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Analytical Solution of Non-Integer Extra-Ordinary Differential Equation Via Adomian Decomposition Method

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Abstract

In the present paper, we obtain the analytical solution of the linear extraordinary fractional equations with constant coefficients by Adomian decomposition method under nonhomgeneous initial value condition, this method is a powerful method which consider the approximate solution as an infinite series usually converges to the exact solution.

Keywords: Extraordinary Fractional differential equation, Adomian decomposition method.

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1 Introduction

Fractional differential equations is a generalization of ordinary differential equations and integration to arbitrary non integer orders. The origin of fractional calculus goes back to Newton and Leibniz in the seventieth century. It is widely and efficiently used to describe many phenomena arising in engineering, physics, economy, and science. Recent investigations have shown that many physical systems can be represented more accurately through fractional derivative formulation [39]. Fractional differential equations, therefore find numerous applications in the field of visco-elasticity, feed back amplifiers, electrical circuits, electro analytical chemistry, fractional multipoles, neuron modelling encompassing different branches of physics, chemistry and biological sciences [43]. There have been many excellent books and monographs available on this field [20, 34, 41, 43, 46, 50]. In [34], the authors gave the most recent and up-to-date developments on fractional differential and fractional integro-differential equations with applications involving many different potentially useful operators of fractional calculus. In a recent work by Jaimini et.al. [33] the authors have given the corresponding Leibnitz rule for fractional calculus. For the history of fractional calculus, interested reader may see the recent review paper by Machado et. al. [38].

Many physical processes appear to exhibit fractional order behavior that may vary with time or space. The fractional calculus has allowed the operations of integration and differentiation to any fractional order. The order may take on any real or imaginary value. Recently theory of fractional differential equations attracted many scientists and mathematicians to work on [16, 28, 29, 43–45, 51]. For the existence of solutions for fractional differential equations, one can see [9, 15, 17–19, 21–25, 30–32, 35, 36, 53] and references therein. The results have been obtained by using fixed point theorems like Picard's, Schauder fixed-point theorem and Banach contraction mapping principle. About the development of existence theorems for fractional functional differential equations, many contribution exists [7, 8, 13, 16, 23, 37, 54]. Many applications of fractional calculus amount to replacing the time derivative in a given evolution equation by a derivative of fractional order. The results of several studies clearly stated that the fractional derivatives seem to arise generally and

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universally from important mathematical reasons. Recently, interesting attempts have been made to give the physical meaning to the initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives proposed in [27, 29, 44, 45].

Ahmed et. al. [10] considered the fractional order predator-prey model and the fractional order rabies model. They have shown the existence and uniqueness of solutions of the model system and also studied the stability of equilibrium points. The motivation behind fractional order system are discussed in [10]. Lakshmikantham and Vatsala in [35, 36] and Lakshmikantham in [37] defined and proved the existence of the solution of fractional initial value problems.

Large classes of linear and nonlinear differential equations, both ordinary as well as partial, can be solved by the Adomian Decomposition Method[3, 4, 6]. This method is much more simpler in computation and quicker in convergence than any other method available in the open literature.

The application of the fractional differential equation in physical problems is available in the book of Bracewell [12]. Recently, the solution of the fractional differential equation has been obtained through the Adomian Decomposition Method by the researchers in [11, 26].

The purpose of this paper is to develop further the applicability of the Decomposition Method to solve extraordinary differential equations of fractional order.

For the sake of convenience, we first of all give definitions of fractional integral and fractional derivative introduced by Riemann-Liouville

Definition 1.1. (*Fractional integral*) ([*11*])

Let $q > 0$ denote a real number. Assuming $f(x)$ to be a function of class $C^{(n)}$ (the class of function with continuous nth derivative), the fractional integral of a function f of order $-q$ is given by

$$\frac{d^{-q}f(x)}{dx^{-q}} = \frac{1}{\Gamma(q)}\int_0^x \frac{f(t)dt}{(x-t)^{1-q}}, \tag{1.1}$$

Definition 1.2. (*Fractional derivative*) [*11*]

Let $q > 0$ denote a real number and n the smallest integer exceeding q such that $n-q>0$ $(n=0 \text{ if } q<0)$. Assuming $f(x)$ to be a function of class $C^{(n)}$ (the class of function with continuous nth derivative), the fractional derivative of a function f of order q is given by

$$\frac{d^q f(x)}{dx^q} = \frac{d^n}{dx^n}\left(\frac{d^{-(n-q)}f(x)}{dx^{-(n-q)}}\right) = \frac{1}{\Gamma(n-q)}\frac{d^n}{dx^n}\int_0^x \frac{f(t)dt}{(x-t)^{1-n+q}} \tag{1.2}$$

Definition 1.3. (*Mittag-Leffler function*) [*2, 43*]

A two-parameter function of the Mittag-Leffler type is defined by the series expansion:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)} \qquad (\alpha>0, \beta>0), \tag{1.3}$$

In particular

$$\begin{aligned} E_{\frac{1}{2},1}(z) &= \sum_{k=0}^{\infty}\frac{z^k}{\Gamma\left(\frac{k}{2}+1\right)} = \exp\left(z^2\right)(1+erf(z)) \\ &= \exp\left(z^2\right)erf(-z) \end{aligned}$$

$$\begin{aligned} E_{\frac{3}{2},2}(z) &= \sum_{k=0}^{\infty}\frac{z^k}{\Gamma\left(\frac{3k}{2}+2\right)} = \exp\left(z^2\right)(1+erf(z)) \\ &= \exp\left(z^2\right)erf(-z). \end{aligned}$$

2 The Decomposition Method

We consider an equation in the form

$$Lu + Ru + Nu = g, \tag{2.4}$$

where L is an easily or trivially invertible linear operator, R is the remaining linear part, and N represents a nonlinear operator.

The general solution of the given equation is decomposed into the sum

$$u = \sum_{n=0}^{\infty} u_n, \tag{2.5}$$

where u_0 is the solution of the linear part.

Our approach will be to write any nonlinear term in terms of the Adomian A_n polynomials. Its has been derived by Adomian that $Nu = \sum_{n=0}^{\infty} A_n$,where the A_n are special polynomials obtained for the particular nonlinearity $Nu = f(u)$ and generated by Adomian [3–5]. These A_n polynomials depends, of course, on the particular nonlinearity.

The A_n are given as

$$\begin{aligned}
A_0 &= f(u_0), \\
A_1 &= u_1 \left(\frac{d}{du_0}\right) f(u_0), \\
A_2 &= u_2 \left(\frac{d}{du_0}\right) f(u_0) + \left(\frac{u_1^2}{2!}\right)\left(\frac{d^2}{du_0^2}\right) f(u_0), \\
A_3 &= u_3 \left(\frac{d}{du_0}\right) f(u_0) + u_1 u_2 \left(\frac{d^2}{du_0^2}\right) f(u_0) \\
&\quad + \left(\frac{u_1^3}{3!}\right)\left(\frac{d^3}{du_0^3}\right) f(u_0), \\
&\quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned} \tag{2.6}$$

and can be found the formula (for $n \geq 1$)

$$A_n = \sum_{\nu=1}^{n} c(\nu, n) f^{(\nu)}(u_0), \tag{2.7}$$

where the $c(\nu, n)$ are products (or sums of products) of ν components of u whose subscripts sum to n, divided by the factorial of the number of repeated subscripts [5].

Therefore, the general solution becomes

$$\begin{aligned}
u &= u_0 - L^{-1} R \sum_{n=0}^{\infty} u_n - L^{-1} N u \\
&= u_0 - L^{-1} R \sum_{n=0}^{\infty} u_n - L^{-1} \sum_{n=0}^{\infty} A_n,
\end{aligned} \tag{2.8}$$

where $u_0 = \phi + L^{-1} g$ and $L\phi = 0$.

To identify the terms in ${}_{n=1}^{\infty} u_n$, it has been derived by Adomian that

$$u_{n+1} = -L^{-1} R u_n - L^{-1} A_n, \qquad n \geq 0. \tag{2.9}$$

From (2.9) , we can write $u_1 = -L^{-1} R u_0 - L^{-1} A_0$. Thus u_1 can be calculated in terms of the known u_0. Now,

$$\begin{aligned}
u_2 &= -L^{-1} R u_1 - L^{-1} A_1, \\
u_3 &= -L^{-1} R u_2 - L^{-1} A_2,
\end{aligned} \tag{2.10}$$

and so on.

Hence all the terms of u are now calculated and the general solution is obtained as

$$u = \sum_{n=0}^{\infty} u_n. \tag{2.11}$$

Recently, the Adomian Decomposition Method was reviewed and a mathematical model of Adomian polynomials was introduced in [1].

3 Solution of An Extraordinary Fractional Differential Equation

A relationship involving one or more derivatives of an unknown function f with respect to its independent variable x is known as an ordinary differential equation. A similair relationships involving at least one differintegral of noninteger order may be termed as an extraordinary differential equation. Such an equation is solved when an explicit expression for f is exhibited. As with ordinary differential equations, the solutions of extraordinary differential equations often involve integrals and contain arbitrary constants as discussed in [42]. These types of equations are also known as fractional differential equations. The application of extraordinary differential equation is now available in many physical and technical areas [43]. It can be mentioned here that the simplified fractional order differential equation appearing in applied problems is of the form

$$\begin{aligned} D^m y(t) + \lambda D^\alpha y(t) &= ty(t), m = 1,2 \text{ and } 0 < \alpha < 1 \\ y(0) &= k_0, y^{'}(0) = k_1. \end{aligned} \tag{3.12}$$

where k_0 and k_1 are constant

Appling $D^{-m}$to both sides of (3.12), we obtain

$$y(t) + = D^{-m} ty(t) - \lambda D^{\alpha - m} y(t) \tag{3.13}$$

According to the above procedure of solving the fractional differential equations and using Adomian decomposition method, Let

$$y_0\left(t\right) = \sum_{i=0}^{1} k_i t^i \tag{3.14}$$

$$\begin{aligned} y_1(t) &= D^{-m} t y_0(t) - \lambda D^{\alpha-m} y_0(t) \\ &= -\left(\lambda - D^{-\alpha} t\right) D^{\alpha-m} y_0(t) \end{aligned} \tag{3.15}$$

$$\begin{aligned} y_2(t) &= D^{-m} t y_1(t) - \lambda D^{\alpha-m} y_1(t) \\ &= (-1)^2 \left(\lambda - D^{-\alpha} t\right)^2 D^{2(\alpha-m)} y_0(t) \end{aligned} \tag{3.16}$$

$$\begin{aligned} y_3(t) &= D^{-m} t y_2(t) - \lambda D^{\alpha-m} y_2(t) \\ &= (-1)^3 \left(\lambda - D^{-\alpha} t\right)^3 D^{3(\alpha-m)} y_0(t) \end{aligned} \tag{3.17}$$

$$\begin{aligned} y_n(t) &= D^{-m} t y_{n-1}(t) - \lambda D^{\alpha-m} y_{n-1}(t) \\ &= (-1)^n \left(\lambda - D^{-\alpha} t\right)^n D^{n(\alpha-m)} y_0(t) \end{aligned} \tag{3.18}$$

Adding all terms,we obtain the solution of the equation by Adomian decomposition method as follows

$$y\left(t\right) = \sum_{n=0}^{\infty} y_n(t) \tag{3.19}$$

$$\begin{aligned} y\left(t\right) &= \sum_{n=0}^{\infty} (-1)^n \left(\lambda - D^{-\alpha} t\right)^n D^{n(\alpha-m)} y_0(t) \\ &= \sum_{i=0}^{1} k_i \sum_{n=0}^{\infty} (-1)^n \left(\lambda - D^{-\alpha} t\right)^n D^{n(\alpha-m)} t^i \end{aligned} \tag{3.20}$$

4 Applications and Results

Our objective in this section is to apply the Adomian Decomposition Method to solve some fractional differential equations in physics. This technique can be carried out to solve the problem defined by the problem (3.12). The method is easily applicable via Matlab 7.9.3 program. So we can give the following example.

Example 4.1. *Consider the following fractional differential equation*

$$\begin{aligned} Dy(t) + D^{\frac{1}{2}}y(t) &= ty(t), \\ y(0) &= 1, \end{aligned} \tag{4.21}$$

In the light of the Adomian Decomposition Method, we assume $y(t) = y_0(t) + y_1(t) + y_2(t) + ...$ to be the solution of (4.21) where

$$y_0(t) = 1$$

$$\begin{aligned} y_1(t) &= D^{-1}ty_0(t) - D^{-\frac{1}{2}}y_0(t) \\ &= -\left(1 - D^{-\frac{1}{2}}t\right)D^{-\frac{1}{2}}y_0(t) \\ &= -\left(1 - \frac{8}{3\sqrt{\pi}}t^{\frac{3}{2}}\right)\frac{\Gamma(1)}{\Gamma\left(\frac{3}{2}\right)}t^{\frac{1}{2}} \\ &= \frac{16}{3\pi}t^2 - \frac{2}{\sqrt{\pi}}t^{\frac{1}{2}}. \end{aligned}$$

$$\begin{aligned} y_2(t) &= D^{-1}ty_1(t) - D^{-\frac{1}{2}}y_1(t) \\ &= (-1)^2\left(1 - D^{-\frac{1}{2}}t\right)^2 D^{2\left(-\frac{1}{2}\right)}y_0(t) \\ &= (-1)^2\left(1 - \frac{8}{3\sqrt{\pi}}t^{\frac{3}{2}}\right)^2\frac{\Gamma(1)}{\Gamma(2)}t \\ &= \frac{32}{9\pi}t^4 - \frac{8}{3\sqrt{\pi}}t^{\frac{5}{2}} + \frac{1}{2}t. \end{aligned}$$

$$\begin{aligned} y_3(t) &= D^{-1}ty_2(t) - D^{-\frac{1}{2}}y_2(t) \\ &= (-1)^3\left(1 - D^{-\frac{1}{2}}t\right)^3 D^{3\left(-\frac{1}{2}\right)}y_0(t) \\ &= (-1)^3\left(1 - \frac{8}{3\sqrt{\pi}}t^{\frac{3}{2}}\right)^3\frac{\Gamma(1)}{\Gamma\left(\frac{5}{2}\right)}t^{\frac{3}{2}} \end{aligned}$$

$$\begin{aligned} y_n(t) &= D^{-1}ty_{n-1}(t) - D^{-\frac{1}{2}}y_{n-1}(t) \\ &= (-1)^n\left(1 - \frac{8}{3\sqrt{\pi}}t^{\frac{3}{2}}\right)^n D^{n\left(-\frac{1}{2}\right)}y_0(t) \\ &= (-1)^n\left(1 - \frac{8}{3\sqrt{\pi}}t^{\frac{3}{2}}\right)^n\frac{\Gamma(1)}{\Gamma\left(\frac{n}{2}+1\right)}t^{\frac{n}{2}} \\ &= E_{\frac{1}{2},1}\left(\frac{8}{3\sqrt{\pi}}t^2 - \sqrt{t}\right) \\ &= \exp\left(\frac{8}{3\sqrt{\pi}}t^2 - \sqrt{t}\right)^2 erf\left(\sqrt{t} - \frac{8}{3\sqrt{\pi}}t^2\right). \end{aligned}$$

Therefore, the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} y_n(t) = \sum_{n=0}^{\infty} (-1)^n \left(1-\frac{8}{3\sqrt{\pi}}t^{\frac{3}{2}}\right)^n \frac{1}{\Gamma\left(\frac{n}{2}+1\right)} t^{\frac{n}{2}} \\ &= \sum_{n=0}^{\infty} \frac{\left(-\left(1-\frac{8}{3\sqrt{\pi}}t^{\frac{3}{2}}\right)t^{\frac{1}{2}}\right)^n}{\Gamma\left(\frac{n}{2}+1\right)} \\ &= \sum_{n=0}^{\infty} \frac{\left(\frac{8}{3\sqrt{\pi}}t^2-\sqrt{t}\right)^n}{\Gamma\left(\frac{n}{2}+1\right)} \\ &= E_{\frac{1}{2},1}\left(\frac{8}{3\sqrt{\pi}}t^2-\sqrt{t}\right) \\ &= \exp\left(\frac{8}{3\sqrt{\pi}}t^2-\sqrt{t}\right)^2 erf\left(\sqrt{t}-\frac{8}{3\sqrt{\pi}}t^2\right). \end{aligned}$$

Example 4.2. *Consider the fractional differential equation*

$$\begin{aligned} D^2y(t)+D^{\frac{1}{2}}y(t) &= ty(t), \\ y(0) &= 0, y^{'}(0)=1. \end{aligned} \tag{4.22}$$

In the light of the Adomian decomposition method, we assume $y(t)=y_0(t)+y_1(t)+y_2(t)+...$ to be the solution of (4.22) where

$$y_0(t)=t$$

$$\begin{aligned} y_1(t) &= D^{-2}ty_0(t)-D^{-\frac{3}{2}}y_0(t) \\ &= -\left(1-D^{-\frac{1}{2}}t\right)D^{-\frac{3}{2}}y_0(t) \\ &= -\left(1-\frac{8}{3\sqrt{\pi}}t^{\frac{3}{2}}\right)\frac{\Gamma(2)}{\Gamma\left(\frac{7}{2}\right)}t^{\frac{5}{2}} \end{aligned}$$

$$\begin{aligned} y_2(t) &= D^{-2}ty_1(t)-D^{-\frac{3}{2}}y_1(t) \\ &= (-1)^2\left(1-D^{-\frac{1}{2}}t\right)^2 D^{2\left(-\frac{3}{2}\right)}y_0(t) \\ &= (-1)^2\left(1-\frac{8}{3\sqrt{\pi}}t^{\frac{3}{2}}\right)^2\frac{\Gamma(2)}{\Gamma(5)}t^4 \end{aligned}$$

$$\begin{aligned} y_3(t) &= D^{-2}ty_2(t)-D^{-\frac{3}{2}}y_2(t) \\ &= (-1)^3\left(1-D^{-\frac{1}{2}}t\right)^3 D^{3\left(-\frac{3}{2}\right)}y_0(t) \\ &= (-1)^3\left(1-\frac{8}{3\sqrt{\pi}}t^{\frac{3}{2}}\right)^3\frac{\Gamma(2)}{\Gamma\left(\frac{5}{2}\right)}t^{\frac{3}{2}} \end{aligned}$$

$$\begin{aligned} y_n(t) &= D^{-2}ty_{n-1}(t)-D^{-\frac{3}{2}}y_{n-1}(t) \\ &= (-1)^n\left(1-\frac{8}{3\sqrt{\pi}}t^{\frac{3}{2}}\right)^n\frac{\Gamma(2)}{\Gamma\left(\frac{3n}{2}+2\right)}t^{\frac{3n}{2}+1} \\ &= t.E_{\frac{3}{2},2}\left(\frac{8}{3\sqrt{\pi}}t^3-t^{\frac{3}{2}}\right) \\ &= t.\exp\left(\frac{8}{3\sqrt{\pi}}t^3-t^{\frac{3}{2}}\right)^2 erf\left(t^{\frac{3}{2}}-\frac{8}{3\sqrt{\pi}}t^3\right). \end{aligned}$$

Therefore, the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} y_n(t) \\ &= \sum_{n=0}^{\infty} (-1)^n \left(1 - \frac{8}{3\sqrt{\pi}} t^{\frac{3}{2}}\right)^n \frac{\Gamma(2)}{\Gamma\left(\frac{3n}{2}+1\right)} t^{\frac{3n}{2}+1} \\ &= t.\sum_{n=0}^{\infty} \frac{\left(-\left(1 - \frac{8}{3\sqrt{\pi}} t^{\frac{3}{2}}\right) t^{\frac{3}{2}}\right)^n}{\Gamma\left(\frac{3n}{2}+2\right)} \\ &= t.\sum_{n=0}^{\infty} \frac{\left(\frac{8}{3\sqrt{\pi}} t^3 - t^{\frac{3}{2}}\right)^n}{\Gamma\left(\frac{3n}{2}+2\right)} \\ &= t.E_{\frac{3}{2},2}\left(\frac{8}{3\sqrt{\pi}} t^3 - t^{\frac{3}{2}}\right) \\ &= t.\exp\left(\frac{8}{3\sqrt{\pi}} t^3 - t^{\frac{3}{2}}\right)^2 erf\left(t^{\frac{3}{2}} - \frac{8}{3\sqrt{\pi}} t^3\right). \end{aligned}$$

Conclusion

In this paper, Adomian's Method is effectively implemented to determine the (approximate) analytic solution of the fractional-order extraordinary differential equation. Such a solution is expressed in the form of a series with easily computable components. An illustrative example is presented.

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