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Some new integral inequalities for k-fractional integrals

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Abstract

The aim of the present paper is to investigate some new integral inequalities for k-fractional integrals. Moreover, special cases of the integral inequalities in this paper have been obtained by Tariboon *et.al*. in [22].

Keywords: k-Reimann-Liouville calculus; fractional integral inequalities; Grüss inequality.

2010 MSC: 26A33, 26D10, 26D15.

1 Introduction and Preliminaries

Integration with weight functions is used in countless mathematical problems such as approximation theory, spectral analysis, statistical analysis and the theory of distributions. Grüss developed an integral inequality [11, p. 236] in 1935. During the last few years, many researchers focused their attention on the study and generalizations of the Grüss inequality [7–9, 14, 18]. The integral inequality that establishes a connection between the integral of the product of two functions and the product of the integrals is known in the literature as the Grüss inequality. The Grüss inequality is as follows:

Theorem 1.1. *Let $f, g : [a,b] \to \mathbb{R}$ be integrable functions such that $\varphi < f(x) < \Phi$ and $\psi < g(x) < \Psi$ for all $x \in [a,b]$, where $\varphi, \Phi, \psi, \Psi$ are constants. Then*

$$\left| \frac{1}{b-a} \int_a^b f(x) g(x) \, dx - \frac{1}{b-a} \int_a^b f(x) \, dx \frac{1}{b-a} \int_a^b g(x) \, dx \right| \leq \frac{1}{4} (\Phi - \varphi)(\Psi - \psi),$$

where the constant $\frac{1}{4}$ is sharp. (see, [11, p. 236])

Fractional calculus and its widely applications have recently been paid more and more attention. For more recent development on fractional calculus, we refer the reader to [1–4, 10, 16, 19, 20, 24]. There are several known forms of the fractional integrals which have been studied extensively for their applications [5, 13, 15, 21, 23].

The first is the Riemann-Liouville fractional integral of order $\alpha \geq 0$ for a continuous function f on $[a,b]$ which is defined by

$$J_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau; \ \alpha \geq 0, \ a \leq t \leq b.$$

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This integral is motivated by the well known Cauchy formula:

$$\int_a^x dt_1 \int_a^{t_1} dt_2 \cdots \int_a^{t_{n-1}} f(t_n)dt_n = \frac{1}{\Gamma(n)} \int_a^x (x-t)^{n-1} f(t)dt;\ n \in \mathbb{N}.$$

The second is the Hadamard fractional integral introduced by Hadamard [12]. It is given by:

$$J_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\log \frac{x}{t}\right)^{\alpha-1} f(t)\frac{dt}{t}, \alpha > 0,\ x > a > 0.$$

The Hadamard integral is based on the generalization of the integral

$$\int_a^x \frac{dt_1}{t_1} \int_a^{t_1} \frac{dt_2}{t_2} \cdots \int_a^{t_{n-1}} \frac{f(t_n)}{t_n} dt_n = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\log \frac{x}{t}\right)^{n-1} f(t)\frac{dt}{t}$$

for $n \in \mathbb{N}$.

Recently, in [6], Diaz and Pariguan have defined new functions called k-gamma and k-beta functions and the Pochhammer k-symbol, that is respectively generalization of the classical gamma and beta functions and the classical Pochhammer symbol:

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n!k^n\,(nk)^{\frac{x}{k}-1}}{(x)_{n,k}},\ (k>0),$$

where $(x)_{n,k}$ is the Pochhammer k-symbol for factorial function defined by

$$(x)_{n,k} = x(x+k)(x+2k)\cdots(x+nk),\ \ k \in \mathbb{R},\ n \in \mathbb{N}.$$

It has been shown that the Mellin transform of the exponential function $e^{-\frac{t^k}{k}}$ is the k-gamma function, explicitly given by

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}} dt,\ x > 0.$$

Clearly,

$$\Gamma(x) = \lim_{k\longrightarrow 1} \Gamma_k(x),\ \Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right) \text{ and } \Gamma_k(x+k) = x\Gamma_k(x).$$

Later, under the above definitions, in [17], Mubeen and Habibullah have introduced the k-fractional integral of the Riemann-Liouville type as follows:

$$_kJ_a^\alpha f(t) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt;\ \alpha > 0,\ x > a. \tag{1.1}$$

Note that when $k \to 1$, then it reduces to the classical Riemann-liouville fractional integral.

Recently in [22] some new fractional integral inequalities of Grüss type were proved, by replacing the constants appeared as bounds of the functions f and g, by four integrable functions. In this paper we extend the results of [22] to k-fractional integral inequalities of Grüss type.

2 Main results

Throughout of this paper, we denote the Riemann-Liouville fractional integral of order α of a function f which have limit zero by $_kJ_0^\alpha f(t) = {_kJ^\alpha} f(t)$

Theorem 2.1. *Let f be an integrable function on $[0,\infty)$. Assume that:*

(H_1) *There exist two integrable functions φ_1, φ_2 on $[0,\infty)$ such that*

$$\varphi_1(t) \le f(t) \le \varphi_2(t) \quad \textit{for all} \quad t \in [0,\infty).$$

Then, for $t > 0$, $\alpha, \beta > 0$, $k > 0$, we have

$$_kJ^\beta \varphi_1(t)\,_kJ^\alpha f(t) + {_kJ^\alpha} \varphi_2(t)\,_kJ^\beta f(t) \ge {_kJ^\alpha} \varphi_2(t)\,_kJ^\beta \varphi_1(t) + {_kJ^\alpha} f(t)\,_kJ^\beta f(t). \tag{2.2}$$

Proof. From (H_1), for all $\tau \geq 0, \rho \geq 0$, we have

$$(\varphi_2(\tau) - f(\tau))\,(f(\rho) - \varphi_1(\rho)) \geq 0.$$

Therefore

$$\varphi_2(\tau)f(\rho) + \varphi_1(\rho)f(\tau) \geq \varphi_1(\rho)\varphi_2(\tau) + f(\tau)f(\rho). \tag{2.3}$$

Multiplying both sides of (2.3) by $(t-\tau)^{\frac{\alpha}{k}-1}/k\Gamma_k(\alpha)$, $\tau \in (0,t)$, we get

$$f(\rho)\frac{(t-\tau)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}\varphi_2(\tau) + \varphi_1(\rho)\frac{(t-\tau)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}f(\tau) \geq \varphi_1(\rho)\frac{(t-\tau)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}\varphi_2(\tau) + f(\rho)\frac{(t-\tau)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}f(\tau). \tag{2.4}$$

Integrating both sides of (2.4) with respect to τ on $(0,t)$, we obtain

$$\begin{aligned} f(\rho)\int_0^t \frac{(t-\tau)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}\varphi_2(\tau)d\tau + \varphi_1(\rho)\int_0^t \frac{(t-\tau)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}f(\tau)d\tau \\ \geq \varphi_1(\rho)\int_0^t \frac{(t-\tau)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}\varphi_2(\tau)d\tau + f(\rho)\int_0^t \frac{(t-\tau)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}f(\tau)d\tau, \end{aligned}$$

which yields

$$f(\rho)\,{}_kJ^{\alpha}\varphi_2(t) + \varphi_1(\rho)\,{}_kJ^{\alpha}f(t) \geq \varphi_1(\rho)\,{}_kJ^{\alpha}\varphi_2(t) + f(\rho)\,{}_kJ^{\alpha}f(t). \tag{2.5}$$

Multiplying both sides of (2.5) by $(t-\rho)^{\frac{\beta}{k}-1}/k\Gamma_k(\beta)$, $\rho \in (0,t)$, we have

$$\begin{aligned} J^{\alpha}\varphi_2(t)\frac{(t-\rho)^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)}f(\rho) + J^{\alpha}f(t)\frac{(t-\rho)^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)}\varphi_1(\rho) \\ \geq J^{\alpha}\varphi_2(t)\frac{(t-\rho)^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)}\varphi_1(\rho) + J^{\alpha}f(t)\frac{(t-\rho)^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)}f(\rho). \end{aligned} \tag{2.6}$$

Integrating both sides of (2.6) with respect to ρ on $(0,t)$, we get

$$\begin{aligned} {}_kJ^{\alpha}\varphi_2(t)\int_0^t \frac{(t-\rho)^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)}f(\rho)d\rho + J^{\alpha}f(t)\int_0^t \frac{(t-\rho)^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)}\varphi_1(\rho)d\rho \\ \geq {}_kJ^{\alpha}\varphi_2(t)\int_0^t \frac{(t-\rho)^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)}\varphi_1(\rho)d\rho + {}_kJ^{\alpha}f(t)\int_0^t \frac{(t-\rho)^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)}f(\rho)d\rho. \end{aligned}$$

Hence, we deduce inequality (2.2) as requested. This completes the proof. □

As special cases, we give the following results for the Theorem 2.1.

Remark 2.1. *If we take $k = 1$ in the Theorem 2.1, we obtain the Theorem 2 in [22].*

Corollary 2.1. *If we take $\alpha = \beta$ in the Theorem 2.1, we obtain*

$${}_kJ^{\alpha}\left(\varphi_1 + \varphi_2\right)(t)\,{}_kJ^{\alpha}f(t) \geq {}_kJ^{\alpha}\varphi_1(t)\,{}_kJ^{\alpha}\varphi_2(t) + \left({}_kJ^{\alpha}f(t)\right)^2.$$

Corollary 2.2. *Let f be an integrable function on $[0,\infty)$ satisfying $m \leq f(t) \leq M$, for all $t \in [0,\infty)$ and $m, M \in \mathbb{R}$. Then for $t > 0$ and $\alpha, \beta > 0$, we have*

$$m\frac{t^{\frac{\beta}{k}}}{\Gamma_k(\beta+k)}\,{}_kJ^{\alpha}f(t) + M\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\,{}_kJ^{\beta}f(t) \geq mM\frac{t^{\frac{\alpha+\beta}{k}}}{\Gamma_k(\alpha+k)\Gamma_k(\beta+k)} + {}_kJ^{\alpha}f(t){}_kJ^{\beta}f(t).$$

Corollary 2.3. *If we take $\alpha = \beta$ in the Corollary 2.2, we obtain*

$$(m+M)\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\,{}_kJ^{\alpha}f(t) \geq mM\frac{t^{\frac{2\alpha}{k}}}{(\Gamma_k(\alpha+k))^2} + \left({}_kJ^{\alpha}f(t)\right)^2.$$

Theorem 2.2. *Let f be an integrable function on $[0,\infty)$ and constants $\theta_1,\theta_2>0$ satisfying $1/\theta_1+1/\theta_2=1$. Suppose that (H_1) holds. Then, for $t>0$, $\alpha,\beta>0$ and $k>0$, we have*

$$\begin{aligned}
&\frac{1}{\theta_1}\frac{t^{\frac{\beta}{k}}}{\Gamma_k(\beta+k)}{}_kJ^{\alpha}\left((\varphi_2-f)^{\theta_1}\right)(t)+\frac{1}{\theta_2}\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}{}_kJ^{\beta}\left((f-\varphi_1)^{\theta_2}\right)(t)\\
&\quad+{}_kJ^{\alpha}\varphi_2(t){}_kJ^{\beta}\varphi_1(t)+{}_kJ^{\alpha}f(t){}_kJ^{\beta}f(t)\\
&\quad\geq {}_kJ^{\alpha}\varphi_2(t){}_kJ^{\beta}f(t)+{}_kJ^{\alpha}f(t){}_kJ^{\beta}\varphi_1(t). \qquad (2.7)
\end{aligned}$$

Proof. According to the well-known Young's inequality

$$\frac{1}{\theta_1}x^{\theta_1}+\frac{1}{\theta_2}y^{\theta_2}\geq xy,\qquad \forall x,y\geq 0,\ \theta_1,\theta_2>0,\ \frac{1}{\theta_1}+\frac{1}{\theta_2}=1,$$

setting $x=\varphi_2(\tau)-f(\tau)$ and $y=f(\rho)-\varphi_1(\rho)$, $\tau,\rho>0$, we have

$$\frac{1}{\theta_1}(\varphi_2(\tau)-f(\tau))^{\theta_1}+\frac{1}{\theta_2}(f(\rho)-\varphi_1(\rho))^{\theta_2}\geq(\varphi_2(\tau)-f(\tau))(f(\rho)-\varphi_1(\rho)). \qquad (2.8)$$

Multiplying both sides of (2.8) by $\dfrac{(t-\tau)^{\frac{\alpha}{k}-1}(t-\rho)^{\frac{\beta}{k}-1}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}$, $\tau,\rho\in(0,t)$, we get

$$\begin{aligned}
&\frac{1}{\theta_1}\frac{(t-\tau)^{\frac{\alpha}{k}-1}(t-\rho)^{\frac{\beta}{k}-1}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}(\varphi_2(\tau)-f(\tau))^{\theta_1}+\frac{1}{\theta_2}\frac{(t-\tau)^{\frac{\alpha}{k}-1}(t-\rho)^{\frac{\beta}{k}-1}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}(f(\rho)-\varphi_2(\rho))^{\theta_2}\\
\geq\ &\frac{(t-\tau)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}(\varphi_2(\tau)-f(\tau))\frac{(t-\rho)^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)}(f(\rho)-\varphi_1(\rho)).
\end{aligned}$$

Integrating the above inequality with respect to τ and ρ from 0 to t, we have

$$\frac{1}{\theta_1}{}_kJ^{\beta}(1)(t){}_kJ^{\alpha}(\varphi_2-f)^{\theta_1}(t)+\frac{1}{\theta_2}{}_kJ^{\alpha}(1)(t){}_kJ^{\beta}(f-\varphi_1)^{\theta_2}(t)\geq {}_kJ^{\alpha}(\varphi_2-f)(t){}_kJ^{\beta}(f-\varphi_1)(t),$$

which implies (2.7). □

Corollary 2.4. *Let f be an integrable function on $[0,\infty)$ satisfying $m\leq f(t)\leq M$, for all $t\in[0,\infty)$ and $m,M\in\mathbb{R}$. Then for $t>0$, $\alpha,\beta>0$ and $k>0$, we have*

$$\begin{aligned}
&(m+M)^2\frac{t^{\frac{\alpha+\beta}{k}}}{\Gamma_k(\alpha+k)\Gamma_k(\beta+k)}+\frac{t^{\frac{\beta}{k}}}{\Gamma_k(\beta+k)}{}_kJ^{\alpha}f^2(t)+\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}{}_kJ^{\beta}f^2(t)+2{}_kJ^{\alpha}f(t){}_kJ^{\beta}f(t)\\
\geq\ &2(m+M)\left(\frac{t^{\frac{\beta}{k}}}{\Gamma_k(\beta+k)}{}_kJ^{\alpha}f(t)+\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}{}_kJ^{\beta}f(t)\right).
\end{aligned}$$

Theorem 2.3. *Let f be an integrable function on $[0,\infty)$ and constants $\theta_1,\theta_2>0$ satisfying $\theta_1+\theta_2=1$. In addition, suppose that (H_1) holds. Then, for $t>0$, $\alpha,\beta>0$ and $k>0$, we have*

$$\begin{aligned}
&\theta_1\frac{t^{\frac{\beta}{k}}}{\Gamma_k(\beta+k)}{}_kJ^{\alpha}\varphi_2(t)+\theta_2\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}{}_kJ^{\beta}f(t)\\
&\quad\geq {}_kJ^{\alpha}(\varphi_2-f)^{\theta_1}(t){}_kJ^{\beta}(f-\varphi_1)^{\theta_2}(t)+\theta_1\frac{t^{\frac{\beta}{k}}}{\Gamma_k(\beta+k)}{}_kJ^{\alpha}f(t)\\
&\quad+\theta_2\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}{}_kJ^{\beta}\varphi_1(t). \qquad (2.9)
\end{aligned}$$

Proof. From the well-known Weighted AM-GM inequality

$$\theta_1x+\theta_2y\geq x^{\theta_1}y^{\theta_2},\qquad \forall x,y\geq 0,\ \theta_1,\theta_2>0,\ \theta_1+\theta_2=1,$$

and setting $x = \varphi_2(\tau) - f(\tau)$ and $y = f(\rho) - \varphi_1(\rho)$, $\tau, \rho > 0$, we have

$$\theta_1(\varphi_2(\tau) - f(\tau)) + \theta_2(f(\rho) - \varphi_1(\rho)) \geq (\varphi_2(\tau) - f(\tau))^{\theta_1}(f(\rho) - \varphi_1(\rho))^{\theta_2}. \tag{2.10}$$

Multiplying both sides of (2.10) by $\dfrac{(t-\tau)^{\frac{\alpha}{k}-1}(t-\rho)^{\frac{\beta}{k}-1}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}$, $\tau, \rho \in (0,t)$, we get

$$\begin{aligned}\theta_1 \frac{(t-\tau)^{\frac{\alpha}{k}-1}(t-\rho)^{\frac{\beta}{k}-1}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}(\varphi_2(\tau) - f(\tau)) + \theta_2 \frac{(t-\tau)^{\frac{\alpha}{k}-1}(t-\rho)^{\frac{\beta}{k}-1}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}(f(\rho) - \varphi_2(\rho)) \\ \geq \frac{(t-\tau)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}(\varphi_2(\tau) - f(\tau))^{\theta_1}\frac{(t-\rho)^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)}(f(\rho) - \varphi_1(\rho))^{\theta_2}.\end{aligned}$$

Integrating the above inequality with respect to τ and ρ from 0 to t, we have

$$\begin{aligned}\theta_{1k}J^{\beta}(1)(t)_kJ^{\alpha}(\varphi_2 - f)(t) + \theta_{2k}J^{\alpha}(1)(t)_kJ^{\beta}(f - \varphi_1)(t) \\ \geq {}_kJ^{\alpha}(\varphi_2 - f)^{\theta_1}(t)_kJ^{\beta}(f - \varphi_1)^{\theta_2}(t).\end{aligned}$$

Therefore, we deduce inequality (2.9). □

Corollary 2.5. *Let f be an integrable function on $[0,\infty)$ satisfying $m \leq f(t) \leq M$, for all $t \in [0,\infty)$ and $m, M \in \mathbb{R}$. Then for $t > 0$, $\alpha, \beta > 0$ and $k > 0$, we have*

$$\begin{aligned}M\frac{t^{\frac{\alpha+\beta}{k}}}{\Gamma_k(\alpha+k)\Gamma_k(\beta+k)} + \frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}{}_kJ^{\beta}f(t) \quad \geq \quad & m\frac{t^{\frac{\alpha+\beta}{k}}}{\Gamma_k(\alpha+k)\Gamma_k(\beta+k)} + \frac{t^{\frac{\beta}{k}}}{\Gamma_k(\beta+k)}{}_kJ^{\alpha}f(t) \\ & +2{}_kJ^{\alpha}(M-f)^{\frac{1}{2}}(t)_kJ^{\beta}(f-m)^{\frac{1}{2}}(t).\end{aligned}$$

Theorem 2.4. *Let f and g be two integrable functions on $[0,\infty)$. Suppose that (H_1) holds and moreover we assume that:*

(H_2) *There exist ψ_1 and ψ_2 integrable functions on $[0,\infty)$ such that*

$$\psi_1(t) \leq g(t) \leq \psi_2(t) \quad \text{for all} \quad t \in [0,\infty).$$

Then, for $t > 0$, α, $\beta > 0$, $k > 0$ the following inequalities hold:

(a) ${}_kJ^{\beta}\psi_1(t)\,{}_kJ^{\alpha}f(t) + {}_kJ^{\alpha}\varphi_2(t)\,{}_kJ^{\beta}g(t) \geq {}_kJ^{\beta}\psi_1(t)\,{}_kJ^{\alpha}\varphi_2(t) + {}_kJ^{\alpha}f(t)\,{}_kJ^{\beta}g(t)$.

(b) ${}_kJ^{\beta}\varphi_1(t)\,{}_kJ^{\alpha}g(t) + {}_kJ^{\alpha}\psi_2(t)\,{}_kJ^{\beta}f(t) \geq {}_kJ^{\beta}\varphi_1(t)\,{}_kJ^{\alpha}\psi_2(t) + {}_kJ^{\beta}f(t)\,{}_kJ^{\alpha}g(t)$.

(c) ${}_kJ^{\alpha}\varphi_2(t)\,{}_kJ^{\beta}\psi_2(t) + {}_kJ^{\alpha}f(t)\,{}_kJ^{\beta}g(t) \geq {}_kJ^{\alpha}\varphi_2(t)\,{}_kJ^{\beta}g(t) + {}_kJ^{\beta}\psi_2(t)\,{}_kJ^{\alpha}f(t)$.

(d) ${}_kJ^{\alpha}\varphi_1(t)J^{\beta}\psi_1(t) + {}_kJ^{\alpha}f(t)\,{}_kJ^{\beta}g(t) \geq {}_kJ^{\alpha}\varphi_1(t)\,{}_kJ^{\beta}g(t) + {}_kJ^{\beta}\psi_1(t)\,{}_kJ^{\alpha}f(t)$.

Proof. To prove (a), from (H_1) and (H_2), we have for $t \in [0,\infty)$ that

$$(\varphi_2(\tau) - f(\tau))\,(g(\rho) - \psi_1(\rho)) \geq 0.$$

Therefore

$$\varphi_2(\tau)g(\rho) + \psi_1(\rho)f(\tau) \geq \psi_1(\rho)\varphi_2(\tau) + f(\tau)g(\rho). \tag{2.11}$$

Multiplying both sides of (2.11) by $(t-\tau)^{\frac{\alpha}{k}-1}/k\Gamma_k(\alpha)$, $\tau \in (0,t)$, we get

$$\begin{aligned}g(\rho)\frac{(t-\tau)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}\varphi_2(\tau) + \psi_1(\rho)\frac{(t-\tau)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}f(\tau) \\ \geq \psi_1(\rho)\frac{(t-\tau)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}\varphi_2(\tau) + g(\rho)\frac{(t-\tau)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}f(\tau).\end{aligned} \tag{2.12}$$

Integrating both sides of (2.12) with respect to τ on $(0,t)$, we obtain

$$g(\rho)\int_0^t \frac{(t-\tau)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}\varphi_2(\tau)d\tau + \psi_1(\rho)\int_0^t \frac{(t-\tau)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}f(\tau)d\tau$$
$$\geq \psi_1(\rho)\int_0^t \frac{(t-\tau)^{\alpha-1}}{k\Gamma(\alpha)}\varphi_2(\tau)d\tau + g(\rho)\int_0^t \frac{(t-\tau)^{\alpha-1}}{k\Gamma(\alpha)}f(\tau)d\tau.$$

Then we have

$$g(\rho)_k J^\alpha \varphi_2(t) + \psi_1(\rho)_k J^\alpha f(t) \geq \psi_1(\rho)_k J^\alpha \varphi_2(t) + g(\rho)_k J^\alpha f(t). \tag{2.13}$$

Multiplying both sides of (2.13) by $(t-\rho)^{\frac{\beta}{k}-1}/k\Gamma_k(\beta)$, $\rho \in (0,t)$, we have

$$\begin{aligned} &{}_kJ^\alpha \varphi_2(t)\frac{(t-\rho)^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)}g(\rho) + {}_kJ^\alpha f(t)\frac{(t-\rho)^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)}\psi_1(\rho) \\ &\geq {}_kJ^\alpha \varphi_2(t)\frac{(t-\rho)^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)}\psi_1(\rho) + {}_kJ^\alpha f(t)\frac{(t-\rho)^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)}g(\rho). \end{aligned} \tag{2.14}$$

Integrating both sides of (2.14) with respect to ρ on $(0,t)$, we get the desired inequality (a).

To prove (b)-(d), we use the following inequalities

(b) $(\psi_2(\tau) - g(\tau))(f(\rho) - \varphi_1(\rho)) \geq 0$.

(c) $(\varphi_2(\tau) - f(\tau))(g(\rho) - \psi_2(\rho)) \leq 0$.

(d) $(\varphi_1(\tau) - f(\tau))(g(\rho) - \psi_1(\rho)) \leq 0$. □

Remark 2.2. *If we take $k = 1$ in the Theorem 2.4, we obtain the Theorem 5 in [22].*

As a special case of Theorem 2.4, we have the following Corollary.

Corollary 2.6. *Let f and g be two integrable functions on $[0,\infty)$. Assume that:*

(H_3) *There exist real constants m, M, n, N such that*

$$m \leq f(t) \leq M \quad \text{and} \quad n \leq g(t) \leq N \quad \text{for all} \quad t \in [0,\infty).$$

Then, for $t > 0$, α, $\beta > 0$, $k > 0$ we have

(a_1) $$\frac{nt^{\frac{\beta}{k}}}{\Gamma_k(\beta+k)}\,{}_kJ^\alpha f(t) + \frac{Mt^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\,{}_kJ^\beta g(t) \geq \frac{nMt^{\frac{\alpha+\beta}{K}}}{\Gamma_k(\alpha+k)\Gamma_k(\beta+k)} + {}_kJ^\alpha f(t)_k J^\beta g(t).$$

(b_1) $$\frac{mt^{\frac{\beta}{k}}}{\Gamma_k(\beta+k)}\,{}_kJ^\alpha g(t) + \frac{Nt^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\,{}_kJ^\beta f(t) \geq \frac{mNt^{\frac{\alpha+\beta}{k}}}{\Gamma_K(\alpha+k)\Gamma_k(\beta+1)} + {}_kJ^\beta f(t)_k J^\alpha g(t).$$

(c_1) $$\frac{MNt^{\frac{\alpha+\beta}{k}}}{\Gamma_k(\alpha+k)\Gamma_k(\beta+k)} + {}_kJ^\alpha f(t)_k J^\beta g(t) \geq \frac{Mt^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\,{}_kJ^\beta g(t) + \frac{Nt^{\frac{\beta}{k}}}{\Gamma_k(\beta+k)}\,{}_kJ^\alpha f(t).$$

(d_1) $$\frac{mnt^{\frac{\alpha+\beta}{k}}}{\Gamma_k(\alpha+k)\Gamma_k(\beta+k)} + {}_kJ^\alpha f(t)_k J^\beta g(t) \geq \frac{mt^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\,{}_kJ^\beta g(t) + \frac{nt^{\frac{\beta}{k}}}{\Gamma_k(\beta+k)}\,{}_kJ^\alpha f(t).$$

Theorem 2.5. *Let f and g be two positive integrable functions on $[0,\infty)$. Assume that (H_1) and (H_2) hold with*

(H_4) *$\varphi_1(t) > 0$ and $\psi_1(t) > 0$ for all $t \in [0,\infty)$.*

Then, for $t > 0$, α, $\beta > 0$ and $k > 0$ the following inequalities holds

$$\frac{{}_kJ^\alpha \varphi_1\varphi_2(t)_k J^\beta \psi_1\psi_2(t)_k J^\alpha f^2(t)_k J^\beta g^2(t)}{\left({}_kJ^\alpha \varphi_1 f(t)_k J^\beta \psi_1 g(t) + {}_kJ^\alpha \varphi_2 f(t)_k J^\beta \psi_2 g(t)\right)^2} \leq \frac{1}{4}. \tag{2.15}$$

Proof. To prove (2.15), using the conditions (H_1)-(H_3), we obtain

$$\left(\frac{\varphi_2(\tau)}{\psi_1(\rho)} - \frac{f(\tau)}{g(\rho)}\right) \geq 0,$$

and

$$\left(\frac{f(\tau)}{g(\rho)} - \frac{\varphi_1(\tau)}{\psi_2(\rho)}\right) \geq 0,$$

which imply that

$$\left(\frac{\varphi_1(\tau)}{\psi_2(\rho)} + \frac{\varphi_2(\tau)}{\psi_1(\rho)}\right)\frac{f(\tau)}{g(\rho)} \geq \frac{f^2(\tau)}{g^2(\rho)} + \frac{\varphi_1(\tau)\varphi_2(\tau)}{\psi_1(\rho)\psi_2(\rho)}. \tag{2.16}$$

Multiplying both sides of (2.16) by $\psi_1(\rho)\psi_2(\rho)g^2(\rho)$, we have

$$\varphi_1(\tau)f(\tau)\psi_1(\rho)g(\rho) + \varphi_2(\tau)f(\tau)\psi_2(\rho)g(\rho) \geq \psi_1(\rho)\psi_2(\rho)f^2(\tau) + \varphi_1(\tau)\varphi_2(\tau)g^2(\rho). \tag{2.17}$$

Multiplying both sides of (2.17) by $\dfrac{(t-\tau)^{\frac{\alpha}{k}-1}(t-\rho)^{\frac{\beta}{k}-1}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}$, and double integrating with respect to τ and ρ from 0 to t, we have

$${}_kJ^\alpha\varphi_1 f(t)\,{}_kJ^\beta\psi_1 g(t) + {}_kJ^\alpha\varphi_2 f(t)\,{}_kJ^\beta\psi_2 g(t) \geq {}_kJ^\alpha f^2(t)\,{}_kJ^\beta\psi_1\psi_2(t) + {}_kJ^\alpha\varphi_1\varphi_2(t)\,{}_kJ^\beta g^2(t).$$

Applying the AM-GM inequality, we get

$${}_kJ^\alpha\varphi_1 f(t)\,{}_kJ^\beta\psi_1 g(t) + {}_kJ^\alpha\varphi_2 f(t)\,{}_kJ^\beta\psi_2 g(t) \geq 2\sqrt{{}_kJ^\alpha f^2(t)\,{}_kJ^\beta\psi_1\psi_2(t)\,{}_kJ^\alpha\varphi_1\varphi_2(t)\,{}_kJ^\beta g^2(t)},$$

which leads to the desired inequality in (2.15). The proof is completed. □

As a special case of Theorem 2.5, we get the following result:

Corollary 2.7. *Let f and g be two positive integrable functions on $[0,\infty)$ satisfying (H_3) with $m,n > 0$. Then for $t > 0$, $\alpha,\beta > 0$ and $k > 0$, we have*

$$\frac{t^{\frac{\alpha+\beta}{k}}}{\Gamma_k(\alpha+k)\Gamma_k(\beta+k)}\frac{{}_kJ^\alpha f^2(t)\,{}_kJ^\beta g^2(t)}{\left({}_kJ^\alpha f(t)\,{}_kJ^\beta g(t)\right)^2} \leq \frac{1}{4}\left(\sqrt{\frac{mn}{MN}} + \sqrt{\frac{MN}{mn}}\right)^2.$$

Lemma 2.1. *Let f be an integrable function on $[0,\infty)$ and φ_1, φ_2 are two integrable functions on $[0,\infty)$. Assume that the condition (H_1) holds. Then, for $t > 0$, $\alpha > 0$, $k > 0$, we have*

$$\begin{aligned}
\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\,{}_kJ^\alpha f^2(t) - \left({}_kJ^\alpha f(t)\right)^2 \;=\;& \left({}_kJ^\alpha\varphi_2(t) - {}_kJ^\alpha f(t)\right)\left({}_kJ^\alpha f(t) - {}_kJ^\alpha\varphi_1(t)\right) \\
& -\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\,{}_kJ^\alpha\left((\varphi_2(t) - f(t))(f(t) - \varphi_1(t))\right) \\
& +\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\,{}_kJ^\alpha\varphi_1 f(t) - {}_kJ^\alpha\varphi_1(t)\,{}_kJ^\alpha f(t) \\
& +\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\,{}_kJ^\alpha\varphi_2 f(t) - {}_kJ^\alpha\varphi_2(t)\,{}_kJ^\alpha f(t) \\
& + {}_kJ^\alpha\varphi_1(t)\,{}_kJ^\alpha\varphi_2(t) - \frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\,{}_kJ^\alpha\varphi_1\varphi_2(t).
\end{aligned} \tag{2.18}$$

Proof. For any $\tau > 0$ and $\rho > 0$, we have

$$\begin{aligned}
&\left(\varphi_2(\rho) - f(\rho)\right)\left(f(\tau) - \varphi_1(\tau)\right) + \left(\varphi_2(\tau) - f(\tau)\right)\left(f(\rho) - \varphi_1(\rho)\right) \\
&+\left(\varphi_2(\tau) - f(\tau)\right)\left(f(\tau) - \varphi_1(\tau)\right) - \left(\varphi_2(\rho) - f(\rho)\right)\left(f(\rho) - \varphi_1(\rho)\right) \\
&= f^2(\tau) + f^2(\rho) - 2f(\tau)f(\rho) + \varphi_2(\rho)f(\tau) + \varphi_1(\tau)f(\rho) - \varphi_1(\tau)\varphi_2(\rho) \\
&+\varphi_2(\tau)f(\rho) + \varphi_1(\rho)f(\tau) - \varphi_1(\rho)\varphi_2(\tau) - \varphi_2(\tau)f(\tau) + \varphi_1(\tau)\varphi_2(\tau) \\
&\varphi_1(\tau)f(\tau) - \varphi_2(\rho)f(\rho) + \varphi_1(\rho)\varphi_2(\rho) - \varphi_1(\rho)f(\rho).
\end{aligned} \tag{2.19}$$

Multiplying (2.19) by $(t-\tau)^{\frac{\alpha}{k}-1}/k\Gamma_k(\alpha)$, $\tau \in (0,t)$, $t>0$ and integrating the resulting identity with respect to τ from 0 to t, we get

$$\begin{aligned}
&(\varphi_2(\rho)-f(\rho))\left({}_kJ^{\alpha}f(t)-{}_kJ^{\alpha}\varphi_1(t)\right)+\left({}_kJ^{\alpha}\varphi_2(t)-{}_kJ^{\alpha}f(t)\right)(f(\rho)-\varphi_1(\rho))\\
&+{}_kJ^{\alpha}\left((\varphi_2(t)-f(t))(f(t)-\varphi_1(t))\right)-(\varphi_2(\rho)-f(\rho))(f(\rho)-\varphi_1(\rho))\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\\
&= {}_kJ^{\alpha}f^2(t)+f^2(\rho)\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}-2f(\rho){}_kJ^{\alpha}f(t)+\varphi_2(\rho){}_kJ^{\alpha}f(t)+f(\rho){}_kJ^{\alpha}\varphi_1(t)\\
&+\varphi_2(\rho){}_kJ^{\alpha}\varphi_1(t)+f(\rho){}_kJ^{\alpha}\varphi_2(t)+\varphi_1(\rho){}_kJ^{\alpha}f(t)-\varphi_1(\rho){}_kJ^{\alpha}\varphi_2(t)-{}_kJ^{\alpha}\varphi_2f(t)\\
&+{}_kJ^{\alpha}\varphi_1\varphi_2(t)-{}_kJ^{\alpha}\varphi_1f(t)-\varphi_2(\rho)f(\rho)\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}+\varphi_1(\rho)\varphi_2(\rho)\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\\
&+\varphi_1(\rho)f(\rho)\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}.
\end{aligned} \tag{2.20}$$

Multiplying (2.20) by $(t-\rho)^{\frac{\alpha}{k}-1}/k\Gamma_k(\alpha)$, $\rho \in (0,t)$, $t>0$ and integrating the resulting identity with respect to ρ from 0 to t, we have

$$\begin{aligned}
&\left({}_kJ^{\alpha}\varphi_2(t)-{}_kJ^{\alpha}f(t)\right)\left({}_kJ^{\alpha}f(t)-{}_kJ^{\alpha}\varphi_1(t)\right)\\
&+\left({}_kJ^{\alpha}\varphi_2(t)-{}_kJ^{\alpha}f(t)\right)\left({}_kJ^{\alpha}f(t)-{}_kJ^{\alpha}\varphi_1(t)\right)\\
&{}_kJ^{\alpha}\left((\varphi_2(t)-f(t))(f(t)-\varphi_1(t))\right)\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\\
&{}_kJ^{\alpha}\left((\varphi_2(t)-f(t))(f(t)-\varphi_1(t))\right)\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\\
&=\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\,{}_kJ^{\alpha}f^2(t)+\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\,{}_kJ^{\alpha}f^2(t)-2{}_kJ^{\alpha}f(t){}_kJ^{\alpha}f(t)+{}_kJ^{\alpha}\varphi_2(t){}_kJ^{\alpha}f(t)\\
&+{}_kJ^{\alpha}\varphi_1(t){}_kJ^{\alpha}f(t)-{}_kJ^{\alpha}\varphi_1(t){}_kJ^{\alpha}\varphi_2(t)+{}_kJ^{\alpha}\varphi_2(t){}_kJ^{\alpha}f(t)+{}_kJ^{\alpha}\varphi_1(t){}_kJ^{\alpha}f(t)\\
&+{}_kJ^{\alpha}\varphi_1(t){}_kJ^{\alpha}\varphi_2(t)-\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\,{}_kJ^{\alpha}\varphi_2f(t)+\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\,{}_kJ^{\alpha}\varphi_1\varphi_2(t)\\
&+\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}{}_kJ^{\alpha}\varphi_1f(t)-\frac{t^{\alpha}}{\Gamma_k(\alpha+1)}\,{}_kJ^{\alpha}\varphi_2f(t)+\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\,{}_kJ^{\alpha}\varphi_1\varphi_2(t)\\
&+\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}{}_kJ^{\alpha}\varphi_1f(t),
\end{aligned} \tag{2.21}$$

which implies (2.18). □

Remark 2.3. *If we take $k=1$ in the Lemma 2.1, we obtain the Lemma 7 in [22].*

If $\varphi_1(t)\equiv m$ and $\varphi_2(t)\equiv M$, $m,M\in\mathbb{R}$, for all $t\in[0,\infty)$, then inequality (2.18) reduces to the following corollary.

Corollary 2.8. *Let f be an integrable function on $[0,\infty)$ satisfying $m\le f(t)\le M$, for all $t\in[0,\infty)$. The for all $t>0$, $\alpha>0$ we have*

$$\begin{aligned}
\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\,{}_kJ^{\alpha}f^2(t)-\left({}_kJ^{\alpha}f(t)\right)^2 &= \left(M\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}-{}_kJ^{\alpha}f(t)\right)\left({}_kJ^{\alpha}f(t)-m\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\right)\\
&\quad-\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\,{}_kJ^{\alpha}\left((M-f(t))(f(t)-m)\right).
\end{aligned} \tag{2.22}$$

Remark 2.4. *If we take $k=1$ in the Corollary 2.8, we obtain the Corollary 8 in [22].*

Theorem 2.6. *Let f and g be two integrable functions on $[0,\infty)$ and φ_1, φ_2, ψ_1 and ψ_2 are four integrable functions on $[0,\infty)$ satisfying the conditions (H_1) and (H_2) on $[0,\infty)$. Then for all $t>0$, $\alpha>0$,$k>0$, we have*

$$\left|\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\,{}_kJ^{\alpha}fg(t)-{}_kJ^{\alpha}f(t)\,{}_kJ^{\alpha}g(t)\right|\le\sqrt{T(f,\varphi_1,\varphi_2)T(g,\psi_1,\psi_2)}, \tag{2.23}$$

where $T(u,v,w)$ is defined by

$$\begin{aligned} T(u,v,w) &= \left({}_kJ^{\alpha}w(t) - {}_kJ^{\alpha}u(t)\right)\left({}_kJ^{\alpha}u(t) - {}_kJ^{\alpha}v(t)\right) + \tfrac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\, {}_kJ^{\alpha}vu(t) - {}_kJ^{\alpha}v(t)\, {}_kJ^{\alpha}u(t) \\ &+ \frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} J^{\alpha}wu(t) - {}_kJ^{\alpha}w(t)\, {}_kJ^{\alpha}u(t) + {}_kJ^{\alpha}v(t)\, {}_kJ^{\alpha}w(t) - \frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\, {}_kJ^{\alpha}vw(t). \end{aligned} \tag{2.24}$$

Proof. Let f and g be two integrable functions defined on $[0,\infty)$ satisfying (H_1) and (H_2). Define

$$H(\tau,\rho) := \left(f(\tau) - f(\rho)\right)\left(g(\tau) - g(\rho)\right), \quad \tau,\rho \in (0,t),\ t > 0.$$

Multiplying both sides of (2.24) by $(t-\tau)^{\frac{\alpha}{k}-1}(t-\rho)^{\frac{\alpha}{k}-1}/k^2\Gamma_k^2(\alpha)$, $\tau,\rho \in (0,t)$ and integrating the resulting identity with respect to τ and ρ from 0 to t, we can state that

$$\frac{1}{k^2\Gamma_k^2(\alpha)}\int_0^t\int_0^t (t-\tau)^{\frac{\alpha}{k}-1}(t-\rho)^{\frac{\alpha}{k}-1}H(\tau,\rho)d\tau d\rho = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha+k)}\, {}_kJ^{\alpha}fg(t) - {}_kJ^{\alpha}f(t)\, {}_kJ^{\alpha}g(t). \tag{2.25}$$

Applying the Cauchy-Schwarz inequality to (2.25), we have

$$\begin{aligned} &\left(\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\, {}_kJ^{\alpha}fg(t) - {}_kJ^{\alpha}f(t)\, {}_kJ^{\alpha}g(t)\right)^2 \\ &\le \left(\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\, {}_kJ^{\alpha}f^2(t) - \left({}_kJ^{\alpha}f(t)\right)^2\right)\left(\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\, {}_kJ^{\alpha}g^2(t) - \left({}_kJ^{\alpha}g(t)\right)^2\right). \end{aligned} \tag{2.26}$$

Since $(\varphi_2(t) - f(t))(f(t) - \varphi_1(t)) \ge 0$ and $(\psi_2(t) - g(t))(g(t) - \psi_1(t)) \ge 0$ for $t \in [0,\infty)$, we have

$$\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\, {}_kJ^{\alpha}\left((\varphi_2(t) - f(t))(f(t) - \varphi_1(t))\right) \ge 0$$

and

$$\frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\, {}_kJ^{\alpha}\left((\psi_2(t) - g(t))(g(t) - \psi_1(t))\right) \ge 0.$$

Thus, from Lemma 2.1, we get

$$\begin{aligned} \frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\, {}_kJ^{\alpha}f^2(t) - \left({}_kJ^{\alpha}f(t)\right)^2 &\le \left({}_kJ^{\alpha}\varphi_2(t) - {}_kJ^{\alpha}f(t)\right)\left({}_kJ^{\alpha}f(t) - {}_kJ^{\alpha}\varphi_1(t)\right) \\ &+ \frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\, {}_kJ^{\alpha}\varphi_1 f(t) - {}_kJ^{\alpha}\varphi_1(t)\, {}_kJ^{\alpha}f(t) \\ &+ \frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\, {}_kJ^{\alpha}\varphi_2 f(t) - {}_kJ^{\alpha}\varphi_2(t)\, {}_kJ^{\alpha}f(t) \\ &+ {}_kJ^{\alpha}\varphi_1(t) J^{\alpha}\varphi_2(t) - \frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\, {}_kJ^{\alpha}\varphi_1\varphi_2(t) \\ &= T(f,\varphi_1,\varphi_2), \end{aligned} \tag{2.27}$$

and

$$\begin{aligned} \frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\, {}_kJ^{\alpha}g^2(t) - \left({}_kJ^{\alpha}g(t)\right)^2 &\le \left({}_kJ^{\alpha}\psi_2(t) - {}_kJ^{\alpha}g(t)\right)\left({}_kJ^{\alpha}g(t) - {}_kJ^{\alpha}\psi_1(t)\right) \\ &+ \frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\, {}_kJ^{\alpha}\psi_1 g(t) - {}_kJ^{\alpha}\psi_1(t)\, {}_kJ^{\alpha}g(t) \\ &+ \frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\, {}_kJ^{\alpha}\psi_2 g(t) - {}_kJ^{\alpha}\psi_2(t)\, {}_kJ^{\alpha}g(t) \\ &+ {}_kJ^{\alpha}\psi_1(t)\, {}_kJ^{\alpha}\psi_2(t) - \frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\, {}_kJ^{\alpha}\psi_1\psi_2(t) \\ &= T(g,\psi_1,\psi_2). \end{aligned} \tag{2.28}$$

From (2.26), (2.27) and (2.28), we obtain (2.23). □

Remark 2.5. *If we take $k = 1$ in the Theorem 2.6, we obtain the Theorem 9 in [22].*

Remark 2.6. *If $T(f, \varphi_1, \varphi_2) = T(f, m, M)$ and $T(g, \psi_1, \psi_2) = T(g, p, P)$, $m, M, p, P \in \mathbb{R}$ then inequality* (2.23) *reduces to*

$$\left| \frac{t^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} \, {}_kJ^{\alpha} fg(t) - {}_kJ^{\alpha} f(t) \, {}_kJ^{\alpha} g(t) \right| \leq \left(\frac{t^{\frac{\alpha}{k}}}{2\Gamma_k(\alpha+k)} \right)^2 (M-m)(P-p). \tag{2.29}$$

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