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Oscillation of First Order Neutral Difference Equations

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Abstract

In this paper, we consider a class of first order neutral difference equations of the form

$$\Delta[r(n)(x(n)+p(n)x(n-\tau))]+q(n)x(n-\sigma)=0, \quad n \geq n_0. \qquad (*)$$

Some sufficient conditions for the oscillation of all solutions of (*) are established. Our result extend and improve some of the previous results in the literature. Some examples are considered to illustrate our results.

Keywords: Oscillation, nonoscillation, neutral, difference equations.

2010 MSC: 39A10, 39A12.

1 Introduction

During the past few decades, neutral difference equations have been studied extensively and the oscillatory theory for these equations is well developed; see [1] [2] [3] [4] [5] [6] [7] [8] [9] [10] [11] [12] [13] [14] [15] [16] [17] [18] and the references cited therein. A survey of the most significant efforts in this theory can be found in the excellent monographs of Győri and Ladas [6] and Agarwal [1], [2].

Consider the first order neutral difference equations of the form

$$\Delta[r(n)(x(n)+p(n)x(n-\tau))]+q(n)x(n-\sigma)=0, \quad n \geq n_0 \qquad (1.1)$$

where $\{p(n)\}$ is a sequence of real numbers, $\{r(n)\}$ and $\{q(n)\}$ are sequences of positive real numbers, τ and σ are positive integers, and Δ is the forward difference operator given by $\Delta x(n) = x(n+1) - x(n)$.

Let us choose a positive integer $n^* > \max\{\tau, \sigma\}$. By a solution of (1.1) on $N(n_0) = \{n_0, n_0+1, ...\}$, we mean a real sequence $\{x(n)\}$ which is defined on $n \geq n_0 - n^*$ and which satisfies (1.1) for $n \in N(n_0)$. A solution $\{x(n)\}$ of (1.1) on $N(n_0)$ is said to be oscillatory if for every positive integer $N_0 > n_0$ there exists $n \geq N_0$ such that $x(n)x(n+1) \leq 0$, otherwise $\{x(n)\}$ is said to be non-oscillatory.

There are numerous numbers of oscillation criteria obtained for oscillation of all solutions of (1.1). In particular, Murugesan and Suganthi [9] investigated the oscillation behavior of (1.1) and obtained some new oscillation results under the condition

$$\sum_{n=n_0}^{\infty} q(n) = \infty. \qquad (1.2)$$

For oscillation of (1.1) when $r(n) \equiv 1$ and $p(n)$ is equal to a constant, we refer the readers to the papers by Lalli [7] and the references cited therein. For further oscillation results on the oscillating behavior of solutions of (1.1), when $r(n) \equiv 1$, we refer the reader to the monographs by Agarwal [1], [2] as well as the papers of

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Ying Gao and Zhang [15], Murugesan and Suganthi [9], Chen et. al [3], Tang et. al [13] and the references cited therein.

Define the sequences $\{z(n)\}$ and $\{w(n)\}$ as follows:

$$z(n) = x(n) + p(n)x(n-\tau), \tag{1.3}$$

$$w(n) = z(n) + p(n)z(n-\tau). \tag{1.4}$$

If $\{x(n)\}$ is an eventually positive solution of the equation

$$\Delta[x(n) + px(n-\tau)] + q(n)x(n-\sigma) = 0, \tag{1.5}$$

where p is a real constant then $\{z(n)\}$ and $\{w(n)\}$ are also solutions of (1.5).

In the sequel, unless otherwise specified, when we write a functional inequality, we assume that it holds for all sufficiently large n.

2 Some Useful Lemmas

In the proof of our main results, we need the following Lemmas. The Lemma 2.2 and 2.3 are discrete analogues of the Lemma 1.5.1 and 1.5.3 respectively in [6].

Lemma 2.1. [9] *Assume that (1.2) holds. Let $\{x(n)\}$ be an eventually positive solution of equation (1.1). Then the following statements are true.*

(i) if $p(n) \leq -1$ then $z(n) < 0$;

(ii) if $-1 \leq p(n) \leq 0$ and $\{r(n)\}$ is a decreasing sequence of positive real numbers, then $z(n) > 0$ and $\lim_{n\to\infty} z(n) = 0$.

Lemma 2.2. *Let $\{f(n)\}$ and $\{g(n)\}$ be sequence of real numbers such that*

$$f(n) = g(n) + \mu g(n-c); \quad n \geq n_0 + \max\{0, c\},$$

where $\mu \in R$, $\mu \neq 1$ and c is a positive integer. Assume that $\lim_{n\to\infty} f(n) = l \in R$ exists and $\liminf_{n\to\infty} g(n) = a \in R$. Then $l = (1+p)a$.

Lemma 2.3. *Let $0 \leq \lambda < 1$, c be a positive integer and $n_0 \in N$ and $\{x(n)\}$ be a sequence of positive real numbers and assume that for every $\in > 0$ there exists a $n_\in \geq n_0$ such that*

$$x(n) \leq (\lambda + \in)x(n-c) + \in \quad for \quad n \geq n_\in.$$

Then

$$\lim_{n\to\infty} x(n) = 0.$$

Lemma 2.4. *Assume that (1.2) holds, p is a real number with $p \neq 1$ and $\{q(n)\}$ is a τ-periodic sequence of positive real numbers. Let $\{x(n)\}$ be an eventually positive solution of (1.5).*

Then

(a) $\{z(n)\}$ is decreasing sequence and either

$$\lim_{n\to\infty} z(n) = -\infty; \tag{2.6}$$

or

$$\lim_{n\to\infty} z(n) = 0. \tag{2.7}$$

(b) The following statements are equivalent:

(i) (2.6) holds;

(ii) $p < -1$;

(iii) $\lim_{n\to\infty} x(n) = \infty$;

(iv) $w(n) > 0$, $\Delta w(n) > 0$.

(c) The following statements are equivalent:

(i) (2.7) holds;

(ii) $p > -1$;

(iii) $\lim_{n\to\infty} x(n) = 0$;

(iv) $w(n) > 0$, $\Delta w(n) < 0$.

Proof. (a) we have

$$\Delta z(n) = -q(n)x(n-\sigma) < 0 \tag{2.8}$$

and so $\{z(n)\}$ is strictly decreasing sequence. If (2.6) is not true, then there exists $l \in R$ such that $\lim_{n\to\infty} z(n) = l$. By summing (2.8) from n_1 to ∞, with n_1 sufficiently large, we find

$$l - z(n) = -\sum_{s=n_1}^{\infty} q(s)x(s-\sigma). \tag{2.9}$$

In view of (1.2) this implies that $\liminf_{n\to\infty} x(n) = 0$ and so by Lemma 2.2, $l = (1+p)0 = 0$. The proof of (a) is complete.

Now we turn to the proofs of (b) and (c). First assume that (2.6) holds. Then p must be negative and $\{x(n)\}$ is unbounded. Therefore there exists a $n^* \geq n_0$ such that $z(n^*) < 0$ and

$$x(n^*) \geq \max_{s\leq n^*} x(s) > 0.$$

Then

$$0 > z(n^*) = x(n^*) + px(n^*-\tau) \geq x(n^*)(1+p)$$

which implies that $p < -1$. Also

$$z(n) = x(n) + px(n-\tau) > px(n-\tau)$$

and (2.6) implies that $\lim_{n\to\infty} x(n) = \infty$. Now assume that (2.7) holds.

If $p \geq 0$, then from (1.3) it follows that $\lim_{n\to\infty} x(n) = 0$. Next assume that $p \in (-1, 0)$. Then by Lemma 2.3, $\lim_{n\to\infty} x(n) = 0$.

Finally if $p \leq -1$, then $x(n) > -px(n-\tau) \geq x(n-\tau)$ which shows that $\{x(n)\}$ is bounded from below by a positive constant, say m. Then (2.9) yields.

$$l - z(n_1) + m\sum_{s=n_1}^{\infty} q(s) \leq 0,$$

which is a contradiction. Therefore, if (2.7) holds $p > -1$. Observe that under the hypothesis (2.6), we have

$$\Delta w(n) = -q(n)z(n-\sigma) > 0. \tag{2.10}$$

If (2.6) holds, then

$$\lim_{n\to\infty} w(n) = \infty. \tag{2.11}$$

From (2.10) and (2.11) we have $w(n) > 0$ eventually. By a similar proof, under the hypothesis (2.7), we have $\Delta w(n) < 0$ and $w(n) > 0$. On the basis of the above discussions, the proof of (b) and (c) are now obvious. □

Lemma 2.5. *Assume that*

$$\limsup_{n\to\infty} \sum_{s=n}^{n+\sigma-1} q(s) > 0. \tag{2.12}$$

If $\{x(n)\}$ is an eventually positive solution of the delay difference equation

$$\Delta x(n) + q(n)x(n-\sigma) = 0, \quad n \geq n_0, \tag{2.13}$$

then

$$\liminf_{n\to\infty} \frac{x(n-\sigma)}{x(n)} < \infty. \tag{2.14}$$

Proof. In view of the assumption there exists a constant $d > 0$ and a sequence $\{n_k\}$ of positive integers such that $n_k \to \infty$ as $k \to \infty$ and

$$\sum_{s=n_k}^{n_k+\sigma-1} q(s) \geq d; \quad k = 1, 2, 3, \ldots.$$

Then there exists a $\xi_k \in \{n_k, n_k+1, \ldots, n_k+\sigma\}$ for each k such that

$$\sum_{s=n_k}^{\xi_k} q(s) \geq \frac{d}{2} \quad \text{and} \quad \sum_{s=\xi_k}^{n_k+\sigma} q(s) \geq \frac{d}{2}. \tag{2.15}$$

Summing the equation (2.13) from n_k to ξ_k and ξ_k to $n_k+\sigma$, we find

$$x(\xi_k+1) - x(n_k) + \sum_{s=n_k}^{\xi_k} q(s)x(s-\sigma) = 0 \tag{2.16}$$

and

$$x(n_k+\sigma+1) - x(\xi_k) + \sum_{s=\xi_k}^{n_k+\sigma} q(s)x(s-\sigma) = 0. \tag{2.17}$$

By omitting the first terms in (2.16) and (2.17) and by using the decreasing nature of $\{x(n)\}$ and (2.15), we find

$$-x(n_k) + \frac{d}{2}x(\xi_k-\sigma) \leq 0$$

and

$$-x(\xi_k) + \frac{d}{2}x(n_k) \leq 0$$

(or)

$$\frac{x(\xi_k-\sigma)}{x(\xi_k)} \leq \left(\frac{2}{d}\right)^2.$$

This completes the proof. □

Lemma 2.6. *If the equation (2.13) has an eventually positive solution, then one has eventually that*

$$\sum_{s=n+1}^{n+\sigma} q(s) \leq 1. \tag{2.18}$$

Proof. Let $\{x(n)\}$ be an eventually positive solution of (2.13). On the contrary, assume that

$$\sum_{s=n+1}^{n+\sigma} q(s) > 1, \tag{2.19}$$

eventually. Summing the equation (2.13) from $n+1$ to $n+\sigma$ and using the decreasing nature of $\{x(n)\}$, we have

$$x(n+\sigma+1) - x(n+1) + \sum_{s=n+1}^{n+\sigma} q(s)x(s-\sigma) \leq 0,$$

or

$$x(n+\sigma+1) - x(n) + x(n)\sum_{s=n+1}^{n+\sigma} q(s) \leq 0,$$

or

$$x(n+\sigma+1) + x(n)\left(\sum_{s=n+1}^{n+\sigma} q(s) - 1\right) \leq 0,$$

eventually.

This is a contradiction and the proof is complete. □

Lemma 2.7. [6] *The delay difference inequality*

$$\Delta x(n) + q(n)x(n-\sigma) \le 0 \tag{2.20}$$

has an eventually positive solution if and only if the delay difference equation

$$\Delta y(n) + q(n)y(n-\sigma) = 0 \tag{2.21}$$

has an eventually positive solution.

Lemma 2.8. [6] *The advanced difference inequality*

$$\Delta x(n) - q(n)x(n+\sigma) \le 0 \tag{2.22}$$

has an eventually negative solution if and only if the advanced difference equation

$$\Delta y(n) - q(n)y(n+\sigma) = 0 \tag{2.23}$$

has an eventually negative solution.

Lemma 2.9. *Assume that*

$$\limsup_{n\to\infty} \sum_{s=n-\sigma+1}^{n-1} q(s) > 0. \tag{2.24}$$

If $\{x(n)\}$ is an eventually negative solution of the advanced difference equation

$$\Delta x(n) - q(n)x(n+\sigma) = 0, \quad n \ge n_0, \tag{2.25}$$

then

$$\liminf_{n\to\infty} \frac{x(n+\sigma)}{x(n+1)} < \infty.$$

Proof. In view of the assumption there exists a constant $d > 0$ and a sequence $\{n_k\}$ of positive integers such that $n_k \to \infty$ as $k \to \infty$ and

$$\sum_{s=n_k-\sigma+1}^{n_k-1} q(s) \ge d, \quad k = 1, 2, 3....$$

Then there exists $\xi_k \in \{n_k - \sigma, n_k - \sigma + 1, ..., n_k\}$ for all k such that

$$\sum_{s=n_k-\sigma+1}^{\xi_k} q(s) \ge \frac{d}{2} \quad \text{and} \quad \sum_{s=\xi_k}^{n_k} q(s) \ge \frac{d}{2}. \tag{2.26}$$

Summing the equation (2.25) from $n_k - \sigma$ to ξ_k and ξ_k to n_k, we find

$$x(\xi_k+1) - x(n_k-\sigma) - \sum_{s=n_k-\sigma+1}^{\xi_k} q(s)x(s+\sigma) = 0 \tag{2.27}$$

and

$$x(\xi_k+1) - x(\xi_k) - \sum_{s=\xi_k}^{n_k} q(s)x(s+\sigma) = 0. \tag{2.28}$$

By omitting the second terms in (2.27) and (2.16) and by using the decreasing nature of $\{x(n)\}$ and (2.26), we find

$$x(\xi_k+1) - x(n_k+1)\frac{d}{2} \le 0$$

and

$$x(n_k+1) - x(\xi_k+\sigma)\frac{d}{2} \le 0$$

or

$$\frac{x(\xi_k+\sigma)}{x(\xi_k+1)} \le \left(\frac{2}{d}\right)^2.$$

This completes the proof. □

Lemma 2.10. *If the equation (2.25) has an eventually negative solution, than one has eventually that*

$$\sum_{s=n-\sigma+1}^{n-1} q(s) \leq 1. \tag{2.29}$$

Proof. Let $\{x(n)\}$ be an eventually negative solution of (2.25). On the contrary, let us assume that

$$\sum_{s=n-\sigma+1}^{n-1} q(s) > 1, \tag{2.30}$$

eventually. Summing the equation (2.25) from $n-\sigma+1$ to n and using the decreasing nature of $\{x(n)\}$, we have

$$x(n+1) - x(n-\sigma+1) - \sum_{s=n-\sigma+1}^{n} q(s)x(s+\sigma) = 0$$

or

$$x(n+1) - x(n-\sigma+1) - x(n+1) \sum_{s=n-\sigma+1}^{n} q(s) \leq 0$$

or

$$-x(n-\sigma+1) + x(n+1)\left(1 - \sum_{s=n-\sigma+1}^{n} q(s)\right) \leq 0.$$

This is a contradiction and the proof is complete. □

3 Main Results

Theorem 3.1. *Assume that (1.2) hold with $-1 \leq p(n) \leq 0$ and $\{r(n)\}$ is a decreasing sequence positive real numbers. Suppose that*

$$\sum_{n=n_0}^{\infty} \left[\frac{q(n)}{r(n-\sigma)} \ln\left(e \sum_{s=n+1}^{n+\sigma} \frac{q(s)}{r(s-\sigma}\right)\right] = \infty. \tag{3.31}$$

Then every solution of (1.1) is oscillatory.

Proof. Assume, for the sake of a contradiction, that (1.1) has an eventually positive solution $\{x(n)\}$. Then there exists an integers $n_1 \geq n_0$ such that $x(n) > 0$, $x(n-\tau) > 0$ and $x(n-\sigma > 0$ for $n \geq n_1$.

Set $z(n)$ to be defined as in (1.3). Then by Lemma 2.1 (ii), it follows that

$$z(n) > 0, \quad eventually. \tag{3.32}$$

As $x(n) > z(n)$, it follows from (1.1) that

$$\Delta(r(n)z(n)) + q(n)z(n-\sigma) \leq 0. \tag{3.33}$$

Dividing the last inequality by $r(n) > 0$, we obtain

$$\Delta z(n) + \frac{\Delta r(n)}{r(n)} z(n+1) + \frac{q(n)}{r(n)} z(n-\sigma) \leq 0. \tag{3.34}$$

Let

$$z(n) = \frac{y(n)}{r(n)}. \tag{3.35}$$

This implies that $y(n) > 0$. Substituting in (3.34) yields

$$\Delta y(n) + \frac{q(n)}{r(n-\sigma)} y(n-\sigma) \leq 0, \quad n \geq n_0 \tag{3.36}$$

So by Lemma 2.5, we have that the delay difference equation

$$\Delta y(n) + \frac{q(n)}{r(n-\sigma)} y(n-\sigma) = 0, \quad n \geq n_0 \tag{3.37}$$

has an eventually positive solution as well. Let

$$\lambda(n) = -\frac{\Delta y(n)}{y(n)} \tag{3.38}$$

Then $\{\lambda(n)\}$ satisfies

$$\lambda(n) \geq \bar{Q}(n) \exp\left\{\sum_{s=n-\sigma}^{n-1} \lambda(s)\right\}, \tag{3.39}$$

where

$$\bar{Q}(n) = \frac{q(n)}{r(n-\sigma)}. \tag{3.40}$$

Let

$$R(n) = \sum_{s=n+1}^{n+\sigma} \bar{Q}(s). \tag{3.41}$$

Therefore

$$\lambda(n) \geq \bar{Q}(n) \exp\left\{\frac{1}{R(n)} R(n) \sum_{s=n-\sigma}^{n-1} \lambda(s)\right\}. \tag{3.42}$$

Applying the inequality

$$e^{ax} \geq x + \frac{\ln(ea)}{a}, \quad \forall \quad x, a > 0, \tag{3.43}$$

to (3.42), we have

$$\lambda(n) \geq \bar{Q}(n) \left\{\frac{1}{R(n)} \sum_{s=n-\sigma}^{n-1} \lambda(s) + \frac{\ln(e(R(n))}{R(n)}\right\}, \tag{3.44}$$

or

$$\lambda(n) \left(\sum_{s=n+1}^{n+\sigma} \bar{Q}(s)\right) - \bar{Q}(n) \sum_{s=n-\sigma}^{n-1} \lambda(s) \geq \bar{Q}(n) \ln\left(e \sum_{s=n+1}^{n+\sigma} \bar{Q}(s)\right). \tag{3.45}$$

Then, for $M > N$, we have

$$\sum_{n=N}^{M-1} \lambda(n) \left(\sum_{s=n+1}^{n+\sigma} \bar{Q}(s)\right) - \sum_{n=N}^{M-1} \bar{Q}(n) \sum_{s=n-\sigma}^{n-1} \lambda(s) \geq \sum_{n=N}^{M-1} \bar{Q}(n) \ln\left(e \sum_{s=n+1}^{n+\sigma} \bar{Q}(s)\right). \tag{3.46}$$

By interchanging the order of summation, we get

$$\sum_{n=N}^{M-1} \bar{Q}(n) \sum_{s=n-\sigma}^{n-1} \lambda(s) \geq \sum_{s=N}^{M-\sigma-1} \lambda(n) \left(\sum_{s=n+1}^{n+\sigma} \bar{Q}(s)\right). \tag{3.47}$$

From (3.45) and (3.47), we find that

$$\sum_{n=M-\sigma}^{M-1} \lambda(n) \left(\sum_{s=n+1}^{n+\sigma} \bar{Q}(s)\right) \geq \sum_{n=N}^{M-1} \bar{Q}(n) \ln\left(e \sum_{s=n+1}^{n+\sigma} \bar{Q}(s)\right). \tag{3.48}$$

By Lemma 2.4, we have

$$\sum_{s=n+1}^{n-\sigma} \bar{Q}(s) \leq 1 \quad \textit{eventually}. \tag{3.49}$$

Therefore, from (3.48) and (3.49), we get

$$\sum_{n=M-\sigma}^{M-1} \lambda(n) \geq \sum_{n=N}^{M-1} \bar{Q}(n) \ln\left(e \sum_{s=n+1}^{n+\sigma} \bar{Q}(s)\right), \tag{3.50}$$

$$\sum_{n=M-\sigma}^{M-1} \left(1 - \frac{y(n+1)}{y(n)}\right) \geq \sum_{n=N}^{M-1} \bar{Q}(n) \ln\left(e \sum_{s=n+1}^{n+\sigma} \bar{Q}(s)\right). \tag{3.51}$$

Using the inequality

$$\log x < x - 1, \quad for \quad 0 < x < 1$$

in (3.51), we get

$$\ln \frac{y(M-\sigma)}{y(M)} \geq \sum_{n=N}^{M-1} \bar{Q}(n) \ln \left(e \sum_{s=n+1}^{n+\sigma} \bar{Q}(s) \right) \tag{3.52}$$

which implies by condition (3.31) that

$$\lim_{n\to\infty} \frac{y(n-\sigma)}{y(n)} = \infty. \tag{3.53}$$

On the other hand, (3.31) implies that there exists a sequence $\{n_k\}$ of positive integers with $n_k \to \infty$ as $k \to \infty$ such that

$$\sum_{s=n_k+1}^{n_k+\sigma} \frac{q(s)}{r(s-\sigma)} \geq \frac{1}{e} \quad \text{for} \quad \text{all} \quad k. \tag{3.54}$$

Hence by Lemma 2.3, we obtain

$$\liminf_{n\to\infty} \frac{y(n-\sigma)}{y(n)} < \infty \tag{3.55}$$

This contradicts (3.53) and completes the proof. □

Theorem 3.2. *Assume that (1.2) hold with $p(n) \equiv p > -1$, $r(n) \equiv r > 0$ and $\sigma > \tau$. Assume further that $\{q(n)\}$ is a τ -periodic and*

$$\sum_{n=n_0}^{\infty} \left[\frac{q(n)}{r(1+p)} \ln \left(e \sum_{s=n+1}^{n+\sigma-\tau} \frac{q(s)}{r(1+p)} \right) \right] = \infty. \tag{3.56}$$

Then every solution of (1.1) is oscillatory.

Proof. Assume the contrary. Without loss of generality we may assume that $\{x(n)\}$ be an eventually positive solution of (1.1). Then there exists an integers $n_1 \geq n_0$ such that

$$x(n) > 0, \quad x(n-\tau) > 0 \quad \text{and} \quad x(n-\sigma) > 0 \quad \text{for} \quad \text{all} \quad n \geq n_1. \tag{3.57}$$

Let $z(n)$ and $w(n)$ be defined as in (1.3) and (1.4). It is easily seen, by direct substituting, that $\{z(n)\}$ and $\{w(n)\}$ are also solutions of (1.1) when p and r are constants, that is

$$r\Delta z(n) + pr\Delta z(n-\tau) + q(n)z(n-\sigma) = 0, \tag{3.58}$$

$$r\Delta w(n) + pr\Delta w(n-\tau) + q(n)w(n-\sigma) = 0. \tag{3.59}$$

By Lemma 2.2, we have that $\{z(n)\}$ is decreasing and $w(n) > 0$. Also we have indeed that

$$\Delta w(n) = \frac{-1}{r} q(n) z(n-\sigma) \geq \frac{-1}{r} q(n) \geq (n-\sigma-\tau)$$

$$= \frac{-1}{r} q(n-\tau) z(n-\sigma-\tau) = \Delta w(n-\tau).$$

Then

$$\Delta w(n) \geq \Delta w(n-\tau) \tag{3.60}$$

Using (3.60) in (3.59) implies that

$$r(1+p)\Delta w(n-\tau) + q(n)w(n-\sigma) \leq 0. \tag{3.61}$$

As $p > -1$, we have $1 + p > 0$. Then

$$\Delta w(n-\tau) + \frac{q(n)}{r(1+p)} w(n-\sigma) \leq 0. \tag{3.62}$$

In view of the τ -periodicity of $q(n)$, (3.62) implies that

$$\Delta w(n) + \frac{q(n)}{r(1+p)} w(n-(\sigma-\tau)) \leq 0. \tag{3.63}$$

As $\{w(n)\}$ is a positive solution, so by Lemma 2.5, the delay difference equation

$$\Delta w(n) + \frac{q(n)}{r(1+p)} w(n-(\sigma-\tau)) = 0 \tag{3.64}$$

has an eventually positive solution as well. Let

$$\lambda(n) = -\frac{\Delta w(n)}{w(n)}. \tag{3.65}$$

Then $\lambda(n)$ satisfies

$$\lambda(n) \geq \bar{Q}_1(n) \exp\left\{\sum_{s=n-\sigma+\tau}^{n-1} \lambda(s)\right\}, \tag{3.66}$$

where

$$\bar{Q}_1(x) = \frac{q(n)}{r(1+p)}.$$

Let

$$R_1(n) = \sum_{s=n+1}^{n+\sigma-\tau} \bar{Q}_1(s). \tag{3.67}$$

Therefore,

$$\lambda(n) \geq \bar{Q}_1(n) \exp\left\{\frac{1}{R_1(n)} R_1(n) \sum_{s=n-\sigma+\tau}^{n-1} \lambda(s)\right\}. \tag{3.68}$$

Applying the inequality (3.43) to (3.68), we have

$$\lambda(n) \geq \bar{Q}_1(n) \left\{\frac{1}{R_1(n)} \sum_{s=n-\sigma+\tau}^{n-1} \lambda(s) + \frac{\ln(eR_1(n))}{R_1(n)}\right\}, \tag{3.69}$$

or

$$\lambda(n) \sum_{s=n+1}^{n+\sigma-\tau} \bar{Q}_1(s) - \bar{Q}_1(n) \sum_{s=n-\sigma+\tau}^{n-1} \lambda(s) \geq \bar{Q}_1(n) \left\{\ln\left(e \sum_{s=n+1}^{n+\sigma-\tau} \bar{Q}_1(s)\right)\right\}.$$

Then, for $M > N$, we have

$$\sum_{n=N}^{M-1} \lambda(n) \left(\sum_{s=n+1}^{n+\sigma-1} \bar{Q}_1(s)\right) - \sum_{n=N}^{M-1} \bar{Q}_1(n) \sum_{s=n-\sigma+\tau}^{n-1} \lambda(s)$$
$$\geq \sum_{n=N}^{M-1} \bar{Q}_1(n) \left(\ln\left(e \sum_{s=n+1}^{n+\sigma-\tau} \bar{Q}_1(s)\right)\right). \tag{3.70}$$

By interchanging the order of summation, we get

$$\sum_{n=N}^{M-1} \bar{Q}_1(n) \sum_{s=n-\sigma+\tau}^{n-1} \lambda(s) \geq \sum_{n=N}^{M-\sigma+\tau-1} \lambda(n) \sum_{s=n+1}^{n+\sigma-\tau} Q_1(s). \tag{3.71}$$

From (3.70) and (3.71), we find that

$$\sum_{n=M-\sigma+\tau}^{M-1} \lambda(n) \sum_{s=n+1}^{n+\sigma-\tau} \bar{Q}_1(s) \geq \sum_{n=N}^{M-1} \bar{Q}_1(n) \left(\ln\left(e \sum_{s=n+1}^{n+\sigma-\tau} \bar{Q}_1(s)\right)\right). \tag{3.72}$$

By Lemma 2.4, we have

$$\sum_{s=n+1}^{n+\sigma-\tau} \bar{Q}_1(s) \leq 1 \quad \text{eventually}. \tag{3.73}$$

Using (3.73) in (3.72), we get

$$\sum_{n=M-\sigma+\tau}^{M-1} \lambda(n) \geq \sum_{n=N}^{M-1} Q_1(n) \left(\ln\left(e \sum_{s=n+1}^{n+\sigma-\tau} \bar{Q}_1(s)\right)\right), \tag{3.74}$$

or

$$\sum_{n=M-\sigma+\tau}^{M-1}\left(1-\frac{y(n+1)}{y(n)}\right)\geq\sum_{n=N}^{M-1}Q_1(n)\left(\ln\left(e\sum_{s=n+1}^{n+\sigma-\tau}\bar{Q}_1(s)\right)\right). \tag{3.75}$$

Using the inequality $\ln x < x-1$ for $0 < x < 1$ in (3.75), we get

$$\ln\left(\frac{y(M-\sigma+\tau)}{y(M)}\right)\geq\sum_{n=N}^{M-1}\bar{Q}_1(n)\left(\ln\left(e\sum_{s=n+1}^{n+\sigma-\tau}\bar{Q}_1(s)\right)\right), \tag{3.76}$$

which implies by condition (3.56) that

$$\lim_{n\to\infty}\frac{y(n-\sigma+\tau)}{y(n)}=\infty. \tag{3.77}$$

On the other hand, (3.56) implies that there exists a sequence $\{n_k\}$ of positive integers with $n_k\to\infty$ as $k\to\infty$ such that

$$\sum_{s=n_k+1}^{n_k+\sigma-\tau}\frac{q(s)}{r(1+p)}\geq\frac{1}{e}\quad\text{for all k}. \tag{3.78}$$

Hence by Lemma 2.3, we obtain

$$\liminf_{n\to\infty}\frac{y(n-\sigma+\tau)}{y(n)}<\infty.$$

This contradicts (3.77) and completes the proof. □

Theorem 3.3. *Assume that (1.2) hold with $p(n)\leq-1$ and $\tau-\sigma>1$. Assume further that*

$$\sum_{n=n_0}^{\infty}\left[\frac{-q(n)}{p(n+\tau-\sigma)r(n+\tau-\sigma)}\ln\left(e\sum_{s=n-\tau+\sigma+1}^{n-1}\frac{-q(s)}{p(s+\tau-\sigma)r(s+\tau-\sigma)}\right)\right]=\infty. \tag{3.79}$$

Then every solution of (1.1) is oscillatory.

Proof. Assume that (1.1) has a nonoscillatory solution. Without loss of generality we may assume that $\{x(n)\}$ is an eventually positive solution of (1.1). Then there exists an integer $n_1\geq n_0$ such that $x(n)>0$, $x(n-\tau)>0$ and $x(n-\sigma)>0$ for $n\geq n_1$.

Set $z(n)$ to be defined as in (1.3). Then by Lemma 2.1, it follows that

$$z(n)<0\quad\text{eventually.}$$

As $z(n)>p(n)x(n-\tau)$, it follows from (1.1) that

$$\Delta(r(n)z(n))+\frac{q(n)}{p(n+\tau-\sigma)}z(n+\tau-\sigma)\leq 0. \tag{3.80}$$

Dividing the last inequality by $r(n)>0$, we obtain

$$\Delta z(n)+\frac{\Delta r(n)}{r(n)}z(n+1)+\frac{q(n)}{p(n+\tau-\sigma)r(n)}z(n+\tau-\sigma)\leq 0. \tag{3.81}$$

Let

$$y(n)=r(n)z(n). \tag{3.82}$$

This implies that $y(n)>0$. Substituting in (3.82) yields

$$\Delta y(n)+\frac{q(n)}{p(n+\tau-\sigma)r(n+\tau-\sigma)}y(n+\tau-\sigma)\leq 0,\quad n\geq n_0. \tag{3.83}$$

So by Lemma 2.8, we have that the advanced difference equation

$$\Delta y(n)+\frac{q(n)}{p(n+\tau-\sigma)r(n+\tau-\sigma)}y(n+\tau-\sigma)=0,\quad n\geq n_0 \tag{3.84}$$

has an eventually negative solution as well as.

Let

$$\lambda(n) = \frac{\Delta y(n)}{y(n+1)}. \tag{3.85}$$

Then $\{\lambda(n)\}$ is positive sequence. Furthermore, $\{\lambda(n)\}$ satisfies

$$\lambda(n) \geq \bar{Q}_2(n) \exp\left(\sum_{s=n+1}^{n+\tau-\sigma-1} \lambda(s)\right), \tag{3.86}$$

where

$$\bar{Q}_2(n) = \frac{-q(n)}{p(n+\tau-\sigma)r(n+\tau-\sigma)} > 0. \tag{3.87}$$

Let

$$R_2(n) = \sum_{s=n-\tau+\sigma+1}^{n-1} \bar{Q}_2(s). \tag{3.88}$$

Therefore

$$\lambda(n) \geq \bar{Q}_2(n) \exp\left\{\frac{1}{R_2(n)} R_2(n) \sum_{s=n+1}^{n+\tau-\sigma-1} \lambda(s)\right\}. \tag{3.89}$$

Applying the inequality (3.43) to (3.89), we have

$$\lambda(n) \geq \bar{Q}_2(n) \left\{\frac{1}{R_2(n)} \sum_{s=n+1}^{n+\tau-\sigma-1} \lambda(s) + \frac{\ln(eR_2(n))}{R_2(n)}\right\}, \tag{3.90}$$

or

$$\lambda(n) \sum_{s=n-\tau+\sigma}^{n-1} \bar{Q}_2(s) - \bar{Q}_2(n) \sum_{s=n+1}^{n+\tau-\sigma-1} \lambda(s) \geq \bar{Q}_2(n) \ln\left(e \sum_{s=n-\tau+\sigma+1}^{n-1} \bar{Q}_2(s)\right).$$

Then for $M > N$, we have

$$\sum_{n=N+1}^{M} \lambda(n) \sum_{s=n-\tau+\sigma+1}^{n-1} \bar{Q}_2(s) - \sum_{n=N+1}^{M} \bar{Q}_2(n) \sum_{s=n+1}^{n+\tau-\sigma-1} \lambda(s)$$
$$\geq \sum_{n=N+1}^{M} \bar{Q}_2(n) \ln\left(e \sum_{s=n-\tau+\sigma+1}^{n-1} \bar{Q}_2(s)\right). \tag{3.91}$$

By interchanging the order of summation we get

$$\sum_{n=N+1}^{M} \bar{Q}_2(n) \sum_{s=n+1}^{n+\tau-\sigma-1} \lambda(s) \geq \sum_{n=N+\tau-\sigma}^{M} \lambda(n) \sum_{s=n-\tau+\sigma+1}^{n-1} \bar{Q}_2(s). \tag{3.92}$$

From (3.91) and (3.92), we find that

$$\sum_{n=N+1}^{n+\tau-\sigma-1} \sum_{s=n-\tau+\sigma+1}^{n-1} \bar{Q}_2(s) \geq \sum_{n=N}^{M-1} \bar{Q}_2(n) \ln\left(e \sum_{s=n-\tau+\sigma+1}^{n-1} \bar{Q}_2(s)\right). \tag{3.93}$$

However, using Lemma 2.10, it follows that

$$\sum_{s=n-\tau+\sigma+1}^{n-1} \bar{Q}_2(s) \leq 1 \quad \text{eventually}. \tag{3.94}$$

Therefore from (3.94) in (3.94), we get

$$\sum_{n=N+1}^{N+\tau-\sigma-1} \lambda(n) \geq \sum_{n=N}^{M-1} \bar{Q}_2(n) \ln\left(e \sum_{s=n-\tau+\sigma+1}^{n-1} \bar{Q}_2(s)\right),$$

or

$$\sum_{n=N+1}^{N+\tau-\sigma-1} \left(1 - \frac{y(n)}{y(n+1)}\right) \geq \sum_{n=N}^{M-1} \bar{Q}_2(n) \ln\left(e \sum_{s=n-\tau+\sigma+1}^{n-1} \bar{Q}_2(s)\right). \tag{3.95}$$

Using the inequality $\ln x < x-1$, for $0 < x < 1$ in (3.95), we get

$$\log \frac{y(N+\tau-\sigma)}{y(N+1)} \geq \sum_{n=N}^{M-1} \bar{Q}_2(n) \ln \left(e \sum_{s=n-\tau+\sigma+1}^{n-1} \bar{Q}_2(s) \right)$$

which implies by condition (3.79) that

$$\lim_{n\to\infty} \frac{y(n+\tau-\sigma)}{y(n+1)} = \infty. \tag{3.96}$$

On the other hand, (3.79) implies that there exists a sequence $\{n_k\}$ of positive integers with $n_k \to \infty$ as $k \to \infty$ such that

$$\sum_{s=n_k-\tau+\sigma+1}^{n_k-1} \frac{-q(s)}{p(s+\tau-\sigma)r(s+\tau-\sigma)} \geq \frac{1}{e}, \quad \text{for all k.} \tag{3.97}$$

Hence by Lemma 2.9, we obtain

$$\liminf_{n\to\infty} \frac{y(n+\tau-\sigma)}{y(n+1)} < \infty. \tag{3.98}$$

This contradicts (3.96) and completes the proof. □

4 Some Examples

Example 4.1. *Consider the equation*

$$\Delta \left[\frac{1}{n+2} \left(x(n) - \frac{n+1}{n+2} x(n-2) \right) \right] + \frac{1}{n+1} x(n-1) = 0, \quad n = 0,1,2,..., \tag{4.99}$$

where

$$r(n) = \frac{1}{n+1}, \quad q(n) = \frac{1}{n+1}, \quad p(n) = -\frac{n+1}{n+2}, \tau = 2 \quad \text{and} \quad \sigma = 1.$$

Observe that

$$\sum_{n=0}^{\infty} q(n) = \infty.$$

Also

$$\sum_{n=0}^{\infty} \left[\frac{q(n)}{r(n-\sigma)} \ln \left(e \sum_{s=n+1}^{n+\sigma} \frac{q(s)}{r(s-\sigma)} \right) \right]$$
$$= \sum_{n=0}^{\infty} \ln(e\sigma) = \infty.$$

All conditions of the Theorem 3.1 are satisfied. Then all solutions of (4.99) oscillate.

Example 4.2. *Consider the equation*

$$\Delta \left[2 \left(x(n) - \frac{1}{2} x(n-2) \right) \right] + (2+(-1)^n) x(n-3) = 0, \quad n = 0,1,2,..., \tag{4.100}$$

where

$$-1 \leq p(n) = \frac{1}{2}, \tau = 2, \sigma = 3, r(n) = 2 \quad \text{and} \quad q(n) = 2+(-1)^n.$$

Observe that

$$\sum_{n=0}^{\infty} \left[\frac{q(n)}{r(1+p)} \ln \left(e \sum_{s=n+1}^{n+\sigma-\tau} \frac{q(s)}{r(1+p)} \right) \right]$$
$$= \sum_{n=0}^{\infty} [(2+(-1)^n) \ln(e(2-(-1)^n))]$$
$$= \infty.$$

Then all conditions of Theorem 3.2 are satisfied and therefore all solution of (4.100) oscillate.

Example 4.3. *Consider the difference equation*

$$\Delta\left[\frac{1}{2^n}(x(n)-2^n x(n-3))\right]+e^n x(n-1)=0, \quad n=0,1,2,..., \tag{4.101}$$

where

$$-1 \geq p(n)=-2^n, r(n)=\frac{1}{2^n}, q(n)=e^n, \tau=3 \quad \sigma=1.$$

Observe that

$$\sum_{n=0}^{\infty}\left[\frac{-q(n)}{p(n+\tau-\sigma)r(n+\tau-\sigma)}\ln\left(e\sum_{s=n-\tau+\sigma+1}^{n-1}\frac{-q(s)}{p(s+\tau-\sigma)r(s+\tau-\sigma)}\right)\right]$$
$$=\sum_{n=0}^{\infty} ne^n=\infty.$$

Then all conditions of Theorem 3.3 are satisfied and therefore all solutions of (4.101) oscillate.

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