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Approximating solutions of nonlinear second order ordinary differential equations via Dhage iteration principle

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Abstract

In this paper the authors prove algorithms for the existence as well as approximation of the solutions for an initial and a periodic boundary value problem of nonlinear second order ordinary differential equations. The main results rely on the Dhage iteration principle embodied in a recent hybrid fixed point theorem of Dhage (2013) in the partially ordered normed linear spaces and the numerical solution of the considered equations is obtained under weaker mixed partial continuity and partial Lipschitz conditions. Our hypotheses and results are also illustrated by some numerical examples.

Keywords: Approximating solutions, Dhage iteration principle, hybrid fixed point theorem, initial value problems, periodic boundary value problems.

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1 Introduction

The study of nonlinear differential equations via successive approximations has been a topic of great interest since long time. It is Picard who first devised a constructive method for the initial value problems of nonlinear ordinary differential equations in terms of a sequence of successive approximations converging to a unique solution of the related differential equations. The method is commonly known as Picard's iteration method in nonlinear analysis and frequently used for nonlinear equations in the literature. It employs the Lipschitz condition of the nonlinearities together with a certain restriction on Lipschitz constant. The Picard's method is further abstracted to metric spaces by Banach which thereby made it possible to relax the condition on Lipschitz constant. Many attempts have been made in the literature to weaken the Lipschitz condition. Nieto and Lopéz [13] weakened Lipschitz condition to partial Lipschitz condition guaranteeing the conclusion of the Picard's method under certain additional conditions. But in any circumstances the hypothesis of Lipschitz condition is unavoidable to guarantee the conclusion of Picard's method. Very recently, the author in [1–4] proved some abstract hybrid fixed point theorems in the setting of a partially ordered metric spaces and to some nonlinear differential and integral equations without using any kind of geometric condition and still the conclusion of Picard's method holds. See Dhage [5–7] and Dhage and Dhage [8–10] and the references therein. However, in this case the order relation and the metric are required to satisfy certain compatibility condition. See [8], [11] and the references therein. In this paper, we use this hybrid fixed point theorem in the study of initial and boundary value problems of nonlinear second order ordinary differential equations and prove a stronger conclusion of Picard method.

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The paper will be organized as follows. In Section 2 we give some preliminaries and key fixed point theorem that will be used in subsequent part of the paper. In Section 3 we discuss the existence result for initial value problems and in Section 4 we discuss the existence result for periodic boundary value problems of ordinary differential equations.

2 Auxiliary Results

Unless otherwise mentioned, throughout this paper that follows, let E denote a partially ordered real normed linear space with an order relation $\preceq$ and the norm $\|\cdot\|$ in which the addition and the scalar multiplication by positive real numbers are preserved by $\preceq$. A few details of a partially ordered normed linear space appear in Dhage [2] and the references therein.

Two elements x and y in E are said to be **comparable** if either the relation $x \preceq$ or $y \preceq x$ holds. A non-empty subset C of E is called a **chain** or **totally ordered** if all the elements of C are comparable. It is known that E is **regular** if $\{x_n\}$ is a nondecreasing (resp. nonincreasing) sequence in E such that $x_n \to x^*$ as $n \to \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. The conditions guaranteeing the regularity of E may be found in Heikkilä and Lakshmikantham [12], Nieto and Lopéz [13] and the references therein.

We need the following definitions in the sequel.

Definition 2.1. *A mapping $\mathcal{T} : E \to E$ is called* **isotone** *or* **monotone nondecreasing** *if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$ for all $x, y \in E$. Similarly, $\mathcal{T}$ is called* **monotone nonincreasing** *if $x \preceq y$ implies $\mathcal{T}x \succeq \mathcal{T}y$ for all $x, y \in E$. Finally, $\mathcal{T}$ is called* **monotonic** *or simply* **monotone** *if it is either monotone nondecreasing or monotone nonincreasing on E.*

The following terminologies may be found in any book on nonlinear analysis and applications.

Definition 2.2. *An operator $\mathcal{T}$ on a normed linear space E into itself is called* **compact** *if $\mathcal{T}(E)$ is a relatively compact subset of E. $\mathcal{T}$ is called* **totally bounded** *if for any bounded subset S of E, $\mathcal{T}(S)$ is a relatively compact subset of E. If $\mathcal{T}$ is continuous and totally bounded, then it is called* **completely continuous** *on E.*

Definition 2.3 (Dhage [2])**.** *A mapping $\mathcal{T} : E \to E$ is called* **partially continuous** *at a point $a \in E$ if for $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\mathcal{T}x - \mathcal{T}a\| < \epsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. $\mathcal{T}$ called partially continuous on E if it is partially continuous at every point of it. It is clear that if $\mathcal{T}$ is partially continuous on E, then it is continuous on every chain C contained in E.*

Definition 2.4 (Dhage [1, 2])**.** *A non-empty subset S of the partially ordered Banach space E is called* **partially bounded** *if every chain C in S is bounded. An operator $\mathcal{T}$ on a partially normed linear space E into itself is called* ***partially bounded*** *if $\mathcal{T}(C)$ is bounded for every chain C in E. $\mathcal{T}$ is called* **uniformly partially bounded** *if all chains $\mathcal{T}(C)$ in E are bounded by a unique constant.*

Definition 2.5 (Dhage [1, 2])**.** *A non-empty subset S of the partially ordered Banach space E is called* **partially compact** *if every chain C in S is compact. A mapping $\mathcal{T} : E \to E$ is called* ***partially compact*** *if $\mathcal{T}(C)$ is a relatively compact subset of E for all totally ordered sets or chains C in E. $\mathcal{T}$ is called* ***uniformly partially compact*** *if $\mathcal{T}$ is a uniformly partially bounded and partially compact operator on E. $\mathcal{T}$ is called* **partially totally bounded** *if for any totally ordered and bounded subset C of E, $\mathcal{T}(C)$ is a relatively compact subset of E. If $\mathcal{T}$ is partially continuous and partially totally bounded, then it is called* **partially completely continuous** *on E.*

Remark 2.1. *Note that every compact mapping on a partially normed linear space is partially compact and every partially compact mapping is partially totally bounded, however the reverse implications do not hold. Again, every completely continuous mapping is partially completely continuous and every partially completely continuous mapping is partially continuous and partially totally bounded, but the converse may not be true.*

Definition 2.6 (Dhage [1])**.** *The order relation $\preceq$ and the metric d on a non-empty set E are said to be* ***compatible*** *if $\{x_n\}$ is a monotone sequence, that is, monotone nondecreasing or monotone nondecreasing sequence in E and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x^* implies that the original sequence $\{x_n\}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are compatible. A subset S of E is called* **Janhavi** *if the order relation $\preceq$ and the metric d or the norm $\|\cdot\|$ are compatible in it. In particular, if $S = E$, then E is called a* **Janhavi metric** *or* **Janhavi Banach space***.*

Clearly, the set $\mathbb{R}$ of real numbers with usual order relation $\leq$ and the norm defined by the absolute value function $|\cdot|$ has this property. Similarly, the finite dimensional Euclidean space $\mathbb{R}^n$ with usual componentwise order relation and the standard norm possesses the compatibility property and so is a **Janhavi Banach space**.

The Dhage iteration principle or method (in short DIP or DIM) developed in Dhage [1–4] may be described as " ***the sequence of successive approximations of a nonlinear equation beginning with a lower or an upper solution as its first or initial approximation converges monotonically to the solution*** " forms a basic and powerful tool in the study of nonlinear differential and integral equations. See Dhage and Dhage [8, 9] and the references therein. The following applicable hybrid fixed point theorem of Dhage [3] containing the DIP is used as a key tool for the work of this paper.

Theorem 2.1 (Dhage [2]). *Let $(E, \preceq, \|\cdot\|)$ be a regular partially ordered complete normed linear space such that the order relation $\preceq$ and the norm $\|\cdot\|$ in E are compatible. Let $\mathcal{T} : E \to E$ be a partially continuous, nondecreasing and partially compact operator. If there exists an element $x_0 \in E$ such that $x_0 \preceq \mathcal{T}x_0$ or $\mathcal{T}x_0 \preceq x_0$, then the operator equation $\mathcal{T}x = x$ has a solution x^* in E and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges monotonically to x^*.*

Remark 2.2. *The conclusion of Theorem 2.1 also remains true if we replace the compatibility of E with respect to the order relation $\leq$ and the norm $\|\cdot\|$ by the weaker condition of the compatibility of every compact chain C in E (cf. [3, 8]). The later condition holds if $\leq$ and $\|\cdot\|$ are compatible in every partially compact subset of E.*

Remark 2.3. *The regularity of E in above Theorem 2.1 may be replaced with a stronger continuity condition of the operator $\mathcal{T}$ on E which is a result proved in Dhage [2].*

3 Initial Value Problems

Given a closed and bounded interval $J = [t_0, t_0 + a]$ of the real line $\mathbb{R}$ for some $t_0, a \in \mathbb{R}$ with $t_0 \geq 0$ and $a > 0$, consider the initial value problem (in short IVP) of second order ordinary nonlinear hybrid differential equation,

$$\left.\begin{aligned} x''(t) &= f(t, x(t)),\ t \in J, \\ x(t_0) &= \alpha_0\,,\ x'(t_0) = \alpha_1, \end{aligned}\right\} \tag{3.1}$$

where $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

By a *solution* of the IVP (3.1) we mean a function $x \in C^2(J, \mathbb{R})$ that satisfies equation (3.1), where $C^2(J, \mathbb{R})$ is the space of twice continuously differentiable real-valued functions defined on J.

The IVP (3.1) is well-known in the literature and discussed at length for existence as well as other aspects of the solutions. In the present paper it is proved that the existence of the solutions may be proved under weaker partially continuity and partially compactness type conditions.

The equivalent integral form of the IVP (3.1) is considered in the function space $C(J, \mathbb{R})$ of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\leq$ in $C(J, \mathbb{R})$ by

$$\|x\| = \sup_{t \in J} |x(t)| \tag{3.2}$$

and

$$x \leq y \iff x(t) \leq y(t) \tag{3.3}$$

for all $t \in J$. Clearly, $C(J, \mathbb{R})$ is a Banach space with respect to above supremum norm and also partially ordered w.r.t. the above partially order relation $\leq$. It is known that the partially ordered Banach space $C(J, \mathbb{R})$ is regular and lattice so that every pair of elements of E has a lower and an upper bound in it. The following lemma follows immediately form Arzellá-Ascoli theorem.

Lemma 3.1. *Let $(C(J, \mathbb{R}), \leq, \|\cdot\|)$ be a partially ordered Banach space with the norm $\|\cdot\|$ and the order relation $\leq$ defined by (3.2) and (3.3) respectively. Then $\|\cdot\|$ and $\leq$ are compatible in every partially compact subset of $C(J, \mathbb{R})$.*

Proof. The proof of the lemma appears in Dhage and Dhage [8] and Dhage [5], but since it is not well-known, we give the details of the proof for the sake of completeness. Let S be a partially compact subset of $C(J,\mathbb{R})$ and let $\{x_n\}$ be a monotone nondecreasing sequence of points in S. Then we have

$$x_1(t) \le x_2(t) \le \cdots \le x_n(t) \le \cdots \tag{ND}$$

for each $t \in J$.

Suppose that a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is convergent and converges to a point x in S. Then the subsequence $\{x_{n_k}(t)\}$ of the monotone real sequence $\{x_n(t)\}$ is convergent. By monotone characterization, the sequence $\{x_n(t)\}$ is convergent and converges to a point $x(t)$ in $\mathbb{R}$ for each $t \in J$. This shows that the sequence $\{x_n(t)\}$ converges point-wise in S. To show the convergence is uniform, it is enough to show that the sequence $\{x_n(t)\}$ is equicontinuous. Since S is partially compact, every chain or totally ordered set and consequently $\{x_n\}$ is an equicontinuous sequence by Arzelá-Ascoli theorem. Hence $\{x_n\}$ is convergent and converges uniformly to x. As a result $\|\cdot\|$ and $\le$ are compatible in S. This completes the proof. □

We need the following definition in what follows.

Definition 3.7. *A function $u \in C^2(J,\mathbb{R})$ is said to be a lower solution of the IVP (3.1) if it satisfies*

$$\left.\begin{aligned} u''(t) \le&\quad f(t,u(t)),\ t \in J, \\ u(t_0) \le&\quad \alpha_0, \quad u'(t_0) \le \alpha_1, \end{aligned}\right\} \tag{$*$}$$

Similarly, an upper solution v to the IVP (3.1) *is defined on J, by reversing the above inequalities.*

We consider the following set of assumptions in what follows:

(H_1) There exists a constant $K > 0$ such that $|f(t,x)| \le K$ for all $t \in J$ and $x \in \mathbb{R}$.

(H_2) The mapping $x \mapsto f(t,x)$ is monotone nondecreasing for each $t \in J$.

(H_3) The IVP (3.1) has a lower solution $u \in C^2(J,\mathbb{R})$.

Lemma 3.2. *For a given integrable function $h : J \to \mathbb{R}$, a function $u \in C^2(J,\mathbb{R})$ is a solution of the IVP*

$$\left.\begin{aligned} x''(t) &= h(t),\ t \in J, \\ x(t_0) &= \alpha_0, \quad x'(t_0) = \alpha_1, \end{aligned}\right\} \tag{3.4}$$

if and only if it is a solution of the nonlinear integral equation,

$$x(t) = \alpha_0 + x_1(t-t_0) + \int_{t_0}^{t} (t-s)h(s)\,ds,\ t \in J. \tag{3.5}$$

Theorem 3.2. *Assume that hypotheses (H_1) through (H_3) hold. Then the IVP (3.1) has a solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by*

$$x_{n+1}(t) = x(t) = \alpha_0 + \alpha_1(t-t_0) + \int_{t_0}^{t} (t-s)f(s,x_n(s))\,ds, \tag{3.6}$$

for all $t \in \mathbb{R}$, where $x_0 = u$, converges monotonically to x^*.

Proof. Set $E = C(J,\mathbb{R})$. Then by Lemma 3.1, every compact chain in E is compatible with respect to the norm $\|\cdot\|$ and order relation $\le$. Define the operator $\mathcal{T}$ by

$$\mathcal{T}x(t) = \alpha_0 + \alpha_1(t-t_0) + \int_{t_0}^{t} (t-s)f(s,x(s))\,ds,\ t \in J. \tag{3.7}$$

From the continuity of the integral, it follows that $\mathcal{T}$ defines the map $\mathcal{T} : E \to E$. Now by Lemma 3.2, the IVP (3.1) is equivalent to the operator equation

$$\mathcal{T}x(t) = x(t),\ \ t \in J. \tag{3.8}$$

We shall show that the operator $\mathcal{T}$ satisfies all the conditions of Theorem 2.1. This is achieved in the series of following steps.

Step I: *$\mathcal{T}$ is nondecreasing on E.*

Let $x, y \in E$ be such that $x \leq y$. Then by hypothesis (H_2), we obtain

$$\begin{aligned} \mathcal{T}x(t) &= \alpha_0 + \alpha_1(t-t_0) + \textstyle\int_{t_0}^{t}(t-s)f(s,x(s))\,ds \\ &\leq \alpha_0 + \alpha_1(t-t_0) + \textstyle\int_{t_0}^{t}(t-s)f(s,y(s))\,ds \\ &= \mathcal{T}y(t), \end{aligned}$$

for all $t \in J$. This shows that $\mathcal{T}$ is nondecreasing operator on E into E.

Step II: *$\mathcal{T}$ is partially continuous on E.*

Let $\{x_n\}$ be a sequence in a chain C in E such that $x_n \to x$ for all $n \in \mathbb{N}$. Then, by dominated convergence theorem, we have

$$\begin{aligned} \lim_{n\to\infty} \mathcal{T}x_n(t) &= \lim_{n\to\infty}\left[\alpha_0 + \alpha_1(t-t_0) + \int_{t_0}^{t}(t-s)f(s,x_n(s))\,ds\right] \\ &= \alpha_0 + \alpha_1(t-t_0) + \int_{t_0}^{t}(t-s)\left[\lim_{n\to\infty} f(s,x_n(s))\right]\,ds \\ &= \alpha_0 + \alpha_1(t-t_0) + \int_{t_0}^{t}(t-s)f(s,x(s))\,ds \\ &= \mathcal{T}x(t), \end{aligned}$$

for all $t \in J$. This shows that $\{\mathcal{T}x_n\}$ converges to $\mathcal{T}x$ pointwise on J.

Next, we will show that $\{\mathcal{T}x_n\}$ is an equicontinuous sequence of functions in E. Let $t_1, t_2 \in J$ be arbitrary with $t_1 < t_2$. Then

$$\begin{aligned} |\mathcal{T}x_n(t_2) - \mathcal{T}x_n(t_1)| &\leq |\alpha_1|\,|t_2-t_1| + \left|\int_{t_0}^{t_2}(t_2-s)f(s,x_n(s))\,ds - \int_{t_0}^{t_1}(t_1-s)f(s,x_n(s))\,ds\right| \\ &\leq |\alpha_1|\,|t_2-t_1| + \left|\int_{t_0}^{t_1}(t_2-t_1)f(s,x_n(s))\,ds\right| + \left|\int_{t_1}^{t_2}(t_2-s)f(s,x_n(s))\,ds\right| \\ &\leq |\alpha_1|\,|t_2-t_1| + K\int_{t_0}^{t_0+a}|t_2-t_1|\,ds + K\int_{t_1}^{t_2}|t_2-s|\,ds \\ &\leq \big(|\alpha_1| + 2a\,K\big)\,|t_2-t_1| \\ &\to 0 \quad as \quad t_1 \to t_2, \end{aligned}$$

uniformly for all $n \in \mathbb{N}$. This shows that the convergence $\mathcal{T}x_n \to \mathcal{T}x$ is uniformly and hence $\mathcal{T}$ is partially continuous on E.

Step III: *$\mathcal{T}$ is partially compact on E.*

Let C be an arbitrary chain in E. We show that $\mathcal{T}(C)$ is a uniformly bounded and equicontinuous set in E. First we show that $\mathcal{T}(C)$ is uniformly bounded. Let $x \in C$ be arbitrary. Then,

$$\begin{aligned} |\mathcal{T}x(t)| &\leq |\alpha_0| + |\alpha_1|(t-t_0) + \left|\int_{t_0}^{t}(t-s)f(s,x(s))\,ds\right| \\ &\leq |\alpha_0| + |\alpha_1|a + \int_{t_0}^{t_0+a}|t-s|\,|f(s,x(s))|\,ds \\ &\leq |\alpha_0| + |\alpha_1|a + a^2\,K \\ &= r, \end{aligned}$$

for all $t \in J$. Taking supremum over t, we obtain $\|\mathcal{T}x\| \le r$ for all $x \in C$. Hence $\mathcal{T}$ is a uniformly bounded subset of E. Next, we will show that $\mathcal{T}(C)$ is an equicontinuous set in E. Let $t_1, t_2 \in J$ be arbitrary with $t_1 < t_2$. Then

$$\begin{aligned}
|\mathcal{T}x(t_2) - \mathcal{T}x(t_1)| &\le |\alpha_1|\,|t_2 - t_1| + \left| \int_{t_0}^{t_2} (t_2 - s) f(s, x(s))\, ds - \int_{t_0}^{t_1} (t_1 - s) f(s, x(s))\, ds \right| \\
&\le |\alpha_1|\,|t_2 - t_1| + \left| \int_{t_0}^{t_1} (t_2 - t_1) f(s, x(s))\, ds \right| + \left| \int_{t_1}^{t_2} (t_2 - s) f(s, x(s))\, ds \right| \\
&\le |\alpha_1|\,|t_2 - t_1| + K \int_{t_0}^{t_0+a} |t_2 - t_1|\; ds + K \int_{t_1}^{t_2} |t_2 - s|\, ds \\
&\le \big(|\alpha_1| + 2a\,K \big)\, |t_2 - t_1| \\
&\to 0 \quad as \quad t_1 \to t_2,
\end{aligned}$$

uniformly for all $x \in C$. Hence $\mathcal{T}(C)$ is compact subset of E and consequently $\mathcal{T}$ is a partially compact operator on E into itself.

Step IV: *u satisfies the operator inequality $u \le \mathcal{T}u$.*

Since the hypothesis (H_3) holds, u is a lower solution of (3.1) defined on J. Then

$$u''(t) \le f(t, u(t)), \tag{3.9}$$

satisfying,

$$u(t_0) \le \alpha_0, \quad u'(t_0) \le \alpha_1, \tag{3.10}$$

for all $t \in J$.

Integrating (3.9) from t_0 to t, we obtain

$$u'(t) \le \alpha_1 + \int_{t_0}^{t} f(s, u(s))\, ds. \tag{3.11}$$

Again integrating (3.11) from t_0 to t,

$$u(t) \le \alpha_0 + \alpha_1 (t - t_0) + \int_{t_0}^{t} (t - s) f(s, u(s))\, ds = \mathcal{T}u(t)$$

for all $t \in J$. This show that u is a lower solution of the operator equation $x = \mathcal{T}x$.

Thus $\mathcal{T}$ satisfies all the conditions of Theorem 2.1 in view of Remark 2.2 and we apply to conclude that the operator equation $\mathcal{T}x = x$ has a solution. Consequently the integral equation and the IVP (3.1) has a solution x^* defined on J. Furthermore, the sequence $\{x_n\}$ of successive approximations defined by (3.1) converges monotonically to x^*. This completes the proof. □

Remark 3.4. *The conclusion of Theorem 3.2 also remains true if we replace the hypothesis (H_3) with the following one:*

(H_3') The IVP (3.1) *has an upper solution $v \in C^2(J, \mathbb{R})$.*

Example 3.1. *Given a closed and bounded interval $J = [0, 1]$, consider the IVP,*

$$\left.\begin{aligned} x''(t) &= \tanh x(t), \; t \in J, \\ x(0) &= 0, \; x'(0) = 1. \end{aligned}\right\} \tag{3.12}$$

Here, $f(t, x) = \tanh x$. Clearly, the functions f is continuous on $J \times \mathbb{R}$. The function f satisfies the hypothesis (H_1) with $K = 1$. Moreover, the function $f(t, x) = \tanh x$ is nonincreasing in x for each $t \in J$ and so the hypothesis (H_2) is satisfied.

Finally, the IVP (3.1) has a lower solution $u(t) = t - \dfrac{t^2}{2}$ defined on J. Thus all hypotheses of Theorem 3.2 are satisfied. Hence we apply Theorem 3.2 and conclude that the IVP (3.12) has a solution x^* defined on J and the sequence $\{x_n\}$ defined by

$$x_{n+1}(t) = t + \int_0^t (t - s) \tanh x_n(s)\, ds, \tag{3.13}$$

for all $t \in J$, where $x_0 = u$, converges monotonically to x^*.

Remark 3.5. *In view of Remark 3.4, the existence of the solutions x^* of the IVP* (3.12) *may be obtained under the assumption of an upper solution $v(t) = \dfrac{t^2}{2} + t$ defined on J and the sequence $\{x_n\}$ defined by*

$$x_{n+1}(t) = t + \int_0^t (t-s)\tanh x_n(s)\,ds, \tag{3.14}$$

for all $t \in J$, where $x_0 = v$, converges monotonically to x^*.

4 Periodic Boundary Value Problems

Given a closed and bounded interval $J = [0, T]$ of the real line $\mathbb{R}$ consider the periodic boundary value problem (in short PBVP) of second order ordinary nonlinear differential equation

$$\left.\begin{aligned} x''(t) = f(t, x(t)),\ t \in J,\\ x(0) = x(T)\,,\ x'(0) = x'(T), \end{aligned}\right\} \tag{4.15}$$

where $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

By a solution of the PBVP (4.15) we mean a function $x \in C^2(J, \mathbb{R})$ that satisfies equation (4.15).

The PBVP (4.15) is well-known in the literature and discussed at length for existence as well as other aspects of the solutions. In the present paper it is proved that the existence as well as algorithm of the solutions may be proved for periodic boundary value problems of nonlinear second order ordinary differential equations under weaker partially continuity and partially compactness type conditions.

We need the following definition in what follows.

Definition 4.8. *A function $u \in C^2(J, \mathbb{R})$ is said to be a lower solution of the of PBVP (4.15) if it satisfies*

$$\left.\begin{aligned} u''(t) \le f(t, u(t)),\ t \in J,\\ u(0) \le u(T),\ u'(0) \le u'(T). \end{aligned}\right\} \tag{$**$}$$

Similarly, an upper solution v to the PBVP (4.15) *is defined on J, by reversing the above inequalities.*

(H_4) The PBVP (4.15) has a lower solution $u \in C^2(J, \mathbb{R})$.

Consider the PBVP

$$\left.\begin{aligned} x''(t) + h(t)x(t) = \tilde{f}(t, x(t)),\ t \in J,\\ x(0) = x(T)\,,\ x'(0) = x'(T), \end{aligned}\right\} \tag{4.16}$$

where $\tilde{f} : J \times \mathbb{R} \to \mathbb{R}$ and

$$\tilde{f}(t, x) = f(t, x) + h(t)x. \tag{4.17}$$

Remark 4.6. *A function $u \in C^2(J, \mathbb{R})$ is a lower solution of the PBVP* (4.15) *if and only if it is a solution of the PBVP (4.16) defined on J. A similar assertion is also true for an upper solution. A function $u \in C^2(J, \mathbb{R})$ is a solution of the PBVP* (4.15) *if and only if it is a lower as well as an upper solution of* (4.15) *defined on J.*

The following useful lemma may be found in Torres [14].

Lemma 4.3. *If h is a continuous function then for every $\sigma \in C[0, T]$, the linear periodic boundary value problem*

$$\left.\begin{aligned} x''(t) + h(t)x(t) = \sigma(t), \quad t \in J,\\ x(0) = x(T), \quad x'(0) = x'(T), \end{aligned}\right\} \tag{4.18}$$

has a unique solution expressed by

$$x(t) = \int_0^T G_h(t, s)\sigma(s)\,ds \tag{4.19}$$

where $G_h \in C([0, T] \times [0, T], \mathbb{R})$ is the Green function of the linear periodic boundary value problem (4.18), which satisfies the positivity $G_h(t, s) > 0$, for every $(t, s) \in [0, T] \times [0, T]$.

The Green's function G_h is continuous on $J \times J$ and therefore, the number

$$M_h := \max\{\, |G_h(t,s)| \;:\; t,s \in [0,T]\,\}$$

exists for all $h \in L^1(J,\mathbb{R}^+)$. In particular, if $h = 1$, then for the sake of convenience we write $G_1(t,s) = G(t,s)$ and $M_1 = M$.

By an application of above Lemma 4.3 we obtain

Lemma 4.4. *Let $f : J \times \mathbb{R} \to \mathbb{R}$ be a given continuous function. Then a function $u \in C(J,\mathbb{R})$ is a solution of the PBVP* (4.15) *if and only if it is a solution of the nonlinear integral equation*

$$x(t) = \int_0^T G(t,s)\tilde{f}(s,x(s))\,ds \tag{4.20}$$

for all $t \in J$.

Theorem 4.3. *Assume that hypotheses (H_1), (H_2) and (H_4) hold. Then the PBVP (4.15) has a solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by*

$$x_{n+1}(t) = \int_0^T G(t,s)\tilde{f}(s,x_n(s))\,ds \tag{4.21}$$

for all $t \in J$, where $x_0 = u$ converges monotonically to x^*.

Proof. Set $E = C(J,\mathbb{R})$. Then by Lemma 3.1, every compact chain in E is compatible with respect to the norm $\|\cdot\|$ and order relation $\leq$. Define the operator $\mathcal{T}$ on E by

$$\mathcal{T}x(t) = \int_0^T G(t,s)\tilde{f}(s,x(s))\,ds, \;\; t \in J. \tag{4.22}$$

From the continuity of the integral, it follows that $\mathcal{T}$ defines the map $\mathcal{T} : E \to E$. Now by Lemma 4.4, the PBVP (4.15) is equivalent to the operator equation

$$\mathcal{T}x(t) = x(t), \;\; t \in J. \tag{4.23}$$

We shall show that the operators $\mathcal{T}$ satisfies all the conditions of Theorem 2.1. This is achieved in the series of following steps.

Step I: *$\mathcal{T}$ is monotone nondecreasing on E.*

Let $x, y \in E$ be such that $x \leq y$. Then by hypothesis (H_1), we obtain

$$\begin{aligned} \mathcal{T}x(t) &= \int_0^T G(t,s)\tilde{f}(s,x(s))\,ds \\ &\leq \int_0^T G(t,s)\tilde{f}(s,y(s))\,ds \\ &= \mathcal{T}y(t), \end{aligned}$$

for all $t \in J$. This shows that $\mathcal{T}$ is nondecreasing operator on E into E.

Step II: *$\mathcal{T}$ is partially continuous on E.*

Let $\{x_n\}$ be a sequence in a chain C in E such that $x_n \to x$ for all $n \in \mathbb{N}$. Then, by dominated convergence theorem, we have

$$\begin{aligned} \lim_{n\to\infty} \mathcal{T}x_n(t) &= \lim_{n\to\infty} \int_0^T G(t,s)\tilde{f}(s,x_n(s))\,ds \\ &= \int_0^T G(t,s)\Big[\lim_{n\to\infty} \tilde{f}(s,x_n(s))\Big]\,ds \\ &= \int_0^T G(t,s)\tilde{f}(s,x(s))\,ds \\ &= \mathcal{T}x(t), \end{aligned}$$

for all $t \in J$. This shows that $\mathcal{T}x_n$ converges to $\mathcal{T}x$ pointwise on J.

Next, we will show that $\{\mathcal{T}x_n\}$ is an equicontinuous sequence of functions in E. Let $t_1, t_2 \in J$ be arbitrary with $t_1 < t_2$. Then

$$\begin{aligned}|\mathcal{T}x_n(t_2) - \mathcal{T}x_n(t_1)| &= \left|\int_0^T G(t_2,s)\tilde{f}(s,x_n(s))\,ds - \int_0^T G(t_1,s)\tilde{f}(s,x_n(s))\,ds\right| \\ &= \left|\int_0^T |G(t_2,s) - G(t_1,s)||\tilde{f}(s,x_n(s))|\,ds\right| \\ &\leq K\int_0^T |G(t_1,s) - G(t_2,s)|\,ds \\ &\to 0 \quad as \quad t_2 - t_1 \to 0\end{aligned}$$

uniformly for all $n \in \mathbb{N}$. This shows that the convergence $\mathcal{T}x_n \to \mathcal{T}x$ is uniformly and hence $\mathcal{T}$ is partially continuous on E.

Step III: *$\mathcal{T}$ is partially compact on E.*

Let C be an arbitrary chain in E. We show that $\mathcal{T}(C)$ is a uniformly bounded and equicontinuous set in E. First we show that $\mathcal{T}(C)$ is uniformly bounded. Let $x \in C$ be arbitrary. Then,

$$\begin{aligned}|\mathcal{T}x(t)| &= \left|\int_0^T G(t,s)\tilde{f}(s,x(s))\,ds\right| \\ &\leq \int_0^T G(t,s)|\tilde{f}(s,x(s))|\,ds \\ &\leq \int_0^T MK\,ds \\ &\leq MKT = r_1,\end{aligned}$$

for all $t \in J$. Taking supremum over t, we obtain $\|\mathcal{T}x\| \leq r_1$ for all $x \in C$. Hence $\mathcal{T}(C)$ is a uniformly bounded subset of E. Next, we will show that $\mathcal{T}(C)$ is an equicontinuous set in E. Let $t_1, t_2 \in J$ with $t_1 < t_2$. Then

$$\begin{aligned}|\mathcal{T}x(t_2) - \mathcal{T}x(t_1)| &= \left|\int_0^T [G(t_2,s) - G(t_1,s)]\tilde{f}(s,x(s))\,ds\right| \\ &\leq \int_0^T |G(t_2,s) - G(t_1,s)||\tilde{f}(s,x(s))|\,ds \\ &\leq \int_0^T |G(t_2,s) - G(t_1,s)|\,K\,ds \\ &\to 0 \quad as \quad t_1 \to t_2\end{aligned}$$

uniformly for all $x \in C$. Hence $\mathcal{T}(C)$ is compact subset of E and consequently $\mathcal{T}$ is a partially compact operator on E into itself.

Step IV: *u satisfies the operator inequality $u \leq \mathcal{T}u$.*

By hypothesis (H_4), the PBVP (4.15) has a lower solution u. Then we have

$$\left.\begin{aligned}&u''(t) \leq f(t,u(t)),\ t \in J, \\ &u(0) \leq u(T),\ u'(0) \leq u'(T).\end{aligned}\right\} \tag{4.24}$$

Integrating (4.24) twice which together with the definition of the operator $\mathcal{T}$ implies that $u(t) \leq \mathcal{T}u(t)$ for all $t \in J$. See lemma 4.5.1 of Heikkilä and Lakshmikantham [12] and references therein. Consequently, u is a lower solution to the operator equation $x = \mathcal{T}x$.

Thus $\mathcal{T}$ satisfies all the conditions of Theorem 2.1 with $x_0 = u$ and we apply it to conclude that the operator equation $\mathcal{T}x = x$ has a solution. Consequently the integral equation and the PBVP (4.15) has a solution x^* defined on J. Furthermore, the sequence $\{x_n\}$ of successive approximations defined by (4.15) converges monotonically to x^*. This completes the proof. □

Remark 4.7. *The conclusion of Theorem 4.3 also remains true if we replace the hypothesis* (H_4) *with the following one:*

(H_4') *The PBVP* (4.15) *has an upper solution* $v \in C^2(J, \mathbb{R})$.

Example 4.2. *Given a closed and bounded interval* $J = [0,1]$ *in* $\mathbb{R}$, *consider the PBVP,*

$$\left.\begin{aligned} x''(t) = f(t,x(t)) - x(t),\ t \in J, \\ x(0) = x(1)\,,\ x'(0) = x'(1), \end{aligned}\right\} \tag{4.25}$$

where $f : J \times \mathbb{R} \to \mathbb{R}$ *is defined as*

$$f(t,x) = \begin{cases} 1, & \text{if } x \le 0, \\ 1 + \dfrac{tx}{1+x}, & \text{if } x > 0. \end{cases}$$

Here, $\tilde{f}(t,x(t)) = f(t,x(t))$. Clearly, the functions f is continuous on $J \times \mathbb{R}$. The function f satisfies the hypothesis (H_1) with $K = 1$.

Again, $\tilde{f}$ is nondecreasing on $J \times \mathbb{R}$ and thus hypothesis (H_2) holds. Finally, the PBVP (4.25) has a lower solution $u(t) = \int_0^T G(t,s)\,ds$ defined on J. Thus all hypotheses of Theorem 4.3 are satisfied. Hence we apply Theorem 4.3 and conclude that the PBVP (4.15) has a solution x^* defined on J and the sequence $\{x_n\}$ defined by

$$x_{n+1}(t) = \int_0^T G(t,s)\tilde{f}(s,x_n(s))\,ds, \tag{4.26}$$

for all $t \in J$, where $x_0 = u$, converges monotonically to x^*.

Remark 4.8. *In view of Remark 4.7, the existence of the solutions* x^* *of the PBVP* (4.15) *may be obtained under the assumption of an upper solution* $v(t) = 2\int_0^T G(t,s)\,ds$ *defined on* J *and the sequence* $\{x_n\}$ *defined by*

$$x_{n+1}(t) = \int_0^T G(t,s)\tilde{f}(s,x_n(s))\,ds, \tag{4.27}$$

for all $t \in J$, *where* $x_0 = v$, *converges monotonically to* x^*.

5 Conclusion

From the foregoing discussion it is clear that unlike Schauder fixed point principle, the proofs of Theorems 3.2 and 4.3 do not invoke the construction of a non-empty, closed, convex and bounded subset of the Banach space of navigation which is mapped into itself by the operators related to the given differential equations. The convexity hypothesis is altogether omitted from the discussion and still we have proved the existence of the solutions for the differential equations considered in this paper with stronger conclusion. Similarly, unlike the use of Banach fixed point theorem, Theorems 3.2 and 4.3 do not make any use of any type of Lipschitz condition on the nonlinearity involved in the differential equations (3.1) and (4.15), but even then the algorithms for the solutions of the differential equations (3.1) and (4.15) are proved in terms of the successive iteration scheme. The nature of the convergence of the algorithms is not geometrical and so we are not able to obtain the rate of convergence of the algorithms to the solutions of the related problems. However, in a way we have been able to prove the existence results for the IVP (3.1) and PBVP (4.15) under much weaker conditions with a stronger conclusion of the monotone convergence of successive approximations to the solutions than those proved in the existing literature on nonlinear differential equations.

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