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n-power Quasi-isometry and n-power Normal Composition Operators on L^2-spaces

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Abstract

In this paper, we give the characterizations of n-power Quasi-isometry and n-power normal composition operators. Further, we also discuss the characterization of the n-power Quasi-isometry composite multiplication operator.

Keywords: n-power quasi-isometry operator, n-power normal operator, composite multiplication operator.

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1 Introduction

Let $B(H)$ be the algebra of all bounded linear operators on an infinite dimensional complex Hilbert space H. An operator A is an n- power quasi-isometry if $A^{n-1}A^{*2}A^2 = A^*AA^{n-1}$ for all $n \in Z^+$[5]. The operator A is normal if $A^*A = AA^*$ and n-power normal if $A^nA^* = A^*A^n$ for all $n \geq 2$[3]. We denote the class of n-power normal operators and n-power quasi-isometry operators by $[nN]$ and $[nQI]$ respectively. The class of normal operators $\subset$ class$[nN]$. Also A is an n- power normal operator if and only if A^n is normal[3].

Let $(X, \sum, \lambda)$ be a sigma-finite measure space and let T be a measurable transformation from X to itself. If T is a measurable transformation then T^n is also a measurable transformation. Further, if T is non-singular then λT^{-1} is absolutely continuous with respect to λ and it follows that $\lambda(T^{-1})^n$ is absolutely continuous with respaect to λ. The Radon-Nikodym derivative of $\lambda(T^{-1})^n$ with respect to λ is denoted by h_n.

Associated with each transformation T is a conditional expectation operator
$E(f|T^{-1}(\sum)) = E(f)$ is defined for each non-negative function $f \in L^p \quad (1 \leq p < \infty)$ and is uniquely determined by the following conditions:
(i)$E(f)$ is $T^{-1}(\sum)$ measurable.
(ii) If B is any $T^{-1}(\sum)$ measurable set for which $\int_B f d\lambda$ converges then we have $\int_B f d\lambda = \int_B E(f) d\lambda$.

The conditional expectation operator E has the following properties:
(i) $E(f.(g \circ T)) = E(f).(g \circ T)$
(ii) E is monotonically increasing. (i.e) if $f \leq g$ a.e then $E(f) \leq E(g)$ a.e.
(iii) $E(1) = 1$.

When E is defined on a possible infinite σ-finite measure space it behaves similarly to expectations on standard probability spaces. As an operator on $L^2(\lambda)$, E is the projection operator onto the closure of the range of C_T.

Let π be an essentially bounded function. The multiplication operator M_π on $L^2(\lambda)$ induced by π, is given by, $M_\pi f = \pi.f$ for $f \in L^2(\lambda)$.

A composition operator C_T on $L^2(\lambda)$ is a bounded linear operator given by composition with a map $T : X \to X$ as, $C_T f = f \circ T$ for all $f \in L^2(\lambda)$ and C_T^* is given by $C_T^* f = hE(f) \circ T^{-1}$ for all $f \in L^2(\lambda)$. A weighted

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composition operator W is a linear transformation acting on a set of complex valued $\sum$- measurable functions f of the form, $Wf = wC_T f$ where w is a complex valued $\sum$- measurable function. When $w = 1$ we say W is a composition operator.

The adjoint W^* is defined as, $W^* f = hE(wf) \circ T^{-1}$ for $f \in L^2(\lambda)$. Also, $w_n = w.(w \circ T).(w \circ T^2).....(w \circ T^{n-1})$. For $f \in L^2(\lambda)$, $W^n f = w_n f \circ T^n$, $W^{*^n} f = h_n E(w_n.f) \circ T^{-n}$.

A composite multiplication operator is a linear transformation acting on a set of complex valued $\sum$ measurable functions f of the form $M_{u,T}(f) = C_T M_u(f) = (uf) \circ T = u \circ T.f \circ T$ where u is a complex valued, $\sum$ measurable function. In case, $u = 1$ almost everywhere, $M_{u,T}$ becomes a composition operator. The adjoint of $M_{u,T}$ is given by $M^*_{u,T} f = uh.E(f) \circ T^{-1}$.

Various properties of composition operators and weighted composition operators on L^2 spaces have been analyzed by many authors. In particular, spectra of composition operators and their generalized Alugthe transformations as weighted composition operators are characterized in[4]. In this paper we study the characterisations of the $[nQI]$ and $[nN]$ class of composition operators. The characterisations of class$[nN]$ operators A are evaluated mainly by the aid of the normality of A^n. In[7], the characterisations of n-power normal and n-power quasinormal composite multiplication operators are studied. We study the characterisations of quasi-isometry and n-power quasi-isometry composite multiplication operators.

2 Characterization of the class$[nQI]$ composition operators

The following Lemmas of [2] and [8] play an important role in the following Theorems:

Lemma 2.1. *[2, 8] Let P be the projection of $L^2(\lambda)$ onto $\overline{R(C_T)}$, where $\overline{R(C_T)}$ denotes the closure of the range of C_T. Then,*

(i) $C_T^* C_T f = hf$ *and* $C_T C_T^* f = h \circ T P f \ \ \forall f \in L^2(\lambda)$.

(ii) $\overline{R(C_T)} = \{ f \in L^2(\lambda) : f \text{ is } T^{-1}(\sum) \text{ measurable} \}$.

(iii) If f is $T^{-1}(\sum)$ measurable, and g and fg belong to $L^2(\lambda)$, then $P(fg) = fP(g)$ (f need not be in $L^2(\lambda)$). Also, for $k \in \mathbb{N}$

(iv) $(C_T^* C_T)^k f = h^k f$.

(v) $(C_T C_T^*)^k f = (h \circ T)^k P f$.

(vi) E is the identity operator on $L^2(\lambda)$ if and only if $T^{-1}(\sum) = \sum$.

The following Theorem gives the characterization of n- power quasi-isometry operators.

Theorem 2.1. *Let $C_T \in B(L^2(\lambda))$. Then C_T is in the class$[nQI]$ if and only if $h \circ T^{n-1}.E(h) \circ T^{n-2} = h$.*

Proof.

$$
\begin{aligned}
C_T \in [nQI] \quad &\Leftrightarrow \quad C_T^{n-1} C_T^{*^2} C_T^2 f = C_T^* C_T^n f, \text{ where } C_T^* f = h.E(f) \circ T^{-1}. \\
&\Leftrightarrow \quad C_T^{n-1} C_T^{*^2} (f \circ T^2) = C_T^* (f \circ T^n) \\
&\Leftrightarrow \quad C_T^{n-1} C_T^* [h.E(f \circ T^2) \circ T^{-1}] = h.E(f \circ T^n) \circ T^{-1} \\
&\Leftrightarrow \quad C_T^{n-1} C_T^* [h.f \circ T] = h.f \circ T^{n-1} \\
&\Leftrightarrow \quad C_T^{n-1} [h.E(h.f \circ T) \circ T^{-1}] = h.f \circ T^{n-1} \\
&\Leftrightarrow \quad C_T^{n-1} [h.E(h) \circ T^{-1}.f] = h.f \circ T^{n-1} \\
&\Leftrightarrow \quad C_T^{n-2} [h.E(h) \circ T^{-1}.f] \circ T = h.f \circ T^{n-1} \\
&\Leftrightarrow \quad C_T^{n-2} [h \circ T.E(h).f \circ T] = h.f \circ T^{n-1} \\
&\Leftrightarrow \quad C_T^{n-3} [h \circ T^2.E(h) \circ T.f \circ T^2] = h.f \circ T^{n-1} \\
&\Leftrightarrow \quad C_T^{n-4} [h \circ T^3.E(h) \circ T^2.f \circ T^3] = h.f \circ T^{n-1} \\
&\Leftrightarrow \quad h \circ T^{n-1}.E(h) \circ T^{n-2}.f \circ T^{n-1} = h.f \circ T^{n-1} \\
&\Leftrightarrow \quad h \circ T^{n-1}.E(h) \circ T^{n-2} = h
\end{aligned}
$$

□

Theorem 2.2. *Let $C_T \in B(L^2(\lambda))$. Then C_T^* is in the class$[nQI]$ if and only if $h.E[h] \circ T^{-1}.E[h] \circ T^{-2}....E[h] \circ T^{-(n-2)}.E[h] \circ T^{-(n-3)}.E[h] \circ T^{-(n-2)}.E(f) \circ T^{-(n-1)} = h \circ T.E[h].E[h] \circ T^{-1}...E[h] \circ T^{-(n-2)}.E(f) \circ T^{-(n-1)}$.*

Proof.

$$C_T^* \in [nQI] \Leftrightarrow C_T^{*^{n-1}} C_T^2 C_T^{*^2} f = C_T C_T^{*^n} f. \tag{2.1}$$

Now,

$$\begin{aligned} C_T^{*^{n-1}} C_T^2 C_T^{*^2} f &= C_T^{*^{n-1}} C_T^2 h.E(h) \circ T^{-1}.E(f) \circ T^{-2} \\ &= C_T^{*^{n-1}} h \circ T^2.E(h) \circ T.E(f) \\ &= C_T^{*^{n-2}} h.E[h \circ T^2.E(h) \circ T.E(f)] \circ T^{-1} \\ &= C_T^{*^{n-2}} h.h \circ T.E[h].E(f) \circ T^{-1} \\ &= C_T^{*^{n-3}} h.E[h] \circ T^{-1}.h.E[h] \circ T^{-1}.E(f) \circ T^{-2} \\ &= C_T^{*^{n-4}} h.E[h] \circ T^{-1}.E[h] \circ T^{-2}.E[h] \circ T^{-1}.E[h] \circ T^{-2}.E(f) \circ T^{-3} \\ &= C_T^{*^{n-5}} h.E[h] \circ T^{-1}.E[h] \circ T^{-2}.E[h] \circ T^{-3}.E[h] \circ T^{-2}.E[h] \circ T^{-3}.E(f) \circ T^{-4} \\ &= h.E[h] \circ T^{-1}.E[h] \circ T^{-2}....E[h] \circ T^{-(n-2)}.E[h] \circ T^{-(n-3)}.E[h] \circ T^{-(n-2)}. \\ &\quad E(f) \circ T^{-(n-1)}. \end{aligned}$$

And

$$\begin{aligned} C_T C_T^{*^n} f &= C_T C_T^{*^{(n-1)}} h.E(f) \circ T^{-1} \\ &= C_T C_T^{*^{(n-2)}} h.E[h.E(f) \circ T^{-1}] \circ T^{-1} \\ &= C_T C_T^{*^{(n-2)}} h.E[h] \circ T^{-1}.E(f) \circ T^{-2} \\ &= C_T C_T^{*^{(n-3)}} h.E[h] \circ T^{-1}.E[h] \circ T^{-2}.E(f) \circ T^{-3} \\ &= C_T h.E[h] \circ T^{-1}.E[h] \circ T^{-2}...E[h] \circ T^{-(n-1)}.E(f) \circ T^{-n} \\ &= h \circ T.E[h].E[h] \circ T^{-1}...E[h] \circ T^{-(n-2)}.E(f) \circ T^{-(n-1)}. \end{aligned}$$

Now (2.1) becomes $C_T^* \in [nQI] \Leftrightarrow h.E[h] \circ T^{-1}.E[h] \circ T^{-2}....E[h] \circ T^{-(n-2)}.E[h] \circ T^{-(n-3)}.E[h] \circ T^{-(n-2)}.E(f) \circ T^{-(n-1)} = h \circ T.E[h].E[h] \circ T^{-1}...E[h] \circ T^{-(n-2)}.E(f) \circ T^{-(n-1)}$. □

Example 2.1. *Let $X = \mathbb{N}$, the set of all natural numbers and λ be a counting measure on it. $T : \mathbb{N} \to \mathbb{N}$ is defined by $T(k) = k + 1, k \in \mathbb{N}$. Since $T^{n-1}(k) = k_1$where $k_1 \in \mathbb{N}$, $h_2 \circ T^{n-1}(k) = 1$ and $h(k) = 1$, C_T is of class$[nQI]$. Here C_T is the unilateral shift operator on l^2.*

3 Weighted composition operators of class$[nQI]$

Theorem 3.3. *Let W be a weighted composition operator then $W \in [nQI]$ if and only if $w_{n-1}.h \circ T^{n-1}.E(w) \circ T^{n-2}.E(h) \circ T^{n-2}.E(w.w_2) \circ T^{n-3}.f \circ T^{n-1} = h.E[w.w_n] \circ T^{-1}.f \circ T^{n-1}$.*

Proof.

$$W \in [nQI] \Leftrightarrow W^{n-1} W^{*^2} W^2 f = W^* W^n f. \tag{3.2}$$

Consider

$$\begin{aligned}
W^{n-1}W^{*^2}W^2f &= W^{n-1}W^{*^2}W[wf\circ T] \\
&= W^{n-1}W^{*^2}w[w.f\circ T]\circ T \\
&= W^{n-1}W^{*^2}w_2.f\circ T^2 \\
&= W^{n-1}W^{*}h.E[w.w_2.f\circ T^2]\circ T^{-1} \\
&= W^{n-1}W^{*}h.E[w.w_2]\circ T^{-1}.f\circ T \\
&= W^{n-1}h.E(w)\circ T^{-1}.E(h)\circ T^{-1}.E(w.w_2)\circ T^{-2}.f \\
&= W^{n-2}w.h\circ T.E(w).E(h).E(w.w_2)\circ T^{-1}.f\circ T \\
&= W^{n-3}w.w\circ T.h\circ T^2.E(w)\circ T.E(h)\circ T.E(w.w_2).f\circ T^2 \\
&= W^{n-4}w.w\circ T.w\circ T^2.h\circ T^3.E(w)\circ T^2.E(h)\circ T^2. \\
&\quad E(w.w_2)\circ T.f\circ T^3 \\
&= w.w\circ T.w\circ T^2...w\circ T^{n-2}.h\circ T^{n-1}.E(w)\circ T^{n-2}.E(h)\circ T^{n-2}. \\
&\quad E(w.w_2)\circ T^{n-3}.f\circ T^{n-1} \\
&= w_{n-1}.h\circ T^{n-1}.E(w)\circ T^{n-2}.E(h)\circ T^{n-2}. \\
&\quad E(w.w_2)\circ T^{n-3}.f\circ T^{n-1}
\end{aligned}$$

And

$$\begin{aligned}
W^{*}W^{n}f &= W^{*}W^{n-1}w.f\circ T \\
&= W^{*}W^{n-2}w.[w.f\circ T]\circ T \\
&= W^{*}W^{n-2}w.w\circ T.f\circ T^2 \\
&= W^{*}w.w\circ T...w\circ T^{n-1}.f\circ T^{n} \\
&= h.E[w.w.w\circ T...w\circ T^{n-1}.f\circ T^{n}]\circ T^{-1} \\
&= h.E[w.w_n]\circ T^{-1}.f\circ T^{n-1}.
\end{aligned}$$

Now (3.2) becomes
$W\in[nQI]\Leftrightarrow w_{n-1}.h\circ T^{n-1}.E(w)\circ T^{n-2}.E(h)\circ T^{n-2}.E(w.w_2)\circ T^{n-3}.f\circ T^{n-1}=h.E[w.w_n]\circ T^{-1}.f\circ T^{n-1}.$ □

Theorem 3.4. *Let W be a weighted composition operator then $W^*\in[nQI]$ if and only if $h.E[w]\circ T^{-1}.E[h]\circ T^{-1}.E[w]\circ T^{-2}.E[h]\circ T^{-2}.E[w]\circ T^{-3}.E[h]\circ T^{-3}....E[w]\circ T^{n-2}.E[h]\circ T^{n-2}.E[w.w_2]\circ T^{-(n-1)}.h\circ T^{-(n-3)}.E[w]\circ T^{-(n-2)}.E[h]\circ T^{-(n-2)}.E(wf)\circ T^{-(n-1)}=w.h\circ T.E[w].E[h].E[w]\circ T^{-1}.E[h]\circ T^{-1}....E[w]\circ T^{-(n-2)}.E[h]\circ T^{-(n-2)}.E(wf)\circ T^{-(n-1)}$.*

Proof.

$$W^*\in[nQI]\Leftrightarrow W^{*^{n-1}}W^2W^{*^2}f=WW^{*^n}f. \tag{3.3}$$

$$
\begin{aligned}
W^{*^{n-1}}W^2W^{*^2}f &= W^{*^{n-1}}W^2 h.E(w)\circ T^{-1}.E(h)\circ T^{-1}.E(wf)\circ T^{-2}\\
&= W^{*^{n-1}}W.w.h\circ T.E[w].E[h].E(wf)\circ T^{-1}\\
&= W^{*^{n-1}}w_2.h\circ T^2.E[w]\circ T.E[h]\circ T.E(wf)\\
&= W^{*^{n-2}}h.E[w.w_2.h\circ T^2.E[w]\circ T.E[h]\circ T.E(wf)]\circ T^{-1}\\
&= W^{*^{n-2}}h.E[w.w_2]\circ T^{-1}.h\circ T.E[w].E[h].E(wf)\circ T^{-1}\\
&= W^{*^{n-3}}h.E[w]\circ T^{-1}.E[h]\circ T^{-1}.E[w.w_2]\circ T^{-2}.h.\\
&\quad E[w]\circ T^{-1}.E[h]\circ T^{-1}.E(wf)\circ T^{-2}\\
&= W^{*^{n-4}}h.E[w]\circ T^{-1}.E[h]\circ T^{-1}.E[w]\circ T^{-2}.E[h]\circ T^{-2}.E[w.w_2]\circ T^{-3}.\\
&\quad h\circ T^{-1}.E[w]\circ T^{-2}.E[h]\circ T^{-2}.E(wf)\circ T^{-3}\\
&= h.E[w]\circ T^{-1}.E[h]\circ T^{-1}.E[w]\circ T^{-2}.E[h]\circ T^{-2}.E[w]\circ T^{-3}.E[h]\circ T^{-3}....\\
&\quad E[w]\circ T^{n-2}.E[h]\circ T^{n-2}.E[w.w_2]\circ T^{-(n-1)}.h\circ T^{-(n-3)}.E[w]\circ T^{-(n-2)}.\\
&\quad E[h]\circ T^{-(n-2)}.E(wf)\circ T^{-(n-1)}.
\end{aligned}
$$

And,

$$
\begin{aligned}
WW^{*^n}f &= WW^{*^{n-1}}h.E(wf)\circ T^{-1}\\
&= WW^{*^{n-2}}h.E[w.h.E(wf)\circ T^{-1}]\circ T^{-1}\\
&= WW^{*^{n-2}}h.E[w]\circ T^{-1}.E[h]\circ T^{-1}.E(wf)\circ T^{-2}\\
&= WW^{*^{n-3}}h.E[w]\circ T^{-1}.E[h]\circ T^{-1}.E[w]\circ T^{-2}.E[h]\circ T^{-2}.E(wf)\circ T^{-3}\\
&= Wh.E[w]\circ T^{-1}.E[h]\circ T^{-1}.E[w]\circ T^{-2}.E[h]\circ T^{-2}\\
&\quad ....E[w]\circ T^{-(n-1)}.E[h]\circ T^{-(n-1)}.E(wf)\circ T^{-n}\\
&= w.h\circ T.E[w].E[h].E[w]\circ T^{-1}.E[h]\circ T^{-1}\\
&\quad ....E[w]\circ T^{-(n-2)}.E[h]\circ T^{-(n-2)}.E(wf)\circ T^{-(n-1)}.
\end{aligned}
$$

Now (3.3) becomes $W^* \in [nQI] \Leftrightarrow h.E[w]\circ T^{-1}.E[h]\circ T^{-1}.E[w]\circ T^{-2}.E[h]\circ T^{-2}.E[w]\circ T^{-3}.E[h]\circ T^{-3}....E[w]\circ T^{n-2}.E[h]\circ T^{n-2}.E[w.w_2]\circ T^{-(n-1)}.h\circ T^{-(n-3)}.E[w]\circ T^{-(n-2)}.E[h]\circ T^{-(n-2)}.E(wf)\circ T^{-(n-1)} = w.h\circ T.E[w].E[h].E[w]\circ T^{-1}.E[h]\circ T^{-1}....E[w]\circ T^{-(n-2)}.E[h]\circ T^{-(n-2)}.E(wf)\circ T^{-(n-1)}$. □

4 Charecterisations of class $[nN]$ composition operators

In this section we discuss the characterization of n-power normal composition operators on L^2- spaces.

Lemma 4.2. *[1] Let α and β be non-negative functions with $S = support\ \ \alpha$. Then the following are equivalent:*

(i) For every $f \in L^2(X, \sum, \lambda)$, $\int_X \alpha\, |f|^2\, d\lambda \geq \int_X |E(\beta f|F)|^2\, d\lambda$, where F is a sub-sigma algebra of $\sum$.

(ii) Support $\beta \subset S$ and $E(\frac{\beta^2}{\alpha}\chi_S|F) \leq 1a.e.$

Theorem 4.5. *$C_T \in B(L^2(\lambda))$ is of class$[nN]$ if and only if $h_n > 0$ and $E(\frac{1}{h_n}) = \frac{1}{h_n\circ T^n}$.*

Proof.

$$
\begin{aligned}
\left\langle C_T^{*^n}C_T^n f, f\right\rangle &= \langle h_n f, f\rangle = \int_X h_n\, |f|^2\, d\lambda\\
\left\langle C_T^n C_T^{*^n} f, f\right\rangle &= \langle h_n\circ T^n E f, f\rangle = \int_X h_n\circ T^n E(f)\overline{f} d\lambda = \int_X \left|E(h_n^{\frac{1}{2}}\circ T^n f)\right|^2 d\lambda.
\end{aligned}
$$

Let $S = support\ \ h_n$. By Lemma 4.2, C_T is of class$[nN]$ if and only if $support\ \ h_n^{\frac{1}{2}}\circ T^n \subset \ support\ \ h_n$ and $E\left(\frac{\chi_S h_n\circ T^n}{h_n}\right) \leq 1$.

As $h_n\circ T^n > 0$, the condition involving supports is true if and only if $h_n > 0$ (so that $\chi_S = 1$). The inequality is then equivalent to $E(\frac{1}{h_n}) = \frac{1}{h_n\circ T^n}$ since $h_n\circ T$ is $T^{-1}(\sum)$ measurable. □

Theorem 4.6. *(i)* C_T *is of class*$[nN] \Leftrightarrow \left\| h^{\frac{1}{2}} h_{n-1}^{\frac{1}{2}} f \right\|^2 = \left\| h_{n-1}^{\frac{1}{2}} h^{\frac{1}{2}} \circ TPf \right\|^2 \Leftrightarrow \left\| h_n^{\frac{1}{2}} f \right\|^2 = \left\| (h_n \circ T^n)^{\frac{1}{2}} Pf \right\|^2.$

(ii) $\left\| h^{\frac{1}{2}} h_{n-1}^{\frac{1}{2}} Pf \right\|^2 = \left\| h_{n-1}^{\frac{1}{2}} h^{\frac{1}{2}} \circ TPf \right\|^2.$

Proof.

$$
\begin{aligned}
0 &= \left\langle C_T^* C_T^n f, f \circ T^{n-1} \right\rangle - \left\langle C_T^n C_T^* f, f \circ T^{n-1} \right\rangle \\
&= \left\langle h C_T^{n-1} f, f \circ T^{n-1} \right\rangle - \left\langle C_T^{n-1} h \circ TPf, f \circ T^{n-1} \right\rangle \\
&= \left\langle h f \circ T^{n-1}, f \circ T^{n-1} \right\rangle - \left\langle h \circ TPf \circ T^{n-1}, f \circ T^{n-1} \right\rangle \\
&= \left\langle h f \circ T^{n-1}, f \circ T^{n-1} \right\rangle - \left\langle Ph \circ Tf \circ T^{n-1}, f \circ T^{n-1} \right\rangle \\
&= \int h\,|f|^2 \circ T^{n-1} d\lambda - \int Ph \circ T\,|f|^2 \circ T^{n-1} d\lambda \\
&= \int h\,|f|^2\, d\lambda \circ T^{n-1} - \int Ph \circ T\,|f|^2\, d\lambda \circ T^{n-1} \\
&= \langle hh_{n-1} f, f \rangle - \langle Ph \circ Th_{n-1} f, f \rangle \\
&= \langle hh_{n-1} f, f \rangle - \langle h \circ Th_{n-1} Pf, f \rangle
\end{aligned}
$$

Since C_T and C_T^n commutes $\forall n \in \mathbb{N}, h, h_n$ also commutes $\forall n \in \mathbb{N}$.

$$\left\| h^{\frac{1}{2}} h_{n-1}^{\frac{1}{2}} f \right\|^2 = \left\| h_{n-1}^{\frac{1}{2}} h^{\frac{1}{2}} \circ TPf \right\|^2$$

Also,

$$\left\langle (C_T^{*^n} C_T^n - C_T^n C_T^{*^n}) f, f \right\rangle = \langle (h_n - h_n \circ T^n P) f, f \rangle$$

And hence it follows that $\left\| h_n^{\frac{1}{2}} f \right\|^2 = \left\| (h_n \circ T^n)^{\frac{1}{2}} Pf \right\|^2.$
(ii) follows directly from (i)

□

Theorem 4.7. *Let C_T be an n-power normal composition operator on $L^2(\lambda)$ then for all $m > n, f \in L^2$ and $i = m - n$ we have*

$$\left\langle C_T^m C_T^{*^m} f, f \right\rangle = \left\langle Ph_i \circ T^i h_n f, f \right\rangle. \tag{4.4}$$

Proof. For $m = n + 1$, we have

$$
\begin{aligned}
\left\langle C_T^{n+1} C_T^{*^{n+1}} f, f \right\rangle &= \left\langle C_T C_T^* C_T^n C_T^{*^n} f, f \right\rangle \\
&= \langle h \circ TPh_n f, f \rangle \\
&= \langle Ph \circ Th_n f, f \rangle
\end{aligned}
$$

Suppose (4.4) holds for $m = n + 1, n + 2, ...n + k, i = 1, 2, ...k$ and all $f \in L^2$. Then

$$
\begin{aligned}
\left\langle C_T^{n+k+1} C_T^{*^{n+k+1}} f, f \right\rangle &= \left\langle C_T^{k+1} C_T^{*^{k+1}} C_T^n C_T^{*^n} f, f \right\rangle \\
&= \left\langle h_{k+1} \circ T^{k+1} Ph_n f, f \right\rangle \\
&= \left\langle Ph_{k+1} \circ T^{k+1} h_n f, f \right\rangle
\end{aligned}
$$

And hence (4.4) follows by induction. □

Theorem 4.8. *Let C_T be an n-power normal composition operator on $L^2(\lambda)$ then for all $m > n, f \in L^2$ and $i = m - n$ we have*

$$\left\langle C_T^{*^m} C_T^m f, f \right\rangle = \langle h_i h_n f, f \rangle . \tag{4.5}$$

Proof. For $m = n + 1$, we have

$$\begin{aligned} \left\langle C_T^{*^{n+1}} C_T^{n+1} f, f \right\rangle &= \left\langle C_T^* C_T C_T^n C_T^{*^n} f, f \right\rangle \\ &= \langle h h_n f, f \rangle \\ &= \langle h h_n f, f \rangle \end{aligned}$$

Suppose (4.5) holds for $m = n + 1, n + 2, ...n + k, i = 1, 2, ...k$ and all $f \in L^2$. Then

$$\begin{aligned} \left\langle C_T^{*^{n+k+1}} C_T^{n+k+1} f, f \right\rangle &= \left\langle C_T^{*^{k+1}} C_T^{k+1} C_T^n C_T^{*^n} f, f \right\rangle \\ &= \langle h_{k+1} h_n f, f \rangle \\ &= \langle h_{k+1} h_n f, f \rangle \end{aligned}$$

And hence (4.5) follows by induction. □

Theorem 4.9. *Let C_{T_1} and C_{T_2} be n-power normal composition operators on $L^2(\lambda)$ then for all $m > n, p > n$, such that m and p are multiples of $n, C_{T_1}^m C_{T_2}^p$ is normal.*

Proof. On applying Theorem 4.8 in the subsequent equations the assertion is proved. Denote $C_{T_1}^{*^k} C_{T_1}^k$ by $M_{h(1)k}$ and $C_{T_2}^{*^k} C_{T_2}^k$ by $M_{h(2)k}$ respectively.

$$\begin{aligned} \left\langle (C_{T_1}^m C_{T_2}^p)^* (C_{T_1}^m C_{T_2}^p) f, f \right\rangle &= \left\langle C_{T_2}^{*^p} C_{T_1}^{*^m} C_{T_1}^m C_{T_2}^p f, f \right\rangle \\ &= \left\langle h_{(1)m-n} h_{(1)n} C_{T_2}^p f, C_{T_2}^p f \right\rangle \\ &= \left\langle h_{(1)m-n} h_{(1)n} C_{T_2}^{*^p} C_{T_2}^p f, f \right\rangle \\ &= \left\langle h_{(1)m-n} h_{(1)n} h_{(2)p-n} h_{(2)n} f, f \right\rangle \\ \left\langle (C_{T_1}^m C_{T_2}^p)(C_{T_1}^m C_{T_2}^p)^* f, f \right\rangle &= \left\langle C_{T_2}^p C_{T_2}^{*^p} C_{T_1}^{*^m} f, C_{T_1}^{*^m} f \right\rangle \\ &= \left\langle h_{(2)p-n} h_{(2)n} C_{T_1}^{*^m} C_{T_1}^m f, f \right\rangle \\ &= \left\langle h_{(1)m-n} h_{(1)n} h_{(2)p-n} h_{(2)n} f, f \right\rangle \end{aligned}$$

From the above equalities it follows that $C_{T_1}^m C_{T_2}^p$ is normal. □

Corollary 4.1. *Let C_{T_1} and C_{T_2} be n-power normal composition operators on $L^2(\lambda)$ then for all $m > n, p > n$, such that m and p are multiples of $n, (C_{T_1}^m C_{T_2}^p)^q, \ \forall q \in \mathbb{N}$ is normal.*
In particular, for $q = n, C_{T_1}^m C_{T_2}^p$ is of class$[nN]$.

Proof. From Theorem 4.9 it is obvious that $(C_{T_1}^m C_{T_2}^p)^q, \ \forall q \in \mathbb{N}$ is normal. And for $q = n$, it follows from the normality of class$[nN]$ operators that $C_{T_1}^m C_{T_2}^p$ is of class$[nN]$ □

Now we establish a close relationship between $\sigma(C_T)$ and $E(h)$ where $E(h)$ denotes the essential range of the Radon-Nikodym derivative h.

Theorem 4.10. *Let C_T be an n-power normal composition operator on $L^2(\lambda)$ then*
$\sigma(C_T) \subset \left\{ \alpha^{\frac{1}{n}} : \alpha^{\frac{1}{n}} \in \mathbb{C} \ \text{and} \ |\alpha|^{\frac{2}{n}} \in E(h_n)^{\frac{1}{n}} \right\}$

Proof. C_T is in class$[nN]$ implies C_T^n is normal and hence by Spectral mapping Theorem, for normal operators, $\sigma(C_T^{*^n} C_T^n) = \left\{ |\alpha|^2 : \alpha \in \sigma(C_T^n) \right\}$.
But $C_T^{*^n} C_T^n = M_{h_n}$.

Therefore $\sigma(M_{h_n}) = \left\{|\alpha|^2 : \alpha \in \sigma(C_T^n)\right\}$. Because $\sigma(M_{h_n}) = E(h_n)$. We have,
$E(h_n) = \left\{|\alpha|^2 : \alpha \in \sigma(C_T^n)\right\} = \left\{|\alpha|^2 : \alpha \in \sigma(C_T)^n\right\}$.
Thus $\sigma(C_T)^n \subset \left\{\alpha : \alpha \in \mathbb{C} \text{ and } |\alpha|^2 \in E(h_n)\right\}$, which implies
$\sigma(C_T) \subset \left\{\alpha^{\frac{1}{n}} : \alpha^{\frac{1}{n}} \in \mathbb{C} \text{ and } |\alpha|^{\frac{2}{n}} \in E(h_n)^{\frac{1}{n}}\right\}$ □

Theorem 4.11. *Let C_T be an n-power normal composition operator on $L^2(\lambda)$ such that 1 does not belong to $E(h_n)^{\frac{1}{n}}$. Then $\sigma(C_T) = \left\{\alpha^{\frac{1}{n}} : \alpha^{\frac{1}{n}} \in \mathbb{C} \text{ and } |\alpha|^{\frac{2}{n}} \in E(h_n)^{\frac{1}{n}}\right\}$ and hence $\sigma(C_T)$ has cyclic symmetry.*

Proof. From the proof of Theorem 4.10, we have, $E(h_n)^{\frac{1}{n}} = \left\{|\alpha|^{\frac{2}{n}} : \alpha^{\frac{1}{n}} \in \sigma(C_T)\right\}$. Thus for every $m \in E(h_n)^{\frac{1}{n}}$ there is an $\alpha^{\frac{1}{n}} \in \sigma(C_T)$ such that $|\alpha|^{\frac{2}{n}} = m$. If $\alpha \in \sigma(C_T)$ and $|\alpha| \neq 1$, then every β such that $|\alpha| = |\beta|$ is in $\sigma(C_T)$[6]. Since by assumption, $1 \notin E(h_n)^{\frac{1}{n}}$ there is no $\alpha^{\frac{1}{n}} \in \sigma(C_T)$ such that $|\alpha|^{\frac{2}{n}} = 1$. Hence $\left\{\alpha^{\frac{1}{n}} : \alpha^{\frac{1}{n}} \in \mathbb{C} \text{ and } |\alpha|^{\frac{2}{n}} \in E(h_n)^{\frac{1}{n}}\right\} \subset \sigma(C_T)$. The opposite inclusion follows from Theorem 4.10 and hence $\sigma(C_T) = \left\{\alpha^{\frac{1}{n}} : \alpha^{\frac{1}{n}} \in \mathbb{C} \text{ and } |\alpha|^{\frac{2}{n}} \in E(h_n)^{\frac{1}{n}}\right\}$. □

5 Quasi-isometry and *n*-power Quasi-isometry Composite multiplication operators

In this section we give a characterization of quasi-isometry and *n*- power quasi-isometry composite multiplication operators.

Theorem 5.12. *Let $M_{u,T}$ on $L^2(\lambda)$ be a composite mulitiplication operator, then for $\lambda \geq 0$, $M_{u,T}$ is a quasi-isometry if and only if $u.h.E[uh] \circ T^{-1}.E[u \circ Tu \circ T^2] \circ T^{-2}.f = u^2.h.f$.*

Proof.

$$
\begin{aligned}
M_{u,T} \text{ is a quasi-isometry} \quad &\Leftrightarrow \quad M_{u,T}^{*^2} M_{u,T}^2 f = M_{u,T}^* M_{u,T} f \\
&\Leftrightarrow \quad M_{u,T}^{*^2} M_{u,T} u \circ T.f \circ T = M_{u,T}^*.u \circ T.f \circ T \\
&\Leftrightarrow \quad M_{u,T}^{*^2}[u \circ T(u \circ T.f \circ T) \circ T] = u.h.E[u \circ T.f \circ T] \circ T^{-1} \\
&\Leftrightarrow \quad M_{u,T}^{*^2}[u \circ T.u \circ T^2.f \circ T^2] = u.h.u \circ T \circ T^{-1}.f \circ T \circ T^{-1} \\
&\Leftrightarrow \quad M_{u,T}^* u.h.E[u \circ T.u \circ T^2.f \circ T^2] \circ T^{-1} = u.h.u.f \\
&\Leftrightarrow \quad M_{u,T}^* u.h.E[u \circ T.u \circ T^2] \circ T^{-1}.f \circ T = u^2.h.f \\
&\Leftrightarrow \quad u.h.E[u.h.E[u \circ T.u \circ T^2] \circ T^{-1}.f \circ T] \circ T^{-1} = u^2.h.f \\
&\Leftrightarrow \quad u.h.E[uh] \circ T^{-1}.E[u \circ T.u \circ T^2] \circ T^{-2}.f = u^2.h.f.
\end{aligned}
$$

□

Corollary 5.2. *$C_T \in B(L^2(\lambda))$ is quasi-isometry if and only if $h.E[h] \circ T^{-1}.f = h.f$.*

Proof. The proof is obtained by putting $u = 1$ in Theorem 5.12. □

Theorem 5.13. *Let $M_{u,T}$ on $L^2(\lambda)$ be a composite mulitiplication operator, then for $\lambda \geq 0$, $M_{u,T}^*$ is a quasi-isometry if and only if $u \circ T.u \circ T^2.u \circ T^2.h \circ T^2.E[u.h] \circ T.E[f] = u \circ T.u \circ T.h \circ T.E[f]$*

Proof.

$$
\begin{aligned}
M^*_{u,T} \text{ is a quasi-isometry } &\Leftrightarrow M^2_{u,T}M^{*^2}_{u,T}f = M_{u,T}M^*_{u,T}f \\
&\Leftrightarrow M^2_{u,T}M^*_{u,T}u.h.E[f]\circ T^{-1} = M_{u,T}u.h.E[f]\circ T^{-1} \\
&\Leftrightarrow M^2_{u,T}u.h.E[u.h.E[f]\circ T^{-1}]\circ T^{-1} = u\circ T.[u.h.E[f]\circ T^{-1}]\circ T \\
&\Leftrightarrow M^2_{u,T}u.h.E[u.h]\circ T^{-1}.E[f]\circ T^{-2} = u\circ T.u\circ T.h\circ T.E[f] \\
&\Leftrightarrow M_{u,T}u\circ T.[u.h.E[u.h]\circ T^{-1}.E[f]\circ T^{-2}]\circ T \\
&\qquad = u\circ T.u\circ T.h\circ T.E[f] \\
&\Leftrightarrow M_{u,T}u\circ T.u\circ T.h\circ T.E[u.h].E[f]\circ T^{-1} \\
&\qquad = u\circ T.u\circ T.h\circ T.E[f] \\
&\Leftrightarrow u\circ T[u\circ Tu\circ T.h\circ T.E[u.h].E[f]\circ T^{-1}]\circ T \\
&\qquad = u\circ T.u\circ T.h\circ T.E[f] \\
&\Leftrightarrow u\circ T.u\circ T^2.u\circ T^2.h\circ T^2.E[u.h]\circ T.E[f] \\
&\qquad = u\circ T.u\circ T.h\circ T.E[f].
\end{aligned}
$$

□

Theorem 5.14. *Let $M_{u,T}$ on $L^2(\lambda)$ be a composite mulitiplication operator, then for $\lambda \geq 0$, $M_{u,T}$ is an n-power quasi-isometry operator if and only if $u\circ T.u\circ T^2.u\circ T^3....u\circ T^{n-1}.u\circ T^{n-1}.h\circ T^{n-1}.E[u.h]\circ T^{n-2}E[u\circ T.u\circ T^2]\circ T^{n-3}.f\circ T^{n-1} = u.h.E[u\circ T.u\circ T^2.u\circ T^3...u\circ T^n]\circ T^{-1}.f\circ T^{n-1}$.*

Proof.

$$M_{u,T} \text{ is } n\text{-power quasi-isometry} \Leftrightarrow M^{n-1}_{u,T}M^{*^2}_{u,T}M^2_{u,T}f = M^*_{u,T}M_{u,T}M^{n-1}_{u,T}f \tag{5.6}$$

Now,

$$
\begin{aligned}
M^{n-1}_{u,T}M^{*^2}_{u,T}M^2_{u,T}f &= M^{n-1}_{u,T}u.h.E[u.h]\circ T^{-1}.E[u\circ T.u\circ T^2]\circ T^{-2}.f \\
&= M^{n-2}_{u,T}u\circ T[u.h.E[u.h]\circ T^{-1}.E[u\circ T.u\circ T^2]\circ T^{-2}.f]\circ T \\
&= M^{n-2}_{u,T}u\circ T.u\circ T.h\circ T.E[u.h].E[u\circ T.u\circ T^2]\circ T^{-1}.f\circ T \\
&= M^{n-3}_{u,T}u\circ T[u\circ T.u\circ T.h\circ T.E[u.h].E[u\circ T.u\circ T^2]\circ T^{-1}.f\circ T]\circ T \\
&= M^{n-3}_{u,T}u\circ T.u\circ T^2.u\circ T^2.h\circ T^2.E[u.h]\circ T.E[u\circ T.u\circ T^2].f\circ T^2.
\end{aligned}
$$

Continuing in a similar manner, we arrive at the following expression,

$$
\begin{aligned}
M^{n-1}_{u,T}M^{*^2}_{u,T}M^2_{u,T}f = \; & u\circ T.u\circ T^2.u\circ T^3....u\circ T^{n-1}.u\circ T^{n-1} \\
& h\circ T^{n-1}.E[u.h]\circ T^{n-2}.E[u\circ T.u\circ T^2]\circ T^{n-3}. \\
& f\circ T^{n-1}.
\end{aligned}
$$

$$
\begin{aligned}
M^*_{u,T}M^n_{u,T}f &= M^*_{u,T}M^{n-1}_{u,T}u\circ T.f\circ T \\
&= M^*_{u,T}M^{n-2}_{u,T}u\circ T[u\circ T.f\circ T]\circ T \\
&= M^*_{u,T}M^{n-2}_{u,T}u\circ T.u\circ T^2.f\circ T^2 \\
&= . \\
&= . \\
&= . \\
&= M^*_{u,T}u\circ T.u\circ T^2.u\circ T^3...u\circ T^n.f\circ T^n \\
&= u.hE[u\circ T.u\circ T^2.u\circ T^3...u\circ T^n.f\circ T^n]\circ T^{-1} \\
&= u.hE[u\circ T.u\circ T^2.u\circ T^3...u\circ T^n]\circ T^{-1}.f\circ T^{n-1}.
\end{aligned}
$$

Hence equation (5.6) becomes,
$M_{u,T}$ is n- power quasi-isometry $\Leftrightarrow u \circ T.u \circ T^2.u \circ T^3....u \circ T^{n-1}.u \circ T^{n-1}$
$h \circ T^{n-1}.E[u.h] \circ T^{n-2}.E[u \circ T.u \circ T^2] \circ T^{n-3}.f \circ T^{n-1} = u.h.E[u \circ T.u \circ T^2.u \circ T^3...u \circ T^n] \circ T^{-1}.f \circ T^{n-1}$. □

Corollary 5.3. *$C_T \in B(L^2(\lambda))$ is n-power quasi-isometry if and only if $h \circ T^{n-1}.E[h] \circ T^{n-2}.f \circ T^{n-1} = h.f \circ T^{n-1}$.*

Proof. The proof is obtained by putting $u = 1$ in Theorem 5.14. □

Theorem 5.15. *Let $M_{u,T}$ on $L^2(\lambda)$ be a composite mulitiplication operator, then for $\lambda \geq 0$, $M^*_{u,T}$ is an n-power quasi-isometry operator if and only if $u.h.E[u.h] \circ T^{-1}.E[u.h] \circ T^{-2}....E[u.h] \circ T^{-(n-2)}E[u \circ T.u \circ T^2.u \circ T^2] \circ T^{-(n-1)}.$ $E[h] \circ T^{-(n-3)}.E[u.h] \circ T^{-(n-2)}.E[f] \circ T^{-(n-1)} = u \circ T.u \circ T.h \circ T.E[u.h].E[u.h] \circ T^{-1}.E[u.h] \circ T^{-2}....E[u.h] \circ T^{-(n-2)}.E[f] \circ T^{-(n-1)}$.*

Proof.

$$M^*_{u,T} \text{ is } n\text{-power quasi-isometry} \Leftrightarrow M^{*^{n-1}}_{u,T} M^2_{u,T} M^{*^2}_{u,T} f = M_{u,T} M^*_{u,T} M^{*^{n-1}}_{u,T} f \tag{5.7}$$

Now,

$$\begin{aligned}
M^{*^{n-1}}_{u,T} M^2_{u,T} M^{*^2}_{u,T} f &= M^{*^{n-1}}_{u,T}[u \circ T.u \circ T^2.u \circ T^2.h \circ T^2.E[u.h] \circ T.E[f]] \\
&= M^{*^{n-2}}_{u,T} u.h.E[u \circ T.u \circ T^2.u \circ T^2.h \circ T^2.E[u.h] \circ T.E[f]] \circ T^{-1} \\
&= M^{*^{n-2}}_{u,T} u.h.E[u \circ T.u \circ T^2.u \circ T^2] \circ T^{-1}.h \circ T.E[u.h].E[f] \circ T^{-1} \\
&= M^{*^{n-3}}_{u,T} u.h.E[u.h.E[u \circ T.u \circ T^2.u \circ T^2] \circ T^{-1}. \\
&\quad h \circ T.E[u.h].E[f] \circ T^{-1}] \circ T^{-1} \\
&= M^{*^{n-3}}_{u,T} u.h.E[u.h] \circ T^{-1}.E[u \circ T.u \circ T^2.u \circ T^2] \circ T^{-2}. \\
&\quad h.E[u.h] \circ T^{-1}.E[f] \circ T^{-2} \\
&= . \\
&= . \\
&= . \\
&= u.h.E[u.h] \circ T^{-1}.E[u.h] \circ T^{-2}....E[u.h] \circ T^{-(n-2)} \\
&\quad E[u \circ T.u \circ T^2.u \circ T^2] \circ T^{-(n-1)}.E[h] \circ T^{-(n-3)} \\
&\quad E[u.h] \circ T^{-(n-2)}.E[f] \circ T^{-(n-1)}.
\end{aligned}$$

And,

$$\begin{aligned}
M_{u,T} M^{*^n}_{u,T} &= M_{u,T} M^{*^{n-1}}_{u,T} u.h.E[f] \circ T^{-1} \\
&= M_{u,T} M^{*^{n-2}}_{u,T} u.h.E[u.h.E[f] \circ T^{-1}] \circ T^{-1} \\
&= M_{u,T} M^{*^{n-2}}_{u,T} u.h.E[u.h] \circ T^{-1}.E[f] \circ T^{-2} \\
&= . \\
&= . \\
&= . \\
&= M_{u,T} u.h.E[u.h] \circ T^{-1}.E[u.h] \circ T^{-2}.E[u.h] \circ T^{-3}.... \\
&\quad E[u.h] \circ T^{-(n-1)}.E[f] \circ T^{-n} \\
&= u \circ T[u.h.E[u.h] \circ T^{-1}.E[u.h] \circ T^{-2}.E[u.h] \circ T^{-3}.... \\
&\quad E[u.h] \circ T^{-(n-1)}.E[f] \circ T^{-n}] \circ T \\
&= u \circ T.u \circ T.h \circ T.E[u.h].E[u.h] \circ T^{-1}.E[u.h] \circ T^{-2}.... \\
&\quad E[u.h] \circ T^{-(n-2)}.E[f] \circ T^{-(n-1)}.
\end{aligned}$$

Hence equation (5.7) becomes,
$M^*_{u,T}$ is n-power quasi-isometry $\Leftrightarrow$ $u.h.E[u.h] \circ T^{-1}.E[u.h] \circ T^{-2}....E[u.h] \circ T^{-(n-2)}.E[u \circ T.u \circ T^2.u \circ T^2] \circ$

$T^{-(n-1)}.E[h] \circ T^{-(n-3)}.E[u.h] \circ T^{-(n-2)}.E[f] \circ T^{-(n-1)} = u \circ T.u \circ T.h \circ T.E[u.h].E[u.h] \circ T^{-1}.E[u.h] \circ T^{-2}$ $....E[u.h] \circ T^{-(n-2)}.E[f] \circ T^{-(n-1)}$. □

Corollary 5.4. *$C_T^* \in B(L^2(\lambda))$ is n-power quasi-isometry if and only if $h.E[h] \circ T^{-1}.E[h] \circ T^{-2}....E[h] \circ T^{-(n-2)}.$ $E[h] \circ T^{-(n-3)}.E[h] \circ T^{-(n-2)}.E[f] \circ T^{-(n-1)} = h \circ T.E[h].E[h] \circ T^{-1}.E[h] \circ T^{-2}....E[h] \circ T^{-(n-2)}.E[f] \circ T^{-(n-1)}$.*

Proof. The proof is obtained by putting $u = 1$ in Theorem 5.15. □

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