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Certain coefficient inequalities for p-valent functions

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Abstract

In the present paper, applying lemmas due to Nunokawa et al. [3] and Jack's lemma we obtain some coefficient inequalities for certain subclass of p-valent functions.

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1 Introduction

Let $\mathcal{A}_p$ denote the class of functions $f(z)$ of the form:

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \qquad p \in \mathbb{N} := \{1, 2, 3, \dots\}, \tag{1.1}$$

which are analytic in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. Put $\mathcal{A}_1 = \mathcal{A}$. The subclass of $\mathcal{A}$ consisting of all univalent functions $f(z)$ in Δ is denoted by $\mathcal{S}$. A function $f \in \mathcal{S}$ is called starlike (with respect to 0), denoted by $f \in \mathcal{S}^*$, if $tw \in f(\Delta)$ whenever $w \in f(\Delta)$ and $t \in [0, 1]$. A function $f \in \mathcal{S}$ that maps Δ onto a convex domain, denoted by $f \in \mathcal{K}$, is called a convex function. A function $f(z)$ in $\mathcal{A}$ is said to be starlike of order $0 \le \gamma < 1$ if it satisfies

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \gamma, \qquad z \in \Delta.$$

We denote by $\mathcal{S}^*(\gamma)$ the subclass of $\mathcal{A}$ consisting of all starlike functions of order γ in Δ. Furthermore, let $\mathcal{M}(\beta)$ be the class of functions $f(z) \in \mathcal{A}$ which satisfy

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < \beta, \qquad z \in \Delta.$$

for some real number β with $\beta > 1$. The class $\mathcal{M}(\beta)$ was investigated by Uralegaddi, Ganigi and Sarangi [6].

Further, let $\mathcal{P}(\gamma, p)$ denote the subclass of $\mathcal{A}_p$ consisting of functions $f(z)$ which satisfy

$$\mathfrak{Re}\left\{\frac{f(z)}{z^p}\right\} > \gamma, \qquad z \in \Delta,$$

for some real $0 \le \gamma < p$. The class $\mathcal{P}(1/2, 1) \equiv \mathcal{P}(1/2)$ was studied by Obradović et al. in [5]. We remark that $\mathcal{K} \subset \mathcal{P}(1/2)$.

Nunokawa, Cho, Kwon and Sokół [3] obtained the following results.

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Lemma 1.1. *Let $B(z)$ and $C(z)$ be analytic in Δ with*

$$|\mathfrak{Im}\{C(z)\}| < \mathfrak{Re}\{B(z)\}.$$

If $p(z)$ is analytic in Δ with $p(0) = 1$, and if

$$|\arg\{B(z)zp'(z) + C(z)p(z)\}| < \pi/2 + t(z),$$

where

$$t(z) = \begin{cases} \arg\{C(z) + iB(z)\} & \textit{when } \arg\{C(z) + iB(z)\} \in [0, \pi/2] \\ \arg\{C(z) + iB(z)\} - \pi/2 & \textit{when } \arg\{C(z) + iB(z)\} \in (\pi/2, \pi], \end{cases}$$

then we have

$$\mathfrak{Re}\{p(z)\} > 0, \qquad z \in \Delta.$$

Lemma 1.2. *Let $B(z)$ and $C(z)$ be analytic in Δ with*

$$\mathfrak{Re}\left\{\frac{C(z)}{B(z)}\right\} \geq -1, \qquad z \in \Delta.$$

If $p(z)$ is analytic in Δ with $p(0) = 0$, and if

$$|B(z)zp'(z) + C(z)p(z)| < |B(z) + C(z)|, \qquad z \in \Delta, \tag{1.2}$$

then we have

$$|p(z)| < 1, \qquad z \in \Delta.$$

In this paper, applying the Lemma 1.1, Lemma 1.2 and Jack's Lemma, we obtain coefficient conditions for some certain subclasses of p-valent functions.

2 Main results

Our first result is contained in the following:

Theorem 2.1. *Assume that $f \in \mathcal{A}_p$. If*

$$\left|\arg\left\{z^{1-p}f'(z) - (p-1)\frac{f(z)}{z^p} - \frac{\gamma}{p}\right\}\right| < \frac{3\pi}{4}, \qquad z \in \Delta, \tag{2.3}$$

then

$$\mathfrak{Re}\left\{\frac{f(z)}{z^p}\right\} > \frac{\gamma}{p}, \qquad z \in \Delta, \tag{2.4}$$

where $0 \leq \gamma < p$, that is $f \in \mathcal{P}(\gamma/p, p)$.

Proof. Let $f(z) \neq 0$ for $z \neq 0$ and let $p(z)$ be defined by

$$\left(1 - \frac{\gamma}{p}\right)p(z) + \frac{\gamma}{p} = \frac{f(z)}{z^p}, \qquad z \in \Delta, \tag{2.5}$$

where $0 \leq \gamma < p$. Then $p(z)$ is analytic in Δ, $p(0) = 1$ and

$$\left(1 - \frac{\gamma}{p}\right)p(z) + \left(1 - \frac{\gamma}{p}\right)zp'(z) = z^{1-p}f'(z) - (p-1)\frac{f(z)}{z^p} - \frac{\gamma}{p}.$$

If we put $B(z) = C(z) = 1 - \frac{\gamma}{p}$, from (2.3) and applying Lemma 1.1 we obtain (2.4) immediately. □

If we take $p = 1$ in Theorem 2.1, then it becomes the result from [4] of the following form:

Corollary 2.1. *Let $f \in \mathcal{A}$. If*

$$|\arg\{f'(z) - \gamma\}| < \frac{3\pi}{4}, \qquad z \in \Delta,$$

then

$$\mathfrak{Re}\left\{\frac{f(z)}{z}\right\} > \gamma, \qquad z \in \Delta.$$

Theorem 2.2. *Assume that $f \in \mathcal{A}_p$. If*

$$\left| z^{1-p} f'(z) - (p-1)\frac{f(z)}{z^p} - \frac{\gamma}{p} \right| < 2\left(1 - \frac{\gamma}{p}\right), \qquad z \in \Delta, \tag{2.6}$$

then $f \in \mathcal{P}(\gamma/p, p)$, where $0 \le \gamma < p$.

Proof. For $0 \le \gamma < p$, let $p(z)$ be defined by (2.5). Then from (2.6) and applying Lemma 1.2, we can obtain the result. □

Putting $p = 1$, in Theorem 2.2, we have:

Corollary 2.2. *Let $f \in \mathcal{A}$. If*

$$|f'(z) - \gamma| < 2(1-\gamma), \qquad z \in \Delta,$$

then

$$\mathfrak{Re}\left\{\frac{f(z)}{z}\right\} > \gamma, \qquad z \in \Delta.$$

Putting $\gamma = 1/2$, in Corollary 2.2, we have:

Corollary 2.3. *Let $f \in \mathcal{A}$. If*

$$\left| f'(z) - \frac{1}{2} \right| < 1, \qquad z \in \Delta,$$

then $f \in \mathcal{P}(1/2)$.

The following Lemma (popularly known *Jack's lemma* (see [1])) will be required on our present investigation.

Lemma 2.3. *Let the (nonconstant) function $w(z)$ be analytic in Δ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a the point $z_0 \in \Delta$, then*

$$z_0 w'(z_0) = c w(z_0),$$

where c is a real number and $c \ge 1$.

Theorem 2.3. *Assume that $f(z)/z^p \neq \gamma$ and that $f \in \mathcal{A}_p$ satisfies the inequality*

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > p + \frac{\gamma - 1}{2(\gamma+1)}, \qquad z \in \Delta, \tag{2.7}$$

then $f \in \mathcal{P}(\frac{1+\gamma}{2}, p)$, where $0 \le \gamma < p$.

Proof. Define the function $w(z)$ by

$$\frac{f(z)}{z^p} = \frac{1 + \gamma w(z)}{1 + w(z)}, \qquad (w(z) \neq -1, |z| < 1), \tag{2.8}$$

where $0 \le \gamma < p$. Because $f(z)/z^p \neq \gamma$, then $w(z)$ is analytic in Δ and $w(0) = 0$. From (2.7), some computation yields

$$\frac{zf'(z)}{f(z)} = p + \frac{\gamma z w'(z)}{1 + \gamma w(z)} - \frac{z w'(z)}{1 + w(z)}. \tag{2.9}$$

Suppose there exists a point $z_0 \in \Delta$ such that

$$|w(z_0)| = 1 \quad \text{and} \quad |w(z)| < 1 \quad \text{when} \quad |z| < |z_0|.$$

Applying Lemma 2.3, then we have

$$z_0 w'(z_0) = c w(z_0) \quad (c \ge 1,\, w(z_0) = e^{i\theta}, \theta \in \mathbb{R}). \tag{2.10}$$

Thus, by using (2.9) and (2.10), it follows that

$$\begin{aligned}\mathfrak{Re}\left\{\frac{z_0f'(z_0)}{f(z_0)}\right\} &= p+\mathfrak{Re}\left\{\frac{c\gamma e^{i\theta}}{1+\gamma e^{i\theta}}\right\}-\mathfrak{Re}\left\{\frac{\gamma e^{i\theta}}{1+e^{i\theta}}\right\}\\ &= p+\frac{c\gamma(\gamma+\cos\theta)}{1+\gamma^2+2\gamma\cos\theta}-\frac{c}{2}\\ &\leq \frac{(2p+1)\gamma+(2p-1)}{2(1+\gamma)},\end{aligned}$$

which contradicts the hypothesis (2.7). It follows that $|w(z)|<1$, that is,

$$\left|\frac{(f(z)/z^p)-1}{\gamma-(f(z)/z^p)}\right|<1, \qquad (z\in\Delta,\ 0\leq\gamma<p).$$

This evidently completes the proof of Theorem 2.3. □

If we take $\gamma=0$ and $p=1$, in Theorem 2.3, we get:

Corollary 2.4. *Let $f\in\mathcal{A}$. If*

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\frac{1}{2}, \qquad z\in\Delta,$$

then $f\in\mathcal{P}(1/2)$, that is $\mathcal{S}^*(1/2)\subset\mathcal{P}(1/2)$.

Theorem 2.4. *Assume that $f\in\mathcal{A}_p$ satisfies the inequality*

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\}<p+\frac{1-\gamma}{2-\gamma}, \qquad z\in\Delta, \tag{2.11}$$

then

$$\left|\frac{f(z)}{z^p}-1\right|<|1-\gamma|, \qquad z\in\Delta,$$

where $0\leq\gamma<p$.

Proof. Let us $f(z)/z^p\neq\gamma$. Consider the function $w(z)$ defined by

$$\frac{f(z)}{z^p}=1+(1-\gamma)w(z), \qquad |z|<1, \tag{2.12}$$

where $0\leq\gamma<p$. Then $w(z)$ is analytic in Δ and $w(0)=0$. From (2.12), some computation yields

$$\frac{zf'(z)}{f(z)}=p+\frac{(1-\gamma)zw'(z)}{1+(1-\gamma)w(z)}. \tag{2.13}$$

Suppose there exists a point $z_0\in\Delta$ such that

$$|w(z_0)|=1 \quad \text{and} \quad |w(z)|<1 \quad \text{when} \quad |z|<|z_0|.$$

Applying Lemma 2.3, then we have

$$z_0w'(z_0)=cw(z_0) \quad (c\geq 1,\ w(z_0)=e^{i\theta},\theta\in\mathbb{R}). \tag{2.14}$$

Thus, by using (2.13) and (2.14), it follows that

$$\begin{aligned}\mathfrak{Re}\left\{\frac{z_0f'(z_0)}{f(z_0)}\right\} &= p+\mathfrak{Re}\left\{\frac{c(1-\gamma)e^{i\theta}}{1+(1-\gamma)e^{i\theta}}\right\}\\ &= p+\frac{c(1-\gamma)(1-\gamma+\cos\theta)}{1+(1-\gamma)^2+2(1-\gamma)\cos\theta}\\ &\geq \frac{(2p+1)-\gamma(p+1)}{2-\gamma},\end{aligned}$$

which contradicts the hypothesis (2.11). It follows that $|w(z)|<1$, that is,

$$\left|\frac{f(z)}{z^p}-1\right|<|1-\gamma|, \qquad (z\in\Delta,0\leq\gamma<p).$$

This evidently completes the proof of Theorem 2.4. □

Corollary 2.5. *Assume that $f \in \mathcal{A}$. If f satisfies the inequalities*

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < 1 + \frac{1-\gamma}{2-\gamma}, \qquad z \in \Delta,$$

then

$$\mathfrak{Re}\left\{\frac{f(z)}{z}\right\} > \gamma, \qquad z \in \Delta,\ 0 \le \gamma < 1.$$

Taking $\gamma = 1/2$ in Corollary 2.5, we have:

Corollary 2.6. *Assume that $f \in \mathcal{A}$. If f satisfies the inequalities*

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < \frac{4}{3}, \qquad z \in \Delta,$$

then

$$f \in \mathcal{P}(1/2), \qquad z \in \Delta,$$

that is, $\mathcal{M}(4/3) \subset \mathcal{P}(1/2)$.

Combining Corollary 2.4 and 2.6, we have:

Corollary 2.7. *Assume that $f \in \mathcal{A}$. If f satisfies the following two-sided inequality*

$$\frac{1}{2} < \mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < \frac{4}{3}, \qquad z \in \Delta,$$

then

$$f \in \mathcal{P}(1/2), \qquad z \in \Delta,$$

that is, $\mathcal{S}(1/2, 3/4) \subset \mathcal{P}(1/2)$, where the class $\mathcal{S}(\alpha, \beta)$, $\alpha < 1$ and $\beta > 1$, was recently considered by K. Kuroki and S. Owa in [2].

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