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A Note on Some Modular Equations Using Theta Functions

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Abstract

In this paper, we have given simple proof to the modular equations using theta function identities.

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1 Introduction

We begin this section by introducing the standard notation

$$(a;q)_0 := 1$$

and

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n), \qquad |q|<1.$$

We now give Ramanujans denition of his general theta-function. For $|ab|<1$, define

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}.$$

We introduce theta-functions that play major roles. In [4, Entry 22, P. 36], they are defined by, for $q=e^{2iz}$,

$$\begin{aligned}
\varphi(q) &:= f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)^2{}_\infty (q^2;q^2)_\infty = \frac{(-q;-q)_\infty}{(q;-q)_\infty},\\
\psi(q) &:= f(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},\\
f(-q) &:= f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty.
\end{aligned}$$

After Ramanujan,we define

$$\chi(q) := (-q;q^2)_\infty,$$

Ramanujan's modular equations involve quotients of the function $f(-q)$ at certain arguments. For example [5, P. 206], let

$$P := \frac{f(-q)}{q^{1/6} f(-q^5)} \qquad \text{and} \qquad Q := \frac{f(-q^2)}{q^{1/3} f(-q^{10})},$$

then

$$PQ + \frac{5}{PQ} = \left(\frac{Q}{P}\right)^3 + \left(\frac{P}{Q}\right)^3. \tag{1.1}$$

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These modular equations are also called Schläfli type. Since the publication of [5] several authors, including N.D. Baruah [1], [2], M. S. M. Naika [8], [9] and K. R. Vasuki [10], [11], have found additional modular equations of the type (1.1). After that several new $P-Q$ eta function identities have been discovered and employed them in finding the explicit evaluation of continued fractions, class invariants and ratio of theta functions by many mathematicians.

There are many definitions of a modular equation in the literature. We now give the definition of a modular equation as given by Ramanujan. A modular equation of degree n is an equation relating α and β that is induced by

$$n\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\alpha\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)}=\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\beta\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)},$$

where

$${}_2F_1(a,b;c;z):=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}z^n \qquad |z|<1,$$

denotes an ordinary hypergeometric function with

$$(a)_k=\frac{\Gamma(a+k)}{\Gamma(a)}.$$

Then, we say that β is of n^{th} degree over α and call the ratio

$$m:=\frac{z_1}{z_n},$$

the multiplier, where $z_1 = {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)$ and $z_n = {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)$.

In Section 2 of this paper, we recall some theta function identities and in Section 3, we prove few modular equations using the same. Recently N. D. Baruah and N. Saikia [3] have obtained few modular equations for the explicit evaluations of Ramanujan's theta functions with two parameters. We are obtaining these modular equations using theta functions. The purpose of this paper is to provide direct proofs of some of P-Q eta function identities. Our proofs use nothing more than theta function identities. However our proofs are much more elementary and can be extended to prove other modular equations.

2 Preliminary Results

For convenience, we denote $f(-q^n)$ by f_n for a positive integer n. It is easy to see that

$$\begin{aligned}\varphi(-q) &= \frac{f_1^2}{f_2}, \qquad \psi(q)=\frac{f_2^2}{f_1}, \qquad \varphi(q)=\frac{f_2^5}{f_1^2f_4^2}, \qquad \psi(-q)=\frac{f_1f_4}{f_2},\\ \chi(q) &= \frac{f_2^2}{f_1f_4}, \qquad \chi(-q)=\frac{f_1}{f_2} \qquad \text{and} \qquad f(q)=\frac{f_2^3}{f_1f_4}.\end{aligned} \tag{2.1}$$

We have

$$\varphi(q)\psi(q^2) = \psi^2(q), \tag{2.2}$$

$$\varphi^2(q)+\varphi^2(-q) = 2\varphi^2(q^2), \tag{2.3}$$

$$\varphi^2(q)-\varphi^2(-q) = 8q\psi^2(q^4), \tag{2.4}$$

$$\varphi^2(q)-\varphi^2(q^5) = 4q\chi(q)\chi(q^5)\psi^2(-q^5), \tag{2.5}$$

$$\varphi^2(q)-5\varphi^2(q^5) = -4\frac{f_2^2\chi(q^5)}{\chi(q)}, \tag{2.6}$$

$$\psi^2(-q)+q\psi^2(-q^5) = \frac{\varphi^2(q^5)}{\chi(q)\chi(q^5)}, \tag{2.7}$$

$$\psi^2(-q)+5q\psi^2(-q^5) = f_1^2\frac{\chi(q)}{\chi(q^5)}. \tag{2.8}$$

The identities (2.2) - (2.4) are due to Ramanujan and for a proof see [4]. Again the identities (2.5) - (2.8) are due to Ramanujan [4], S. -Y. Kang [7], has given the proof of (2.5) - (2.8) by employing theta function identities. Recently S. Bhargava, K. R. Vasuki and Rajanna [6] deduced (2.5) - (2.8) from Ramanujan's ${}_1\psi_1$ summation formula.

3 Main Results

Theorem 3.1. *If*

$$P := \frac{\psi(q)}{q^{1/8}\psi(q^2)} \qquad \text{and} \qquad Q := \frac{\psi(q^2)}{q^{1/4}\psi(q^4)}$$

then

$$P^2 - \left(\frac{2}{PQ}\right)^2 - \left(\frac{Q}{P}\right)^2 = 0.$$

Proof. Consider

$$\begin{aligned} Q^2 + \frac{4}{Q^2} &= \frac{\psi^4(q^2) + 4q\psi^4(q^4)}{q^{1/2}\psi^2(q^2)\psi^2(q^4)} \\ &= \frac{1}{q^{1/2}\psi^2(q^2)}\left\{\frac{\psi^4(q^2)}{\psi^2(q^4)} + 4q\psi^2(q^2)\right\} \\ &= \frac{1}{q^{1/2}\psi^2(q^2)}\left\{\varphi^2(q^2) + 4q\psi^2(q^2)\right\} \\ &= \frac{\varphi^2(q)}{q^{1/2}\psi^2(q^2)} = P^4. \end{aligned}$$

where we used (2.2) after changing q to q^2, (2.3), (2.4) consecutively. Further on employing (2.2) again we have the result. □

Theorem 3.2. *If*

$$P := \frac{\psi(-q)}{q^{1/2}\psi(-q^5)} \qquad \text{and} \qquad Q := \frac{\varphi(q)}{\varphi(q^5)}$$

then

$$PQ - \frac{5}{PQ} = \frac{P}{Q} - \frac{Q}{P}.$$

Proof. From the definition of P and Q, it is easy to see that

$$PQ = \frac{\psi(-q)\varphi(q)}{q^{1/2}\psi(-q^5)\varphi(q^5)} \qquad \text{and} \qquad \frac{P}{Q} = \frac{\psi(-q)\varphi(q^5)}{q^{1/2}\varphi(q)\psi(-q^5)}.$$

Thus in order to prove the result, it suffices to prove the following identity:

$$\psi^2(-q)\left\{\varphi^2(q) - \varphi^2(q^5)\right\} = q\psi^2(q^5)\left\{5\varphi^2(q^5) - \varphi^2(q)\right\}.$$

Using (2.5) and (2.6), we have

$$\frac{\psi^2(-q)}{q\psi^2(-q^5)} - \frac{f_2^2}{q\chi^2(q)\psi^2(-q^5)} = 0.$$

Further, on using (2.1) we have the result. □

Theorem 3.3. *If*

$$P := \frac{\psi(-q)}{q^{1/2}\psi(-q^5)} \qquad \text{and} \qquad Q := \frac{\varphi(-q^2)}{\varphi(q^5)}$$

then

$$PQ + \frac{Q}{P} = \frac{5}{PQ} + \frac{P}{Q}.$$

Proof. Consider

$$\begin{aligned}\frac{P^2+5}{P^2+1} &= \frac{\psi^2(-q)+5q\psi^2(-q^5)}{\psi^2(-q)+q\psi^2(-q^5)} \\ &= f_1^2\frac{\chi^2(q)}{\varphi^2(q^5)} \\ &= \frac{\varphi^2(-q^2)}{\varphi^2(q^5)} = Q^2.\end{aligned}$$

where we employed (2.7), (2.8) and then (2.1). Hence the result. □

Theorem 3.4. *If*

$$P := \frac{\varphi(q)}{\varphi(q^5)} \qquad \text{and} \qquad Q := \frac{\varphi(-q)}{\varphi(-q^5)}$$

then

$$PQ + \frac{5}{PQ} - 4 = \frac{Q}{P} + \frac{P}{Q}.$$

Proof. Consider

$$\begin{aligned}\frac{5P-P^3}{Q^3-5Q} &= \frac{\varphi(q)\varphi^3(-q^3)\left\{5\varphi^2(q^5)-\varphi^2(q)\right\}}{\varphi(-q)\varphi^3(q^5)\left\{\varphi^2(-q)-5\varphi^2(-q^5)\right\}} \\ &= -\frac{\varphi(q)\varphi^2(-q^3)\chi(-q)\chi(q^5)}{\varphi(-q)\varphi^3(q^5)\chi(q)\chi(-q^5)} \\ &= \frac{P^2-1}{Q^2-1}.\end{aligned}$$

Where we used (2.6) twice and (2.1). Now on factorizing and on dividing throughout by PQ we have the result. □

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