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ORTHOGONAL STABILITY OF THE NEW GENERALIZED QUADRATIC FUNCTIONAL EQUATION

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Abstract

In this paper, the authors investigate the Hyers - Ulam - Rassias stability and J. M. Rassias mixed type product- sum of powers of norms stability of a orthogonally generalized quadratic functional equation of the form

$$f(nx+y)+f(nx-y)=n[f(x+y)+f(x-y)]+2n(n-1)f(x)-2(n-1)f(y).$$

Where $f : A \to B$ be a mapping from a orthogonality normed space A into a Banach Space B, $\perp$ is orthogonality in the sense of Ratz with $x \perp y$ for all $x, y \in A$.

Keywords: : Hyers - Ulam - Rassias stability, J. M. Rassias mixed type product - sum of powers of norms stability,Example, Orthogonally quadratic functional equation, Orthogonality space, Quadratic mapping.

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1 Introduction

The stability problem of functional equations originated from the following question of Ulam[19]: Under what condition does there exist an additive mapping near an approximately additive mapping? In 1941, Hyers [8] gave a partial armative answer to the question of Ulam in the context of Banach spaces. In 1978, Th.M. Rassias [14] extended the theorem of Hyers by considering the unbounded Cauchy dierence.

The idea of generalized Hyers-Ulam stability is extended to various functional equations like additive equations, Jensen's equations, Hosszu's equations, homogeneous equations, logarithmic equations, exponential equations, multiplicative equations, trigonometric and gamma functional equations .

It is easy to see that the quadratic function $f(x) = kx^2$ is a solution of each of the following functional equations

$$f(x+y)+f(x-y)=2f(x)+2f(y), \tag{1.1}$$

$$f(x+y+z)+f(x)+f(y)+f(z)=f(x+y)+f(y+z)+f(z+x), \tag{1.2}$$

$$f(x-y-z)+f(x)+f(y)+f(z)=f(x-y)+f(y+z)+f(z-x), \tag{1.3}$$

$$f(x+y+z)+f(x-y+z)+f(x+y-z)+f(x-y-z)=4f(x)+4f(y)+4f(z). \tag{1.4}$$

So it is natural that each equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.1) is said to be quadratic function. It is well known that a function f between real vector spaces is quadratic if and only if there exists a unique symmetric biadditive function B such that $f(x) = B(x, x)$ for all x(see [1, 9]). The bi-additive function B is given by

$$B(x,y)=\frac{1}{4}[f(x+y)-f(x-y)]. \tag{1.5}$$

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Hyers-Ulam-Rassias stability problem for the quadratic functional equation (1.1) was first treated by F. Skof for functions $f : A \to B$ where A is a normed space and B is a Banach space (see [17]). Cholewa [2] noticed that the theorem of Skof is still true if relevent domain A is replaced by abelian group. Czerwik [3] proved the Hyers-Ulam-Rassias stability of the equation (1.1).

In 1982-1984, J.M. Rassias [12, 13] proved the following theorem in which he generalized the Hyers stability result by presenting a weaker condition controlled by a product of different powers of norms.

Theorem 1.1. *[12, 13] Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon \|x\|^p \|y\|^q$$

for all $x, y \in E$, where ϵ and p, q are constants with $\epsilon > 0$ and $r = p + q \neq 1$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{\epsilon}{2 - 2^r} \|x\|^r$$

for all $x \in E$. If, in addition, for every $x \in E$, $f(tx)$ is continuous in real t for each fixed x, then L is linear.

The above-mentioned stability involving a product of different powers of norms is called Ulam-Gavruta-Rassias stability. Later, J.M.Rassias [15] discused the stability of quadratic functional equation

$$f(mx+y) + f(mx-y) = 2f(x+y) + 2f(x-y) + 2(m^2-2)f(x) - 2f(y)$$

for any arbitrary but fixed real constant m with $m \neq 0; m \neq \pm 1; m \neq \pm\sqrt{2}$ using the mixed powers of norms.

Now we present the results connected with functional equation in orthogonal space. The orthogonal Cauchy functional equation

$$f(x+y) = f(x) + f(y), x \perp y \tag{1.6}$$

in which $\perp$ is an abstract orthogonally was first investigated by S. Gudder and D. Strawther . R. Ger and J. Sikorska discussed the orthogonal stability of the equation (1.6) in [7]. The orthogonally quadratic functional equation (1.1) was first investigated by F. Vajzovic [20] when X is a Hilbert space, Y is the scalar field, f is continuous and $\perp$ means the Hilbert space orthogonality. This result was then generalized by H. Drljevic [4], M. Fochi [5], M.Moslehian [10, 11] and G. Szabo [18].

Definition 1.1. *A vector space X is called an orthogonality vector space if there is a relation $x \perp y$ on X such that*
(i) totality of $\perp$ for zero: $x \perp 0$, $0 \perp x$ for all $x \in X$;
(ii) independence: if $x \perp y$ and $x, y \neq 0$, then x, y are linearly independent;
(iii) homogeneity: if $x \perp y$, then $ax \perp by$ for all $a, b \in \mathbb{R}$;
(iv) the Thalesian property: if P is a two-dimensional subspace of X ; then
(a) for every $x \in P$ there exists $0 \neq y \in P$ such that $x \perp y$;
(b) there exists vectors $x, y \neq 0$ such that $x \perp y$ and $x + y \perp x - y$.
Any vector space can be made into an orthogonality vector space if we define $x \perp 0, 0 \perp x$ for all x and for non zero vector x, y define $x \perp y$ iff x, y are linearly independent. The relation $\perp$ is called symmetric if $x \perp y$ implies that $y \perp x$ for all x, $y \in X$.

Definition 1.2. *The pair $(x, \perp)$ is called an orthogonality space. It becomes orthogonality normed space when the orthogonality space is equipped with a norm.*

Definition 1.3. *Let X be an orthogonality space and Y be a real Banach space. A mapping $f : X \to Y$ is called orthogonally quadratic if it satisfies the so called orthogonally Euler-Lagrange (or Jordan - von Neumann) quadratic functional equation*

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.7}$$

for all $x, y \in X$ with $x \perp y$.

In this paper, we obtain the general solution of new quadratic functional equation

$$f\,(nx+y)+f\,(nx-y)=n[f\,(x+y)+f\,(x-y)]+2n(n-1)f(x)-2(n-1)f(y) \tag{1.8}$$

and study the Hyers - Ulam - Rassias stability and J. M. Rassias mixed type product-sum of powers of norms stability in the concept of orthogonality.

Definition 1.4. *A mapping $f : A \to B$ is called orthogonal quadratic if it satisfies the quadratic functional equation (1.8) for all $x, y \in A$ with $x \perp y$ where A be an orthogonality space and B be a real Banach space.*

Through out this paper, let $(A, \perp)$ denote an orthogonality normed space with norm $\|\cdot\|_A$ and $(B, \|\cdot\|_B)$ is a Banach space. We define

$$\begin{aligned} D\,f\,(x,y) &= f\,(nx+y)+f\,(nx-y) \\ &-n[f\,(x+y)+f\,(x-y)]-2n(n-1)f(x)+2(n-1)f(y). \end{aligned} \tag{1.9}$$

for all $x, y \in A$ with $x \perp y$.

Now we proceed to find the general solution of the functional equation (1.8).

2 The General Solution of the Functional Equation (1.8)

In this section, we obtain the general solution of the functional equation (1.8). Through out this section, let X and Y be real vector spaces.

Theorem 2.2. *Let X and Y be real vector spaces. A function $f : X \to Y$ satisfies the functional equation*

$$\begin{aligned} f\,(nx+y)+f\,(nx-y) &= n[f\,(x+y) \\ &+f\,(x-y)]+2n(n-1)f(x)-2(n-1)f(y) \end{aligned} \tag{2.1}$$

for all $x, y \in X$ if and only if it satisfies the quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{2.2}$$

for all $x, y \in X$.

Proof. Suppose a function $f : X \to Y$ satisfies (2.1). Putting $x = y = 0$ in (2.1), we get $f(0) = 0$. Let $x = 0$ and $y = 0$ in (2.1), we obtain $f(-y) = f(y)$ and $f(nx) = n^2 f(x)$, respectively. Setting $(x, y) = (x, x+y)$ in (2.1), we obtain

$$f((n+1)x+y)+f((n-1)x-y)=n[f(2x+y)+f(-y)]+2f(nx)-2nf(x) \tag{2.3}$$

for all $x, y \in X$. Replacing y by $-y$ in (2.3) and adding the resultant with (2.3), we obtain

$$\begin{aligned} &f((n+1)x+y)+f((n+1)x-y)+f((n-1)x+y)+f((n-1)x-y) \\ &=n[f(2x+y)+f(2x-y)]+2n[f(x+y)+f(x-y)]+2[f(x+y)+f(x-y)] \\ &+2nf(y)+4f(nx)-4nf(x) \end{aligned} \tag{2.4}$$

for all $x, y \in X$. Setting $n = n+1, n = n-1$ and $n = 2$ respectively in (2.1), we obtain the following equations

$$\begin{aligned} &f((n+1)x+y)+f((n+1)x-y) \\ &=(n+1)[f(x+y)+f(x-y)]+2n^2f(x)+2nf(x)-2nf(y) \end{aligned} \tag{2.5}$$

$$\begin{aligned} f((n-1)x+y)+f((n-1)x-y) &= (n-1)[f(x+y)+f(x-y)] \\ &+2n^2f(x)-6nf(x)+4f(x)-2nf(y)+4f(y) \end{aligned} \tag{2.6}$$

$$f(2x+y)+f(2x-y)=2[f(x+y)+f(x-y)]+4f(x)-2f(y) \tag{2.7}$$

for all $x, y \in X$. Substitute (2.5), (2.6) and (2.7) in (2.4), we arrive (2.2).

Conversely, assume f satisfies the functional equation (2.2). Letting (x,y) by $(0,0)$ in (2.2), we get $f(0)=0$. Putting $x=0$ in (2.2), we obtain $f(-y)=f(y)$ for all $y \in X$. Thus f is an even function. Substituting (x,y) by (x,x) and $(x,2x)$ in (2.2), we get

$$f(2x)=4f(x), f(3x)=9f(x) \tag{2.8}$$

respectively for all $x \in X$. Setting $(x,y)=(nx+y,nx-y)$ in (2.2), we obtain

$$f(nx+y)+f(nx-y)=2n^2f(x)+2f(y) \tag{2.9}$$

for all $x,y \in X$. Multiplying (2.2) by n and subtracting the resultant from (2.9), we arrive (2.1). □

3 Hyers - Ulam - Rassias Stability of (1.8)

In this section, we present the Hyers - Ulam - Rassias stability of the functional equation (1.8) involving sum of powers of norms.

Theorem 3.3. *Let μ and $s(s<2)$ be non-negative real numbers. Let $f: A \to B$ be a mapping fulfilling*

$$\|D\,f\,(x,y)\|_B \le \mu\left\{\|x\|_A^s+\|y\|_A^s\right\} \tag{3.1}$$

for all $x,y \in A$ with $x \perp y$. Then there exists a unique orthogonally quadratic mapping $Q: A \to B$ such that

$$\|f(x)-Q(x)\|_B \le \frac{\mu}{2(n^2-n^s)}\|x\|_A^s \tag{3.2}$$

for all $x \in A$. The function $Q(x)$ is defined by

$$Q(x)=\lim_{k\to\infty}\frac{f(n^kx)}{(n^2)^k} \tag{3.3}$$

for all $x \in A$.

Proof. Replacing (x,y) by $(0,0)$ in (3.1) we get $f(0)=0$. Setting (x,y) by $(x,0)$ in (3.1), we obtain

$$\left\|f(nx)-n^2f(x)\right\|_B \le \frac{\mu}{2}(\|x\|_A^s) \tag{3.4}$$

for all $x \in A$. Since $x \perp 0$, we have

$$\left\|\frac{f(nx)}{n^2}-f(x)\right\|_B \le \frac{\mu}{2n^2}\|x\|_A^s \tag{3.5}$$

for all $x \in A$. Now replacing x by nx and dividing by n^2 in (3.5) and summing resulting inequality with (3.5), we arrive

$$\left\|\frac{f(n^2x)}{(n^2)^2}-f(x)\right\|_B \le \frac{\mu}{2n^2}\left\{1+\frac{n^s}{n^2}\right\}\|x\|_A^s \tag{3.6}$$

for all $x \in A$. In general, using induction on a positive integer n we obtain that

$$\begin{aligned}\left\|\frac{f(n^kx)}{(n^2)^k}-f(x)\right\|_B &\le \frac{\mu}{2n^2}\sum_{t=0}^{k-1}\frac{n^{s\,t}}{(n^2)^t}\|x\|_A^s \\ &\le \frac{\mu}{2n^2}\sum_{t=0}^{\infty}\frac{n^{s\,t}}{(n^2)^t}\|x\|_A^s\end{aligned} \tag{3.7}$$

for all $x \in A$. In order to prove the convergence of the sequence $\{f(n^kx)/(n^2)^k\}$ replace x by n^mx and divide by $(n^2)^m$ in (3.7), for any $k,m>0$, we obtain

$$\begin{aligned}\left\|\frac{f\left(n^kn^mx\right)}{(n^2)^{(k+m)}}-\frac{f(n^mx)}{(n^2)^m}\right\|_B &= \frac{1}{(n^2)^m}\left\|\frac{f\left(n^kn^mx\right)}{(n^2)^k}-f\left(n^mx\right)\right\|_B \\ &\le \frac{1}{(n^2)^m}\frac{\mu}{2n^2}\sum_{t=0}^{k-1}\frac{n^{s\,t}}{(n^2)^t}\|n^mx\|_A^s \\ &\le \frac{\mu}{2n^2}\sum_{t=0}^{\infty}\frac{1}{n^{(2-s)(t+m)}}\|x\|_A^s\,.\end{aligned} \tag{3.8}$$

As $s < 2$, the right hand side of (3.8) tends to 0 as $m \to \infty$ for all $x \in A$. Thus $\{f(n^k x)/(n^2)^k\}$ is a Cauchy sequence. Since B is complete, there exists a mapping $Q : A \to B$ such that

$$Q(x) = \lim_{k\to\infty} \frac{f(n^k x)}{(n^2)^k} \quad \forall x \in A.$$

Letting $k \to \infty$ in (3.7), we arrive the formula (3.2) for all $x \in A$. To prove Q satisfies (1.8), replace (x, y) by $(n^k x, n^k y)$ in (3.1) and divide by $(n^2)^k$ then it follows that

$$\frac{1}{(n^2)^k}\|f(n^k(nx+y)) + f(n^k(nx-y)) - n[f(n^k(x+y)) - f(n^k(x-y))] \\ - 2n(n-1)f(n^k x) - 2(n-1)f(n^k y)\|_B \le \frac{\mu}{(n^2)^k}\left\{\left\|n^k x\right\|_A^s + \left\|n^k y\right\|_A^s\right\}.$$

Taking limit as $n \to \infty$ in the above inequality, we get

$$\|Q(nx+y) + Q(nx-y) - n[Q(x+y) - Q(x-y)] \\ - 2n(n-1)Q(x) + 2(n-1)Q(y)\|_B \le 0,$$

which gives

$$Q(nx+y) + Q(nx-y) = n[Q(x+y) - Q(x-y)] + 2n(n-1)Q(x) - 2(n-1)Q(y)$$

by taking limit as $k \to \infty$ in (3.7), we obtain

$$\| f(x) - Q(x)\|_B \le \frac{\mu}{2(n^2 - n^s)} \|x\|_A^s \tag{3.9}$$

for all $x, y \in A$ with $x \perp y$. Therefore $Q : A \to B$ is an orthogonally quadratic mapping which satisfies (1.8). To prove the uniqueness: Let Q' be another orthogonally quadratic mapping satisfying (1.8) and the inequality (3.2). Then

$$\begin{aligned}\|Q(x) - Q'(x)\|_B &= \frac{1}{(n^2)^k}\left\|Q\left(n^k x\right) - Q'\left(n^k x\right)\right\|_B \\ &\le \frac{1}{(n^2)^k}\left(\|Q(k^t x) - f(k^t x)\|_B + \left\|f\left(n^k x\right) - Q'\left(n^k x\right)\right\|_B\right) \\ &\le \frac{\mu}{n^2 - n^s}\frac{1}{n^{k(2-s)}}\|x\|_A^s \\ &\to 0 \text{ as } k \to \infty\end{aligned}$$

for all $x \in A$. Therefore Q is unique. This completes the proof of the theorem. □

Theorem 3.4. *Let μ and $s (s > 2)$ be nonnegative real numbers. Let $f : A \to B$ be a mapping satisfying (3.1) for all $x, y \in A$ with $x \perp y$. Then there exists a unique orthogonally quadratic mapping $Q : A \to B$ such that*

$$\|f(x) - Q(x)\|_B \le \frac{\mu}{2(n^s - n^2)} \|x\|_A^s \tag{3.10}$$

for all $x \in A$. The function $Q(x)$ is defined by

$$Q(x) = \lim_{k\to\infty} (n^2)^n f\left(\frac{x}{n^k}\right) \tag{3.11}$$

for all $x \in A$.

Proof. Replacing x by $\frac{x}{2}$ in (3.4), the rest of the proof is similar to that of Theorem 3.1. □

4 J.M. Rassias Mixed Type Product - Sum of Powers of Norms Stability of (1.8)

In this section, we discuss the J.M. Rassias mixed type product - sum of powers of norms stability of the functional equation (1.8).

Theorem 4.5. *Let $f : A \to B$ be a mapping satisfying the inequality*

$$\|D\, f\,(x,y)\|_B \leq \mu \left\{ \|x\|_A^{2s} + \|y\|_A^{2s} + \|x\|_A^{s}\,\|y\|_A^{s} \right\} \tag{4.1}$$

for all $x, y \in A$ where μ and s are constants with, $\mu, s > 0$ and $s < 1$. Then the limit

$$Q(x) = \lim_{k\to\infty} \frac{f\left(n^k x\right)}{(n^2)^k} \tag{4.2}$$

exists for all $x \in A$ and $Q : A \to B$ is the unique quadratic mapping such that

$$\|f(x) - Q(x)\|_B \leq \frac{\mu}{2(n^2 - n^{2s})}\,\|x\|_A^{ns} \tag{4.3}$$

for all $x \in A$.

Proof. Letting (x, y) by $(0, 0)$ in (4.1), we get $f(0) = 0$. Again substituting (x, y) by $(x, 0)$ in (4.1), we obtain

$$\left\| \frac{f\,(nx)}{n^2} - f\,(x) \right\|_B \leq \frac{\mu}{2n^2}\,\|x\|_A^{ns} \tag{4.4}$$

for all $x \in A$. Now replacing x by nx and dividing by n^2 in (4.4) and summing resulting inequality with (4.4), we arrive

$$\left\| \frac{f\,(n^2 x)}{(n^2)^2} - f\,(x) \right\|_B \leq \frac{\mu}{2n^2}\left\{1 + \frac{n^{2s}}{n^2}\right\}\|x\|_A^{2s} \tag{4.5}$$

for all $x \in A$. Using induction on a positive integer k, we obtain that

$$\begin{aligned} \left\| \frac{f\left(n^k x\right)}{(n^2)^k} - f\,(x) \right\|_B &\leq \frac{\mu}{2n^2}\sum_{t=0}^{k-1}\left(\frac{n^{2s}}{n^2}\right)^t \|x\|_A^{2s} \\ &\leq \frac{\mu}{2n^2}\sum_{t=0}^{\infty}\left(\frac{n^{2s}}{n^2}\right)^t \|x\|_A^{2s} \end{aligned} \tag{4.6}$$

for all $x \in A$. In order to prove the convergence of the sequence $\{f(n^k x)/4^k\}$ replace x by $n^m x$ and divide by $(n^2)^m$ in (4.6), for any $k, m > 0$, we obtain

$$\begin{aligned} \left\| \frac{f\left(n^k n^m x\right)}{(n^2)^{(k+m)}} - \frac{f\,(n^m x)}{(n^2)^m} \right\|_B &= \frac{1}{(n^2)^m}\left\| \frac{f\left(n^k n^m x\right)}{(n^2)^k} - f(n^m x) \right\|_B \\ &\leq \frac{1}{(n^2)^m}\frac{\mu}{2n^2}\sum_{t=0}^{k-1}\left(\frac{n^{2s}}{n^2}\right)^t \|n^m x\|_A^{2s} \\ &\leq \frac{\mu}{2n^2}\sum_{t=0}^{\infty}\frac{1}{n^{(n-2s)(t+m)}}\|x\|_A^{2s} \end{aligned} \tag{4.7}$$

As $s < 1$, the right hand side of (4.7) tends to 0 as $m \to \infty$ for all $x \in A$. Thus $\{f(n^k x)/(n^2)^k\}$ is a Cauchy sequence. Since B is complete, there exists a mapping $Q : A \to B$ such that

$$Q(x) = \lim_{k\to\infty}\frac{f(n^k x)}{(n^2)^k} \quad \forall x \in A.$$

Letting $n \to \infty$ in (4.6), we arrive the formula (4.2) for all $x \in A$. To show that Q is unique and it satisfies (1.8), the rest of the proof is similar to that of theorem 3.1 □

Theorem 4.6. *Let $f : A \to B$ be a mapping satisfying the inequality (4.1) for all $x, y \in A$ where μ and s are constants with, $\mu, s > 0$ and $s > 2$. Then the limit*

$$Q(x) = \lim_{k\to\infty} (n^2)^k f\left(\frac{x}{n^k}\right) \tag{4.8}$$

exists for all $x \in A$ and $Q : A \to B$ is the unique quadratic mapping such that

$$\|f(x) - Q(x)\|_B \leq \frac{\mu}{2(n^{2s} - n^2)} \|x\|_A^{2s} \tag{4.9}$$

for all $x \in A$.

Proof. Replacing x by $\frac{x}{3}$ in (4.4), the proof is similar to that of Theorem 4.5. □

Now we will provide an example to illustrate that the functional equation (1.8) is not stable for $s = 2$.

Example 4.1. *Let $\varphi : X \to X$ be a function defined by*

$$\phi(x) = \begin{cases} \mu \|x\|^2 & , \|x\| < 1 \\ \mu & otherwise \end{cases} \tag{4.10}$$

where $\mu > 0$ is a constant and we define a function $f : X \to Y$ by

$$f(x) = \sum_{m=0}^{\infty} \frac{\varphi(n^m x)}{(n^2)^m} \tag{4.11}$$

for all $x \in X$. Then f satisfies the functional inequality

$$\|D(f(x, y))\| \leq \frac{2n^2}{(n-1)} \mu (\|x\|^2 + \|y\|^2) \tag{4.12}$$

for all $x, y \in X$. Then there exist any quadratic mapping $Q : X \to Y$ satisfying

$$\|f(x) - Q(x)\| \leq \eta \|x\|^2 \tag{4.13}$$

for $x \in X$.

Proof. From the equation (4.10) and (4.11) , we obtain

$$f(x) \leq \sum_{m=0}^{\infty} \frac{\phi(n^m x)}{n^{2m}} = \sum_{k=0}^{\infty} \frac{\mu}{n^{2m}} \leq \mu(\frac{n^2}{n^2 - 1}) \tag{4.14}$$

for all $x \in X$. Therefore we see that f is bounded. We are going to prove that f satisfies (4.12).
If $(\|x\|^2 + \|y\|^2) \geq 1$ then the left hand side of (4.12) is less than

$$\frac{2n^2}{(n-1)}$$

. Now we suppose that $0 \leq \|x\|^2 + \|y\|^2 \leq 1$. Then there exist a positive integer k such that

$$\frac{1}{n^{2k-1}} \leq \|x\|^2 + \|y\|^2 < \frac{1}{n^{2k}} \tag{4.15}$$

for all $x \in X$. so that

$$n^{2k}\|x\|^2 < 1, n^{2k}\|y\|^2 < 1$$

and consequently, $n^{k-1}\|x\| < 1$, $n^{k-1}\|y\| < 1$, $n^{k-1}\|x+y\| < 1$, $n^{k-1}\|x-y\| < 1$, $n^{k-1}\|nx+y\| < 1$, $n^k\|nx-y\| < 1$ for all $m \in 0, 1, 2, ..., k-1$

$$n^{k-1}\|x\| < 1, n^{k-1}\|y\| < 1, n^{k-1}(\|x+y\|) < 1,$$
$$n^{k-1}(\|x-y\|) < 1, n^{k-1}(\|nx+y\|) < 1, n^{k-1}(\|nx-y\|) < 1.$$

for all $x \in \{0,1,2,.....k-1\}$.

$$\|D(f(x,y))\| \leq \sum_{m=k}^{\infty} \frac{2n(n+1)}{n^{2m}}\ \mu$$
$$\leq \frac{2n(n+1)}{n^{2m}} \left(\frac{n^2}{n^2-1}\right) \mu$$
$$\leq \frac{2n^2}{n-1}\, \mu \left(\|x\|^2 + \|y\|^2\right)$$

Thus f satisfies the inequality (4.12) Let us consider the an orthogonally quadratic mapping satisfying $Q : X \to Y$ and a constant $\eta > 0$ such that

$$\|f(x) - Q(x)\| \leq \eta \|x\|^2$$

for all $x \in X$.Since f is bounded,Q is also bounded on any open interval containing the origin zero. we have

$$Q(x) = c\|x\|^2$$

for all $x \in X$ and c is constant.Thus we obtain

$$\left\| f(x) - c\|x\|^2 \right\| \leq \eta \|x\|^2$$
$$\|f(x)\| \leq (\|c\| + \eta)\, \|x\|^2 \tag{4.16}$$

for all $x \in X$. But we can choose a positive integer

$$p, p\mu > \eta + |c|$$

. If $x \in \left(0, \frac{1}{n^{p-1}}\right)$, then $n^m x \in (0,1)$ for all $m = 0,1,\ldots,p-1$. For this x, we get $f(x) = \sum_{m=0}^{\infty} \frac{\phi(n^m x)}{n^{2m}} \geq \sum_{m=0}^{\infty} \frac{\mu(n^{2m})\|x\|^2}{n^{2m}} = p\mu\|x\|^2 > (\eta + |c|)\, \|x\|^2$ which contradicts (4.16). Therefore the functional equation (1.8) is not stable in sense of Ulam, Hyers and Rassias if $s = 2$, assumed in the inequality (4.16). □

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