{n-1}(\xi)) + G(\xi, W_{n-1}(\xi)))d\xi| \\
= |\int_0^t (F_1(\xi, V_{n-1}(\xi)) + F_2(\xi, W_{n-1}(\xi)))d\xi \\
- \int_0^s (F_1(\xi, V_{n-1}(\xi)) + F_2(\xi, W_{n-1}(\xi)))d\xi| \\
\leq |\int_s^t (F_1(\xi, V_{n-1}(\xi)) + F_2(\xi, W_{n-1}(\xi)))d\xi| \\
\leq M|t - s|.
\end{aligned}$$

Here we utilized the properties of integral, together with the fact F_1 and F_2 are bounded and $\{V_n\}$,$\{W_n\}$ are uniformly bounded. Hence $\{V_n(t)\}$ is equicontinuous on I. The corresponding Ascoli's Theorem, Theorem 2.9, gives a subsequence $\{V_{n_k}\}$ which converges uniformly to $\rho(t) \in \mathbb{R}^{n \times n}, t \in I$, and since $\{V_n(t)\}$ is nondecreasing sequence, the entire sequence $\{V_n(t)\}$ converges uniformly to $\rho(t)$ on I.

Similar arguments apply to the sequence $\{W_n(t)\}$ and we obtain $W_n(t) \to R(t)$ uniformly on I. It therefore follows, using the integral representation of (3.9), (3.10) that $\rho(t)$, R(t) satisfy

$$\rho'(t) = F_1(t, \rho(t)) + F_2(t, R(t)), \quad \rho(0) = \rho(T);$$

$$R'(t) = F_1(t, R(t)) + F_2(t, \rho(t)), \quad R(0) = R(T).$$

and that

$$V_0 \leq \rho \leq R \leq W_0, \;\; t \in I.$$

Next we claim that (ρ, R) are coupled minimal and maximal solutions of (3.4), that is, if U(t) is any solution of (3.4) such that

$$V_0 \leq U \leq W_0, \;\; on\ I, \tag{3.25}$$

then

$$V_0 \leq \rho \leq U \leq R \leq W_0, \;\; on\ I. \tag{3.26}$$

Suppose that for some n,

$$V_n \leq U \leq W_n \quad on\ I. \tag{3.27}$$

Then we have using the monotone nature of $F_1,\ F_2$ and (3.4)

$$U' = F_1(t,U) + F_2(t,U) \geq F_1(t,V_n) + F_2(t,W_n), \quad U(0) = U(T).$$

$$V'_{n+1} = F_1(t,V_n) + F_2(t,W_n), \quad V_{n+1}(0) = V_n(T).$$

Now we show that $V_{n+1} \leq U$ on I. Set $P(t) = V_{n+1}(t) - U(t)$, $P'(t) = F_1(t,V_n) + F_2(t,W_n) - F_1(t,V_n) - F_2(t,W_n) \leq 0$ and $P(0) = V_{n+1}(0) - U(0) = V_n(T) - U(T) \leq 0$. Then by Lemma 3.2, we get $P(t) \leq 0$. Thus $V_{n+1} \leq U$ on I. Similarly $W_{n+1} \geq U$ on I.
Hence by induction (3.27) is true for all $n \geq 1$. Now taking the limit as $n \to \infty$, we get (3.26), proving the claim. The proof is complete.

In the following theorem we use a different set of iterative scheme to form the existence a result for the MDE (3.4).

Theorem 3.3. *Let the hypothesis of Theorem 3.4 hold with $V_0 \leq W_0$ on I. Then the iterative scheme given by*

$$V'_{n+1} = F_1(t,W_n) + F_2(t,V_n), \tag{3.28}$$

$$V_{n+1}(0) = W_n(T). \tag{3.29}$$

and

$$W'_{n+1}(t) = F_1(t,V_n) + F_2(t,W_n), \tag{3.30}$$

$$W_{n+1}(0) = V_n(T). \tag{3.31}$$

yield alternating sequences $\{V_{2n}, W_{2n+1}\}$ converging to ρ and $\{W_{2n}, V_{2n+1}\}$ converging to R uniformly on I such that the relation

$$V_0 \leq W_1 \leq ... \leq V_{2n} \leq W_{2n+1} \leq U \leq V_{2n+1} \leq W_{2n} \leq ... \leq V_1 \leq W_0. \tag{3.32}$$

holds on I. Further ρ and R are coupled minimal and maximal solutions of Type II for the MDE (3.4).

Proof.

Clearly the IVPs (3.28), (3.29), (3.30) and (3.31) have unique solutions for each $n = 0, 1, 2, ...$ denoted by V_{n+1}(t) and W_{n+1}(t) respectively. Setting $n = 0$ in the iterative scheme we obtain that V_1 and W_1 are solutions of the IVPs for MDEs given by

$$V'_1 = F_1(t,W_0) + F_2(t,V_0), \tag{3.33}$$

$$V_1(0) = W_0(T), \tag{3.34}$$

and

$$W'_1(t) = F_1(t,V_0) + F_2(t,W_0), \tag{3.35}$$

$$W_1(0) = V_0(T). \tag{3.36}$$

First we show that $V_0 \leq W_1$ on I. Set $P(t) = V_0(t) - W_1(t)$, then $P'(t) \leq 0$, due to the fact that $V_0 \leq W_0$ and F_1 and F_2 are monotone in E. Also $P(0) \leq 0$. Then by Lemma 3.2, we get $P(t) \leq 0$. Thus $V_0 \leq W_1$ on I. A similar argument shows that $V_1 \leq W_0$, $W_1 \leq V_1$ on I. Thus, $V_0 \leq W_1 \leq V_1 \leq W_0$ on I.
We now proceed to prove that $V_0 \leq W_1 \leq V_2 \leq W_3 \leq V_3 \leq W_2 \leq V_1 \leq W_0$ on I. To do this, set $n = 1$ in (3.28), (3.29) then

$$V'_2 = F_1(t,W_1) + F_2(t,V_1), \quad V_2(0) = W_1(T). \tag{3.37}$$

Now we show that $W_1 \leq V_2$ on J. Put $P(t) = W_1(t) - V_2(t)$, then by using $V_0 \leq W_0$ and monotone nature of F_1 and F_2, we arrive at $P'(t) \leq 0$ and also $P(0) \leq 0$. Hence by Lemma 3.2, we get $P(t) \leq 0$. Thus $W_1 \leq V_2$ on I. Working in a similar fashion we shows that $W_2 \leq V_1$ on I.
To prove $V_2 \leq W_3$, set $n = 1$ in (3.28), (3.29) and $n = 2$ in (3.30), (3.31) then

$$V'_2 = F_1(t,W_1) + F_2(t,V_1),$$

$$V_2(0) = W_1(T).$$

and

$$W'_3 = F_1(t,V_2) + F_2(t,W_2),$$

$$W_2(0) = V_2(T).$$

We now proceed to prove that $V_2 \leq W_3$ on I. Consider $P(t) = V_2(t) - W_3(t)$. Since $W_1 \leq V_2$, $W_2 \leq V_1$ on I and using the monotone nature of F_1 and F_2, gives $P'(t) \leq 0$ and also we get $P(0) \leq 0$. Then by Lemma 3.2, we have $P(t) \leq 0$. Thus $V_2 \leq W_3$ on I. Working in a similar fashion we shows that $W_2 \leq V_1$ on I. Working as earlier, it can be easily shown that $W_3 \leq V_3$ on I.
Now assume that the relation (3.32) holds for some integer $n = k$ such that

$$W_{2k-1} \leq V_{2k} \leq W_{2k+1} \leq U \leq V_{2k+1} \leq W_{2k} \leq V_{2k-1}. \tag{3.38}$$

To apply mathematical induction we need to prove that

$$W_{2k+1} \leq V_{2k+2} \leq W_{2k+3} \leq U \leq V_{2k+3} \leq W_{2k+2} \leq V_{2k+1} \ \ on\ I. \tag{3.39}$$

For this, set $n = 2k + 1$ in (3.28), (3.29) and $n = 2k$ in (3.30), (3.31). Then,

$$V'_{2k+2} = F_1(t, W_{2k+1}) + F_2(t, V_{2k+1}), \quad V_{2k+2}(0) = W_{2k+1}(T). \tag{3.40}$$

and

$$W'_{2k+1} = F_1(t, V_{2k}) + F_2(t, W_{2k}), \quad W_{2k+1}(0) = V_{2k}(T). \tag{3.41}$$

Further, we show that $V_{2k+2} \leq W_{2k+3}$ on I. Set $P(t) = V_{2k+2}(t) - W_{2k+3}(t)$. Then $P'(t) \leq 0$, due to the fact that $W_1 \leq V_2$, $W_2 \leq V_1$ and monotone nature of F_1 and F_2. Also $P(0) \leq 0$. Applying by Lemma 3.2, we get $P(t) \leq 0$. Thus $V_2 \leq W_3$ on I. Similarly, $V_{2k+3} \leq W_{2k+2}$, $W_{2k+2} \leq V_{2k+1}$ all hold on I.
Thus we are in a position to apply mathematical induction and claim that the relation (3.32) holds. Working as in Theorem 3.3, we can show that the sequences $\{V_{2n}\}, \{V_{2n+1}\}, \{W_{2n}\}, \{W_{2n+1}\}$ are equicontinuous and uniformly bounded. Thus from Theorem 2.9, which is the Arzela-Ascoli Theorem, we conclude that they are uniformly convergent and that $V_{2n} \to \rho, W_{2n+1} \to \rho$ and $W_{2n} \to R$ and $V_{2n+1} \to R$ as $n \to \infty$.

The proof is complete if we show that ρ and R are coupled minimal and maximal solutions of the MDE (3.4). This follows by considering the corresponding Hukuhara integral and using the properties of uniform continuity of F_1 and F_2 and uniform convergence of the sequences $\{V_{2n}\}, \{W_{2n+1}\}$ and $\{V_{2n+1}\}, \{W_{2n}\}$. As the details are routine, we omit them and the proof of the theorem is complete.

Now in order to extend our results to GDEs, we define the various notions of lower and upper solutions of (3.3) and use the results obtained earlier to obtain solutions of the PBVP (3.3).

Definition 3.10. *Let $X_0, Y_0 \in C^1[I, D_N]$ be graph functions then we say that X_0 and Y_0 are*
(a) natural lower and upper solutions of (3.3) *if*

$$\left.\begin{array}{r} X'_0 \leq G_1(t, X_0) + G_2(t, X_0), \ \ X_0(0) \leq X_0(T). \\ Y'_0 \geq G_1(t, Y_0) + G_2(t, Y_0), \ Y_0(0) \geq Y_0(T), \ \ t \in I; \end{array}\right\} \tag{3.42}$$

(b) coupled lower and upper solutions of Type I of (3.3) *if*

$$\left.\begin{array}{r} X'_0 \leq G_1(t, X_0) + G_2(t, Y_0), \ \ X_0(0) \leq X_0(T), \\ Y'_0 \geq G_1(t, Y_0) + G_2(t, X_0), \ \ Y_0(0) \geq Y_0(T), \ \ t \in I; \end{array}\right\} \tag{3.43}$$

(c) coupled lower and upper solutions of Type II of (3.3) *if*

$$\left.\begin{array}{r} X'_0 \leq G_1(t, Y_0) + G_2(t, X_0), \ \ X_0(0) \leq X_0(T), \\ Y'_0 \geq G_1(t, X_0) + G_2(t, Y_0), \ \ Y_0(0) \geq Y_0(T), \ \ t \in I; \end{array}\right\} \tag{3.44}$$

(d) coupled lower and upper solutions of Type III of (3.3) *if*

$$\left.\begin{array}{r} X'_0 \leq G_1(t, Y_0) + G_2(t, Y_0), \ \ X_0(0) \leq X_0(T), \\ Y'_0 \geq G_1(t, X_0) + G_2(t, X_0), \ \ Y_0(0) \geq Y_0(T), \ \ t \in I. \end{array}\right\} \tag{3.45}$$

Theorem 3.4. *Assume that*
(A_1) $X_0, Y_0 \in C^1[I, D_N]$ are coupled lower and upper solutions of Type I relative to (3.3) *with $X_0(t) \leq Y_0(t)$, $t \in I$;*

(A_2) $G_1, G_2 \in C^1[I \times D_N, D_N]$, $G_1(t, E)$ is nondecreasing in D for each $t \in I$ and $G_2(t, D)$ is nonincreasing in D for

each $t \in I$;

(A_3) G_1 and G_2 map bounded sets into bounded sets in D_N.

Then there exists solutions $G_\rho(t)$ and $G_R(t)$ where (G_ρ, G_R) are the coupled minimal and maximal solutions of (3.3)*, that is, they satisfy*

$$G'_\rho(t) = G_1(t, G_\rho) + G_2(t, G_R), \quad G_\rho(0) = G_\rho(T),$$
$$G'_R(t) = G_1(t, G_R) + G_2(t, G_\rho), \quad G_R(0) = G_R(T),$$

Proof.

Consider the given graph differential equation (3.3). and its corresponding MDE is (3.4).
By hypothesis we have that X_0, Y_0 are coupled lower and upper solutions of type I of GDE (3.3). with $X_0(t) \leq Y_0(t), t \in I$
By using the isomorphism that exists between graphs and matrices we note that corresponding to $X_0, Y_0 \in C[I, D_N]$ there exists $V_0, W_0 \in C^1[I, \mathbb{R}^{n\times n}]$ such that V_0 *and* W_0 are coupled lower and upper solutions of MDE (3.4) with $V_0(t) \leq W_0(t), t \in I$. Hence hypothesis (A_1) of Theorem 3.3 holds.
Now here $G_1(t, D) \in C[I \times D_N, D_N]$ is nondecreasing in D and there exists $F_1(t, E) \in C[I \times \mathbb{R}^{n\times n}, \mathbb{R}^{n\times n}]$ which is nondecreasing in E and similarly $F_2(t, E)$ is nonincreasing in E for each t. Therefore the hypothesis $(A_2), (A_3)$ of Theorem 3.3 holds. Clearly F_1 and F_2 map bounded sets into bounded sets. Since G_1 *and* G_2 map bounded sets into bounded sets.

Thus we conclude that there exists $\rho(t)$, $R(t) \in C[I, \ \mathbb{R}^{n\times n}]$ such that $(\rho(t), \ R(t))$ are the coupled minimal and maximal solution of the MDE (3.4). By using isomorphism between are obtained that there exist $G_\rho(t), G_R(t) \in C^1[I, \ D_N]$ such that $(G_\rho(t), G_R(t))$ are the coupled minimal and maximal solution of GDE (3.3).

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