Malaya J. Mat. 4(1)(2016) 155-168

Malaya Journal of Matematik

MJM an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

Sobolev type fractional stochastic integro-differential evolution

Kerboua Mourad[a,*]

[a]Department of Mathematics, Guelma University, 24000 Guelma, Algeria.

Abstract

In this paper, we prove the existence of $\alpha-$mild solutions for a class of fractional stochastic integro-differential evolution equations of sobolev type with fractional sobolev stochastic nonlocal conditions in a real separable Hilbert space. To establish our main results, we use the Banach contraction mapping principle, fractional calculus, stochastic analysis and an analytic semigroup of linear operators. An example is given to illustrate the feasibility of our abstract result..

Keywords: Fractional stochastic evolution equations, Fixed point technique, fractional stochastic nonlocal condition.

2010 MSC: 26A33, 46E39, 34K50.

1 Introduction

Let (Ω, Γ, P) be a complete probability space equipped with a normal filtration $\Gamma_t, t \in J$ satisfying the usual conditions (i.e., right continuous and Γ_0 containing all P-null sets). We consider three real separable spaces X, Y and E, and Q-Wiener process on (Ω, Γ_b, P) with the linear bounded covariance operator Q such that $trQ < \infty$. We assume that there exist complete orthonormal systems $\{e_{1,n}\}_{n\geq 1}, \{e_{2,n}\}_{n\geq 1}$ in E, bounded sequences of non-negative real numbers $\{\lambda_{1,n}\}, \{\lambda_{2,n}\}$ such that $Qe_{1,n} = \lambda_{1,n}e_{1,n}, Qe_{2,n} = \lambda_{2,n}e_{2,n}, n = 1, 2, ...,$ and sequences $\{\beta_{1,n}\}_{n\geq 1}, \{\beta_{2,n}\}_{n\geq 1}$ of independent Brownian motions such that

$$\langle w_1(t), e_1\rangle = \sum_{n=1}^{\infty} \sqrt{\lambda_{1,n}}\, \langle e_{1,n}, e_1\rangle\, \beta_{1,n}(t), \quad e_1 \in E,\ t \in J,$$

$$\langle w_2(t), e_2\rangle = \sum_{n=1}^{\infty} \sqrt{\lambda_{2,n}}\, \langle e_{2,n}, e_2\rangle\, \beta_{2,n}(t), \quad e_2 \in E,\ t \in J,$$

and $\Gamma_t = \Gamma_t^{w_1,w_2}$, where $\Gamma_t^{w_1,w_2}$ is the sigma algebra generated by $\{(w_1(s), w_2(s)) : 0 \leq s \leq t\}$. Let $L_2^0 = L_2(Q^{1/2}E; X)$ be the space of all Hilbert–Schmidt operators from $Q^{1/2}$ E to X with the inner product $\langle \psi, \pi\rangle\, L_2^0 = tr[\psi Q \pi^*]$.

In this paper we consider the following Sobolev type fractional stochastic integro-differential evolution equations with fractional sobolev stochastic nonlocal conditions of the form

$$^{C}D_t^q \left[Lx\left(t\right)\right] = Ax\left(t\right) + \sigma_1\left(t, x\left(t\right), Hx\left(t\right)\right)\frac{dw_1\left(t\right)}{dt}, \quad t \in J, \tag{1.1}$$

$$^{L}D_t^{1-q}\left[Mx(0)\right] = \sigma_2(t, x(t))\frac{dw_2(t)}{dt}. \tag{1.2}$$

where $^{C}D_t^q$ and $^{L}D_t^{1-q}$ are the Caputo and Riemann–Liouville fractional derivatives with $0 < q \leq 1$, the state $x\left(\cdot\right)$ takes its values in the Hilbert space X.

*Corresponding author.

E-mail address: kerbouamourad@gmail.com (Kerboua Mourad).

The operators $A : D(A) \subset X \to X$, $L : D(L) \subset X \to X$ and $M : D(M) \subset X \to X$ are closed linear operators in X, σ_1 and σ_2 are given functions to be specified later, $J = [0, b]$, $b > 0$ is a constant. The term $Hx(t)$ is given by

$$Hx(t) = \int_0^t K(t,s)\, x(s)\, ds,$$

where $K \in C(\Delta, \mathbb{R}^+)$, $\Delta = \left\{(t,s) \in \mathbb{R}^2 : 0 \leq s \leq t \leq b\right\}$.

During the past three decades, fractional differential equations and their applications have gained a lot of importance, mainly because this field has become a powerful tool in modeling several complex phenomena in numerous seemingly diverse and widespread fields of science and engineering ([2], [3], [10], [11], [23], [24]) . Recently, there has been a significant development in the existence results for boundary value problems of nonlinear fractional differential equations ([1], [7]).

The problem with nonlocal condition, which is a generalization of the problem of classical condition, was motivated by physical problems. The pioneering work on nonlocal conditions is due to Byszewski see ([4], [5], [6]). Since it is demonstrated that the nonlocal problems have better effects in applications than the classical Cauchy problems, differential equations with nonlocal conditions have been studied by many authors and some basic results on nonlocal problems have been obtained, see ([12], [17], [22]) and the references therein for more comments and citations, A. Debbouche, D. Baleanu and R. P. Agarwal [13] proved the existence of mild and strong solutions for fractional nonlocal nonlinear integro-differential equations of Sobolev type using Schauder fixed point theorem, Gelfand-Shilov principles combined with semigroup theory. A. Debbouche and J.J. Nieto [11] studied the existence and uniqueness of mild solutions for a class of Sobolev type fractional nonlocal abstract evolution equations with nonlocal conditions and optimal multi-controls in Banach spaces by using fractional calculus, semigroup theory, a singular version of Gronwall inequality and Leray–Schauder fixed point theorem.

Stochastic differential and integro-differential equations have attracted great interest due to its applications in various fields of science and engineering. There are many interesting results on the existence, uniqueness and asymptotic stability of solutions to stochastic differential equations, see ([8], [14], [18], [19], [28], [29], [30], [35], [36]) and the references therein. In particular, fractional stochastic differential equations have also been studied by several authors, see ([9], [31], [32]).

More recently, El-Borai [16] studied the existence of mild solutions for a class of semilinear stochastic fractional differential equations by using Leray-Schauder fixed point theorem. Cui and Yan [9] studied the existence of mild solutions for a class of fractional neutral stochastic integro-differential equationswith infinite delay in Hilbert spaces by means of Sadovskii's fixed point theorem. Sakthivel et al. [31] investigated the existence and asymptotic stability in pth moment of mild solutions to a class of nonlinear fractional neutral stochastic differential equations with infinite delays in Hilbert spaces by using semigroup theory and fixed point technique. The existence of mild solutions for impulsive fractional stochastic differential equations with infinite delay has also been established in [32].

For our best knowledge, there is no work reported on the existence of α-mild solutions for Sobolev type fractional stochastic integro-differential evolution equations with fractional sobolev stochastic nonlocal conditions in fractional power space X_α.

Motivated by the above works, we introduce here a new nonlocal fractional stochastic condition of Sobolev type, and we prove the existence of α-mild solutions for the problem (1.1)-(1.2) by using a new strategy which relies on the compactness of the operator semigroup generated by $T = AL^{-1}$, Schauder fixed point theorem and approximating techniques. The rest of this paper is organized as follows. In Section 2 we present some essential facts in fractional calculus, semigroup theory, stochastic analysis that will be used to obtain our main results. In Section 3, we state and prove existence results on α-mild solutions for Sobolev type fractional stochastic system (1.1)-(1.2). Finally, in Section 4, as an example, a fractional stochastic parabolic partial differential equation with a sobolev type fractional stochastic nonlocal condition is considered.

2 Preliminaries

Throughout this paper, $(X, ||\cdot||)$ is a separable Hilbert space.

The operators $A : D(A) \subset X \to X$, $L : D(L) \subset X \to X$ and $M : D(M) \subset X \to X$ satisfy the following conditions:

$(A1)$ $L,\ A$ and M are closed linear operators.

$(A2)$ $D(M) \subset D(L) \subset D(A)$ and L and M are bijective.

$(A3)$ $L^{-1} : X \to D(L) \subset X$ and $M^{-1} : X \to D(M) \subset X$ are linear, bounded, and compact operators.

From $(A3)$, we deduce that L^{-1} is bounded operators. Note $(A3)$ also implies that L is closed since the fact: L^{-1} is closed and injective, then its inverse is also closed. It comes from $(A1) - (A3)$ and the closed graph theorem, we obtain the boundedness of the linear operator $AL^{-1} : X \to X$. Consequently, $-AL^{-1}$ generates a semigroup $\left\{S(t) = e^{AL^{-1}t},\ t \geq 0\right\}$. We suppose that $K_0 = \sup_{t\geq 0} ||S(t)|| < \infty$, and for short, we denote by $C_1 = \left|\left|L^{-1}\right|\right|, C_2 = \left|\left|M^{-1}\right|\right|$ and $T = AL^{-1}$.

$(A4)$ The resolvent $R(\lambda, T)$ is compact for some $\lambda \in \rho(T)$, the resolvent set of T.

Without loss of generality, we assume that $0 \in \rho(T)$. Then it is possible to define the fractional power T^{α} for $0 < \alpha \leq 1$, as a closed linear operator on its domain $D(T^{\alpha})$ with inverse $T^{-\alpha}$. Hereafter, we denote by X_{α} the Banach space $D(T^{\alpha})$ normed with $\|x\|_{\alpha}$.

Lemma 2.1 (See [26]). *Let T be the infinitesimal generator of an analytic semigroup $S(t)$. If $0 \in \rho(T)$, then*

(a) *$D(T^{\alpha})$ is a Hilbert space with the norm $||x||_{\alpha} = ||T^{\alpha}x||$ for $x \in D(T^{\alpha})$.*

(b) *$S(t) : X \to D(T^{\alpha})$ for each $t > 0$ and $\alpha \geq 0$.*

(c) *$S(t)T^{\alpha}x = T^{\alpha}S(t)x$ for each $x \in D(T^{\alpha})$ and $t \geq 0$.*

(d) *If $0 < \alpha \leq \beta \leq 1$, then $D(T^{\beta}) \hookrightarrow D(T^{\alpha})$.*

(e) *For every $t > 0$, $T^{\alpha}S(t)$ is bounded on X and there exist $K_{\alpha} > 0$ and $\delta > 0$ such that*

$$||T^{\alpha}S(t)|| \leq \frac{K_{\alpha}}{t^{\alpha}}e^{-\delta t} \leq \frac{K_{\alpha}}{t^{\alpha}}.$$

(vi) *$T^{-\alpha}$ is a bounded linear operator in X with $D(T^{\alpha}) = Im\left(T^{-\alpha}\right)$.*

Definition 2.1 *The fractional integral of order $\alpha > 0$ of a function $f \in L^1([a,b], R^+)$ is given by*

$$I_a^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds,$$

where Γ is the gamma function. If $a = 0$, we can write $I^{\alpha} f(t) = (g_{\alpha} * f)(t)$, where

$$g_{\alpha}(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} t^{\alpha-1}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

and as usual, $*$ denotes the convolution of functions. Moreover, $\lim\limits_{\alpha\to 0} g_{\alpha}(t) = \delta(t)$, with δ the delta Dirac function.

Definition 2.2 *The Riemann–Liouville derivative of order $n-1 < \alpha < n,\ n \in N$, for a function $f \in C([0,+\infty))$ is given by*

$$^{L}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}} ds,\ \ t > 0.$$

Definition 2.3 *The Caputo derivative of order $n-1 < \alpha < n,\ n \in N$, for a function $f \in C([0,+\infty))$ is given by*

$$^{C}D^{\alpha} f(t) = ^{L} D^{\alpha}\left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0)\right),\ \ t > 0.$$

Remark 2.1 *The following properties hold. Let $n-1 < \alpha < n,\ n \in N$*

(i) *If $f(t) \in C^n([0,\infty))$, then*

$$^{C}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds = I^{n-\alpha} f^{(n)}(s),\ \ t > 0.$$

(ii) The Caputo derivative of a constant is equal to zero.

(iii) The Riemann–Liouville derivative of a constant function is given by

$$ {}^{L}D_{a^{+}}^{\alpha}C=\frac{C}{\Gamma\left(1-\alpha\right)}\left(x-a\right)^{-\alpha}. $$

If f is an abstract function with values in X, then integrals which appear in Definitions 2.1-2.3 are taken in Bochner's sense.

According to previous definitions, it is suitable to rewrite problem (1.1)-(1.2) as the equivalent integral equation

$$ \begin{array}{rcl} x(t) & = & x\left(0\right)+\frac{1}{\Gamma(q)}\int_{0}^{t}L^{-1}\left(t-s\right)^{q-1}Ax\left(s\right)ds \\ & + & \frac{1}{\Gamma(q)}\int_{0}^{t}L^{-1}\left(t-s\right)^{q-1}\sigma_{1}(s,x(s),Hx\left(s\right))dw_{1}(s) \end{array} \tag{2.1} $$

Remark 2.2 *We note that:*

(a) For the nonlocal condition, the function $x(0)$ is dependent on t.

(b) ${}^{L}D_{t}^{1-q}\left[Mx(0)\right]$ is well defined, i.e., if $q=1$ and M is the identity, then (1.2) reduces to the usual nonlocal condition.

(c) The function $x(0)$ takes the form

$$ M^{-1}x_{0}+\frac{1}{\Gamma(1-q)}\int_{0}^{t}(t-s)^{-q}M^{-1}\sigma_{2}(s,x(s))dw_{2}(s), $$

where $Mx(0)|_{t=0}=x_{0}$.

(d) The explicit and implicit integrals given in (2.1) exist (taken in Bochner's sense).

Let $L^{2}(\Gamma_{b},X_{\alpha})$ be the Banach space of all Γ_{b}-measurable square integrable random variables with values in the Hilbert space X_{α}. Let $E(\cdot)$ denotes the expectation with respect to the measure P. An important subspace of $L^{2}(\Gamma_{b},X_{\alpha})$ is given by $L_{2}^{0}(\Gamma_{b},X_{\alpha})=\left\{x\in L^{2}(\Gamma_{b},X_{\alpha}):x\text{ is }\Gamma_{0}-\text{measurable}\right\}.$

Let $C(J,L^{2}(\Gamma,X_{\alpha}))$ be the Hilbert space of continuous maps from J into $L^{2}(\Gamma,X_{\alpha})$ satisfying $\sup_{t\in J}E\left\|x(t)\right\|_{\alpha}^{2}<\infty$. Let $\mathcal{H}_{2}(J,X_{\alpha})$ is a closed subspace of $C(J,L^{2}(\Gamma,X_{\alpha}))$ consisting of measurable and Γ_{t}-adapted X_{α}-valued process $x\in C(J,L^{2}(\Gamma,X_{\alpha}))$ endowed with the norm $\left\|x\right\|_{\mathcal{H}_{2}}=(\sup_{t\in J}E\left\|x(t)\right\|_{\alpha}^{2})^{1/2}$.

Then it easy to check that $(\mathcal{H}_{2},\|\cdot\|_{\mathcal{H}_{2}})$ is a Hilbert space. For any constant $\tau>0$, let $B_{\tau}=\left\{x\in\mathcal{H}_{2}:\|x\|_{\mathcal{H}_{2}}\leq\tau\right\}$, clearly that B_{τ} is a bounded closed conves set in $\mathcal{H}_{2}$.

Definition 2.4 *By the α-mild solution of the problem (1.1)-(1.2), we mean that the Γ_{t}-adapted stochastic process $x\in H_{2}$ which satisfies*

1. *$x(0)\in L^{2}(\Gamma,X_{\alpha})$, where $x\left(0\right)=M^{-1}x_{0}+\frac{1}{\Gamma(1-q)}\int_{0}^{t}(t-s)^{-q}M^{-1}\sigma_{2}(s,x(s))dw_{2}(s)$ and $Mx(0)|_{t=0}=x_{0}$;*
2. *$x(t)\in X_{\alpha}$ has cádlág paths on $t\in J$ almost surely and for each $t\in J$, $x(t)$ satisfies the integral equation*

$$ \begin{array}{rcl} x(t) & = & \mathcal{S}_{q}(t)M^{-1}\left[x_{0}+\frac{1}{\Gamma(1-q)}\int_{0}^{t}(t-s)^{-q}\sigma_{2}(s,x(s))dw_{2}(s)\right]2.2 \\ & & +\int_{0}^{t}\left(t-s\right)^{q-1}L^{-1}\mathcal{T}_{q}(t-s)\sigma_{1}(s,x(s),Hx\left(s\right))dw_{1}(s) \end{array} \tag{2.1} $$

where $S_{q}(t)x=\int_{0}^{+\infty}h_{q}(s)S\left(t^{q}s\right)xds$ and $T_{q}(t)x=q\int_{0}^{+\infty}sh_{q}(s)S\left(t^{q}s\right)xds$.

Here, $S(t)$ is a C_{0}-semigroup generated by a linear operator $T=AL^{-1}:X\to X$, h_{q} is a probability density function defined on $(0,\infty)$, that is $h_{q}(s)\geq0,\ s\in(0,\infty)$ and $\int_{0}^{\infty}h_{q}(s)ds=1$.

The following lemma follows from the results in ([15], [16], [21], [33], [34]) and will be used throughout this paper.

Lemma 2.2. *The operators $S_{q}(t)$ and $T_{q}(t)$ have the following properties:*

(1) *For any fixed $t\geq0$, $S_{q}(t)$ and $T_{q}(t)$ are linear and bounded operators in X_{α},*

$$ \text{i.e.for any } x\in X_{\alpha},\quad \left\|\mathcal{S}_{q}(t)x\right\|\leq K_{0}\left\|x\right\|_{\alpha},\qquad \left\|\mathcal{T}_{q}(t)x\right\|\leq\frac{qK_{0}}{\Gamma\left(1+q\right)}\left\|x\right\|_{\alpha}. $$

(2) *The operators* $\{S_q(t) : t \geq 0\}$ *and* $\{T_q(t) : t \geq 0\}$ *are strongly continuous.*

(3) *For every* $t > 0$, $S_q(t)$ *and* $T_q(t)$ *are compact operators in* X, *and hence they are norm-continuous.*

(4) *For every* $t > 0$, *the restriction of* $S_q(t)$ *to* X_α *and the restriction of* $T_q(t)$ *to* X_α *are compact operators in* X_α.

(5) *The restriction of* $S_q(t)$ *to* X_α *and the restriction of* $T_q(t)$ *to* X_α *are continuous in* $(0, +\infty)$ *by the operator norm* $\|\cdot\|_\alpha$.

(6) *For any* $x \in X$ *and* $t \in J$, $\left|\left|T^\alpha \mathcal{T}_q(t)x\right|\right| \leq A_\alpha t^{-\alpha q} \|x\|$, *where* $A_\alpha = \frac{qK_\alpha \Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))}$.

3 Main results

In this section, we give the existence of α-mild solutions for the problem (1.1)-(1.2). We impose the following assumptions on the data of the problem (1.1)-(1.2).

$(H1)$ The functions $\sigma_1 : J \times X_\alpha \times X_\alpha \to L_2^0$ satisfies the Carathéodory type conditions, i.e. $\sigma_1(\cdot, x, Hx)$ is strongly measurable for all $x \in X_\alpha$, and $\sigma_1(t,\cdot,\cdot)$ is continuous for a.e. $t \in J$.

$(H2)$ For some $\tau > 0$, there exist constants $q_1 \in [\frac{1}{2}, q(1-\alpha))$, $\rho_1 > 0$ and functions $\varphi_\tau \in L^{\frac{1}{2q_1-1}}(J, \mathbb{R}^+)$ such that for a.e. $t \in J$,

$$\sup_{\|x\|_{\mathcal{H}_2}^2 \leq \tau} E\|\sigma_1(t,x,Hx)\|^2 \leq \varphi_\tau(t) \quad \text{and} \quad \liminf_{\tau\to+\infty} \frac{\|\varphi_\tau\|_{L^{\frac{1}{2q_1-1}}[0,b]}}{\tau} = \rho_1 < +\infty.$$

$(H3)$ The nonlocal function $\sigma_2 : J \times X_\alpha \to L_2^0$ is continuous, and there exist a constant $\rho_2 > 0$ and a nondecreasing continuous function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that for some $\tau > 0$ and all $x \in B_\tau$,

$$E\|\sigma_2(t,x)\|_\alpha^2 \leq \Phi(\tau) \quad \text{and} \quad \liminf_{\tau\to+\infty} \frac{\Phi(\tau)}{\tau} = \rho_2 < +\infty.$$

$(H4)$ There exists a constant $\delta \in (0,b)$ such that $\sigma_2(t,x) = \sigma_2(t,y)$ for any $x, y \in \mathcal{H}_2$ with $x(t) = y(t)$, $t \in [\delta, b]$.

Theorem 3.1 *If the assumptions (H1)–(H4) are satisfied, then the problem (1.1)-(1.2) has at least one α-mild solution provided that*

$$\begin{aligned} &\left\{ \left(\tfrac{C_2 K_0}{\Gamma(1-q)}\right)^2 Tr(Q) \tfrac{1}{2q-1} b^{2q-1} \rho_2 \right. \\ &\left. + (C_1 A_\alpha)^2 Tr(Q) b^{2(q-q_1-\alpha q)} \left(\tfrac{1-q_1}{q-q_1-\alpha q}\right)^{2-2q_1} \rho_1 \right\} < \tfrac{1}{2}. \end{aligned} \tag{3.1}$$

Proof Let $\{\delta_n : n \in \mathbb{N}\}$ be a decreasing sequence in $(0,b)$ such that $\lim_{n\to\infty} \delta_n = 0$. We first prove the following problem

$${}^C D_t^q [Lx(t)] = Ax(t) + \sigma_1(t, x(t), Hx(t)) \frac{dw_1(t)}{dt}, \quad t \in J, \tag{3.2}$$

$$x(0) = S(\delta_n) \left(M^{-1} x_0 + \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} M^{-1} \sigma_2(s, x(s)) dw_2(s) \right) \tag{3.3}$$

has at least one α-mild solution $x_n \in \mathcal{H}_2$. To this end, for fixed $n \in \mathbb{N}$, we define an operator $\Psi_n : \mathcal{H}_2 \to \mathcal{H}_2$ by

$$\begin{aligned} (\Psi_n x)(t) &= \mathcal{S}_q(t) S(\delta_n) M^{-1} \left[x_0 + \tfrac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} \sigma_2(s, x(s)) dw_2(s) \right] \\ &+ \int_0^t (t-s)^{q-1} L^{-1} \mathcal{T}_q(t-s) \sigma_1(s, x(s), Hx(s)) dw_1(s), \quad t \in J. \end{aligned} \tag{3.4}$$

By direct calculation, we know that Ψ_n is well defined. From Definition 2.4, it is easy to see that the α-mild solution of the problem (3.2)-(3.3) is equivalent to the fixed point of the operator Ψ_n.

In what follows, we prove that there exists a positive constant R, such that $\Psi_n(B_R) \subset B_R$.

If this is not true, then for each $\tau > 0$, there would exist $x_\tau \in B_\tau$ and $t_\tau \in J$ such that $E\left\|(\Psi_n x_\tau)(t_\tau)\right\|_\alpha^2 > \tau$. It follows from Lemma 2.2 (1) and (6), the assumption (H2) and Hölder inequality that

$$\begin{aligned}
\tau &< E\left\|(\Psi_n x_\tau)(t_\tau)\right\|_\alpha^2 \\
&\leq 2E\left\|\mathcal{S}_q(t_\tau)S\left(\delta_n\right)M^{-1}\left[x_0+\frac{1}{\Gamma(1-q)}\int_0^{t_\tau}(t_\tau-s)^{-q}\sigma_2(s,x_\tau(s))dw_2(s)\right]\right\|_\alpha^2 \\
&\quad +2E\left\|\int_0^{t_\tau}(t_\tau-s)^{q-1}L^{-1}\mathcal{T}_q(t_\tau-s)\sigma_1(s,x_\tau(s),Hx_\tau(s))dw_1(s)\right\|_\alpha^2 \\
&\leq 2\left\|M^{-1}\right\|^2\left\|\mathcal{S}_q(t_\tau)\right\|^2 Tr\left(Q\right)\left(\frac{1}{\Gamma(1-q)}\right)^2 \\
&\quad \times\int_0^{t_\tau}(t_\tau-s)^{-2q}E\left\|\sigma_2(s,x_\tau(s)\right\|^2 ds3.5 \\
&\quad +2\left\|L^{-1}\right\|^2 Tr\left(Q\right)\int_0^{t_\tau}(t_\tau-s)^{2(q-1)}E\left\|T^\alpha\mathcal{T}_q(t_\tau-s)\sigma_1(s,x_\tau(s),Hx_\tau(s))\right\|^2 ds \\
&\leq 2C_2^2K_0^2 Tr\left(Q\right)\left(\frac{1}{\Gamma(1-q)}\right)^2\Phi\left(\tau\right)\left(\int_0^{t_\tau}(t_\tau-s)^{-2q}ds\right) \\
&\quad +2C_1^2 Tr\left(Q\right)A_\alpha^2\left(\int_0^{t_\tau}(t_\tau-s)^{\frac{2q-2-2\alpha q}{2-2q_1}}ds\right)^{2-2q_1}\times\left(\int_0^{t_\tau}\varphi_{\tau_1}^{\frac{1}{2q_1-1}}\left(s\right)ds\right)^{2q_1-1} \\
&\leq 2C_2^2K_0^2 Tr\left(Q\right)\left(\frac{1}{\Gamma(1-q)}\right)^2\Phi\left(\tau\right)\frac{1}{-2q+1}b^{-2q+1} \\
&\quad +2C_1^2 Tr\left(Q\right)A_\alpha^2 b^{2(q-q_1-\alpha q)}\left(\frac{1-q_1}{q-q_1-\alpha q}\right)^{2-2q_1}\left\|\varphi_{\tau_1}\right\|_{L^{\frac{1}{2q_1-1}}[0,b]}.
\end{aligned} \tag{3.2}$$

Dividing both side of (3.5) by τ, then taking the lower limit as $\tau\to+\infty$, we get

$$\begin{Bmatrix} 2C_2^2K_0^2 Tr\left(Q\right)\left(\frac{1}{\Gamma(1-q)}\right)^2\frac{1}{-2q+1}b^{-2q+1}\rho_2 \\ +2C_1^2 Tr\left(Q\right)A_\alpha^2 b^{2(q-q_1-\alpha q)}\left(\frac{1-q_1}{q-q_1-\alpha q}\right)^{2-2q_1}\rho_1 \end{Bmatrix}\geq 1.$$

which contradicts (3.1).

Next, we prove that Ψ_n is continuous in B_R. To this end, let $\{x_m\}_{m=1}^\infty\subset B_R$ be a sequence such that $\lim\limits_{m\to\infty}x_m = x$ in B_R. By the Carathéodory continuity of σ_1 and σ_2, we have

$$\lim_{m\to\infty}\sigma_1(s,x_m(s),Hx_m\left(s\right)) = \sigma_1(s,x(s),Hx\left(s\right)),\quad a.e.\ s\in J. \tag{3.6}$$

$$\lim_{m\to\infty}\sigma_2(s,x_m(s)) = \sigma_2(s,x(s)),\quad a.e.\ s\in J. \tag{3.7}$$

From the assumption (H2), we get that for each $t\in J$,

$$\begin{aligned}
&(t-s)^{2(q-1-\alpha q)}E\left\|\sigma_1(s,x_m(s),Hx_m\left(s\right))-\sigma_1(s,x(s),Hx\left(s\right))\right\|^2 \\
&\leq (t-s)^{2(q-1-\alpha q)}\Big(2E\left\|\sigma_1(s,x_m(s),Hx_m\left(s\right))\right\|^2 \\
&+2E\left\|\sigma_1(s,x_m(s),Hx_m\left(s\right))\right\|^2\Big)\leq 4\left(t-s\right)^{2(q-1-\alpha q)}\varphi_R\left(s\right).
\end{aligned} \tag{3.8}$$

From the assumption (H3), we get that for each $t\in J$,

$$\begin{aligned}
&(t-s)^{-2q}E\left\|\sigma_2(s,x_m(s))-\sigma_2(s,x(s))\right\|^2 \\
&\leq 4\left(t-s\right)^{-2q}\varphi_R\left(s\right).
\end{aligned} \tag{3.9}$$

Using the fact that the functions $s\to 4\left(t-s\right)^{2(q-1-\alpha q)}\varphi_R\left(s\right)$ and $s\to 4\left(t-s\right)^{-2q}\varphi_R\left(s\right)$ are Lebesgue integrables for $s\in[0,t]$, $t\in J$, by Lemma 2.2 (1) and (6), (3.6), (3.7), (3.8), (3.9) and the Lebesgue dominated convergence theorem, we get that

$$
\begin{aligned}
& E\left\|\left(\Psi_n x_m\right)(t)-\left(\Psi_n x\right)(t)\right\|_\alpha^2 3.10 \qquad (3.3)\\
\leq\; & 2\left\|M^{-1}\right\|^2\left\|\mathcal{S}_q(t)\right\|^2 Tr\,(Q)\left(\frac{1}{\Gamma(1-q)}\right)^2 \int_0^t (t-s)^{-2q} \\
& \times E\left\|\sigma_2(s, x_m(s))-\sigma_2(s, x(s)\right\|^2 ds \\
& +2\left\|L^{-1}\right\|^2 Tr\,(Q) \int_0^t (t-s)^{2(q-1)} \\
& \times E\left\|T^\alpha \mathcal{T}_q(t-s)\left(\sigma_1(s, x_m(s), H x_m(s))-\sigma_1(s, x(s), H x(s))\right)\right\|^2 ds \\
\leq\; & 2 C_2^2 K_0^2 Tr\,(Q)\left(\frac{1}{\Gamma(1-q)}\right)^2 \frac{1}{-2q+1} t^{-2q+1} \\
& \times E\left\|\sigma_2(s, x_m(s))-\sigma_2(s, x(s))\right\|^2 \\
& +C_1^2 Tr\,(Q)\, A_\alpha^2 \int_0^t (t-s)^{2(q-1-\alpha q)} \\
& \times E\left\|\sigma_1(s, x_m(s), H x_m(s))-\sigma_1(s, x(s), H x(s))\right\|^2 ds \\
\rightarrow\; & 0 \quad \text{as} \quad m \rightarrow \infty .
\end{aligned}
$$

Therefore, by (3.10) we know that

$$
E\left\|\left(\Psi_n x_m\right)-\left(\Psi_n x\right)\right\|_{H_2}^2=\sup_{t \in J} E\left\|\left(\Psi_n x_m\right)(t)-\left(\Psi_n x\right)(t)\right\|_\alpha^2 \rightarrow 0 \quad \text{as} \quad m \rightarrow \infty .
$$

which means that Ψ_n is continuous in B_R.

Now, we demonstrate that $\Psi_n : B_R \rightarrow B_R$ is a compact operator. We first prove that $\{(\Psi_n x)(t) : x \in B_R\}$ is relatively compact in X_α for all $t \in J$. For $t = 0$, since the compactness of $S(t)$ for every $t > 0$ implies that the restriction of $S(t)$ to X_α is compact semigroup in X_α, for $\forall n \in \mathbb{N}$ we can deduce, by the assumption (H3), that $\{(\Psi_n x)(0) : x \in B_R\}$ is relatively compact in X_α. For $0 < t \leq b$, $\epsilon \in (0, t)$, arbitrary $\delta > 0$ and $x \in B_R$, we define the operator $\Psi_n^{\epsilon,\delta}$ by

$$
\begin{aligned}
\left(\Psi_n^{\epsilon, \delta} x\right)(t) &= \mathcal{S}_q(t) S\left(\delta_n\right) M^{-1}\left[x_0+\frac{1}{\Gamma(1-q)} \int_0^t(t-s)^{-q} \sigma_2(s, x(s)) d w_2(s)\right] 3.11 \qquad (3.4)\\
&+\int_0^{t-\epsilon} \int_\delta^{\infty} q \tau\,(t-s)^{q-1} L^{-1} h_q(\tau)\left(S\,(t-s)^q \tau\right) \sigma_1(s, x(s), H x(s)) d \tau d w_1(s) \\
&= \mathcal{S}_q(t) S\left(\delta_n\right) M^{-1}\left[x_0+\frac{1}{\Gamma(1-q)} \int_0^t(t-s)^{-q} \sigma_2(s, x(s)) d w_2(s)\right] \\
&+L^{-1} S\left(\epsilon^q \delta\right) \int_0^{t-\epsilon} \int_\delta^{\infty} q \tau\,(t-s)^{q-1} h_q(\tau)\left(S\,(t-s)^q \tau-\epsilon^q \delta\right) \sigma_1(s, x(s), H x(s)) d \tau d w_1(s)
\end{aligned}
$$

Since the restriction of $S\left(\epsilon^q \delta\right)$ $\left(\epsilon^q \delta>0\right)$ to X_α is compact semigroup in X_α, by Lemma 2.2 (4) we know that the set $\left\{\left(\Psi_n^{\epsilon, \delta} x\right)(t) : x \in B_R\right\}$ is relatively compact in X_α for $\forall \epsilon \in (0, t)$ and $\forall \delta > 0$. Moreover, for every

$x \in B_R$, by assumption (H2), Lemma 2.2 (6) and Hölder inequality we know that

$$\begin{aligned}
& E\left\|\left(\Psi_n x\right)(t)-\left(\Psi_n^{\epsilon,\delta} x\right)(t)\right\|_\alpha^2 \\
\leq\ & 2E\left\|\int_0^t \int_0^\delta q\tau\,(t-s)^{q-1} L^{-1} h_q(\tau)\left(S\,(t-s)^q\,\tau\right)\sigma_1(s,x(s),Hx\,(s))d\tau dw_1(s)\right\|_\alpha^2 \\
& +2E\left\|\int_{t-\epsilon}^t \int_\delta^\infty q\tau\,(t-s)^{q-1} L^{-1} h_q(\tau)\left(S\,(t-s)^q\,\tau\right)\sigma_1(s,x(s),Hx\,(s))d\tau dw_1(s)\right\|_\alpha^2 \\
\leq\ & 2\left\|L^{-1}\right\|^2 Tr\,(Q)\int_0^t (t-s)^{2(q-1)}\left\|\int_0^\delta q\tau h_q(\tau)\,T^\alpha\left(S\,(t-s)^q\,\tau\right)d\tau\right\|^2 \\
& \times E\left\|\sigma_1(s,x(s),Hx\,(s))\right\|^2 ds \\
& +2\left\|L^{-1}\right\|^2 Tr\,(Q)\int_{t-\epsilon}^t (t-s)^{2(q-1)}\left\|\int_\delta^\infty q\tau h_q(\tau)\,T^\alpha\left(S\,(t-s)^q\,\tau\right)d\tau\right\|^2 \\
& \times E\left\|\sigma_1(s,x(s),Hx\,(s))\right\|^2 ds \\
\leq\ & 2C_1^2 Tr\,(Q)\,M_\alpha^2\left\|\varphi_R\right\|_{L^{\frac{1}{2q_1-1}}[0,b]}\left(\frac{1-q_1}{q-q_1-\alpha q}\right)^{2-2q_1} \\
& \times t^{2(q-q_1-\alpha q)}\left(\int_0^\delta q\tau^{1-\alpha} h_q(\tau)\,d\tau\right)^2 \\
& +2C_1^2 Tr\,(Q)\,A_\alpha^2\left\|\varphi_R\right\|_{L^{\frac{1}{2q_1-1}}[0,b]}\left(\frac{1-q_1}{q-q_1-\alpha q}\right)^{2-2q_1}\epsilon^{2(q-q_1-\alpha q)}.
\end{aligned}$$

Therefore, letting $\delta,\epsilon \to 0$, we see that there are relatively compact sets arbitrarily close to the set $\{(\Psi_n x)(t) : x \in B_R\}$ in X_α for $0 < t \leq b$. Hence, the set $\{(\Psi_n x)(t) : x \in B_R\}$ is also relatively compact in X_α for $0 < t \leq b$. And since $\{(\Psi_n x)(t) : x \in B_R\}$ is relatively compact in X_α, we have the relatively compactness of $\{(\Psi_n x)(t) : x \in B_R\}$ in X_α for all $t \in J$.

Next, we prove that $\Psi_n(B_R)$ is equicontinuous. For $t = 0$, since $S(\delta_n)$ is a compact operator for $\forall n \in \mathbb{N}$, we know that the functions $\{(\Psi_n x)(t) : x \in B_R\}$ are equicontinuous at $t = 0$. For any $x \in B_R$ and $0 < t_1 < t_2 \leq b$, we get that

$$\begin{aligned}
& E\left\|(\Psi_n x)(t_2)-(\Psi_n x)(t_1)\right\|_\alpha^2 \\
\leq\ & 4E\left\|\mathcal{S}_q(t_2)S(\delta_n)M^{-1}\left[x_0+\frac{1}{\Gamma(1-q)}\int_0^{t_2}(t-s)^{-q}\sigma_2(s,x(s))dw_2(s)\right]\right. \\
& \left.-\mathcal{S}_q(t_1)S(\delta_n)M^{-1}\left[x_0+\frac{1}{\Gamma(1-q)}\int_0^{t_1}(t-s)^{-q}\sigma_2(s,x(s))dw_2(s)\right]\right\|_\alpha^2 \\
& +4E\left\|\int_{t_1}^{t_2}(t_2-s)^{q-1}L^{-1}\mathcal{T}_q(t_2-s)\sigma_1(s,x(s),Hx\,(s))dw_1(s)\right\|_\alpha^2 \\
& +4E\left\|\int_0^{t_1}\left((t_2-s)^{q-1}-(t_1-s)^{q-1}\right)L^{-1}\mathcal{T}_q(t_2-s)\sigma_1(s,x(s),Hx\,(s))dw_1(s)\right\|_\alpha^2 \\
& +4E\left\|\int_0^{t_1}(t_1-s)^{q-1}L^{-1}\left(\mathcal{T}_q(t_2-s)-\mathcal{T}_q(t_1-s)\right)\sigma_1(s,x(s),Hx\,(s))dw_1(s)\right\|_\alpha^2 \\
:\ & = I_1+I_2+I_3+I_4.
\end{aligned}$$

where

$$\begin{aligned}
I_1 \ =\ & 4E\left\|\mathcal{S}_q(t_2)S(\delta_n)M^{-1}\left[x_0+\frac{1}{\Gamma(1-q)}\int_0^{t_2}(t-s)^{-q}\sigma_2(s,x(s))dw_2(s)\right]\right. \\
& \left.-\mathcal{S}_q(t_1)S(\delta_n)M^{-1}\left[x_0+\frac{1}{\Gamma(1-q)}\int_0^{t_1}(t-s)^{-q}\sigma_2(s,x(s))dw_2(s)\right]\right\|_\alpha^2,
\end{aligned}$$

$$I_2 = 4E\left\|\int_{t_1}^{t_2}(t_2-s)^{q-1}L^{-1}\mathcal{T}_q(t_2-s)\sigma_1(s,x(s),Hx\,(s))dw_1(s)\right\|_\alpha^2,$$

$$I_3 = 4E\left\|\int_0^{t_1}\left((t_2-s)^{q-1}-(t_1-s)^{q-1}\right)L^{-1}\mathcal{T}_q(t_2-s)\sigma_1(s,x(s),Hx(s))dw_1(s)\right\|_\alpha^2,$$

$$I_4 = 4E\left\|\int_0^{t_1}(t_1-s)^{q-1}L^{-1}\left(\mathcal{T}_q(t_2-s)-\mathcal{T}_q(t_1-s)\right)\sigma_1(s,x(s),Hx(s))dw_1(s)\right\|_\alpha^2.$$

Therefore, we only need to check $I_i \to 0$ independently of $x \in B_R$ when $t_2 - t_1 \to 0$, $i = 1, 2, \cdots, 4$.
For I_1, by Lemma 2.2 (1) and (5) and the assumption (H3), we know that

$$\begin{aligned}
I_1 \quad \leq \quad & 4E\left\|\left(\mathcal{S}_q(t_2)-\mathcal{S}_q(t_1)\right)S(\delta_n)M^{-1}\left[\frac{1}{\Gamma(1-q)}\int_0^{t_1}(t_1-s)^{-q}\sigma_2(s,x(s))dw_2(s)\right]\right\|_\alpha^2 \\
& +4E\left\|\mathcal{S}_q(t_2)S(\delta_n)M^{-1}\left[\frac{1}{\Gamma(1-q)}\int_0^{t_1}\left((t_2-s)^{-q}-(t_1-s)^{-q}\right)\sigma_2(s,x(s))dw_2(s)\right]\right\|_\alpha^2 \\
& +4E\left\|\mathcal{S}_q(t_2)S(\delta_n)M^{-1}\left[\frac{1}{\Gamma(1-q)}\int_{t_1}^{t_2}(t_2-s)^{-q}\sigma_2(s,x(s))dw_2(s)\right]\right\|_\alpha^2 \\
\leq \quad & 4\left\|M^{-1}\right\|^2 K_0^2 Tr(Q)\left(\frac{1}{\Gamma(1-q)}\right)^2\frac{t_1^{-2q+1}}{-2q+1}\Phi(\tau)E\left\|\mathcal{S}_q(t_2)-\mathcal{S}_q(t_1)\right\|_\alpha^2 \\
& +4\left\|M^{-1}\right\|^2 K_0^2 Tr(Q)\left\|\mathcal{S}_q(t_2)\right\|^2\Phi(\tau)\left(\int_0^{t_1}\left((t_2-s)^{-q}-(t_1-s)^{-q}\right)^2 ds\right) \\
& +4\left\|M^{-1}\right\|^2 K_0^2 Tr(Q)\left\|\mathcal{S}_q(t_2)\right\|^2\left(\frac{1}{\Gamma(1-q)}\right)^2\frac{(t_2-t_1)^{-2q+1}}{-2q+1}\Phi(\tau) \\
\longrightarrow \quad & 0 \quad \text{as} \quad t_2-t_1 \longrightarrow 0.
\end{aligned}$$

For I_2, by the assumption (H2), Lemma 2.2 (6) and Hölder inequality, we have

$$\begin{aligned}
I_2 \quad \leq \quad & 4\left\|L^{-1}\right\|^2 Tr(Q_1)\int_{t_1}^{t_2}(t_2-s)^{q-1}\left\|T^\alpha\mathcal{T}_q(t_2-s)\right\|^2 E\left\|\sigma_1(s,x(s),H_1x(s))\right\|^2 ds \\
\leq \quad & 4C_1^2 Tr(Q_1)A_\alpha^2\int_{t_1}^{t_2}(t-s)^{2q--2-2\alpha q}\varphi_R(s)\,ds \\
& 4C_1^2 Tr(Q_1)A_\alpha^2\left\|\varphi_R\right\|_{L^{\frac{1}{2q_1-1}}[0,b]}\left(\frac{1-q_1}{q-q_1-\alpha q}\right)^{2-2q_1}(t_2-t_1)^{2(q-q_1-\alpha q)} \\
\longrightarrow \quad & 0 \quad \text{as} \quad t_2-t_1 \longrightarrow 0.
\end{aligned}$$

For I_3, by the assumption (H2), Lemma 2.2 (6) and Hölder inequality, we get that

$$\begin{aligned}
I_3 \quad \leq \quad & 4\left\|L^{-1}\right\|^2 Tr(Q)\int_0^{t_1}\left((t_2-s)^{q-1}-(t_1-s)^{q-1}\right)\left\|T^\alpha\mathcal{T}_q(t_2-s)\right\|^2 E\left\|\sigma_1(s,x(s),Hx(s))\right\|^2 ds \\
\leq \quad & 4C_1^2 Tr(Q)A_\alpha^2\int_0^{t_1}\left((t_2-s)^{2q--2-2\alpha}-(t_1-s)^{2q--2-2\alpha}\right)\varphi_R(s)\,ds \\
\leq \quad & 4C_1^2 Tr(Q)A_\alpha^2\left(\int_0^{t_1}\left((t_2-s)^{\frac{2q--2-2\alpha}{2-2q_1}}-(t_1-s)^{\frac{2q--2-2\alpha}{2-2q_1}}\right)ds\right)^{2-2q_1}\times\left\|\varphi_R\right\|_{L^{\frac{1}{2q_1-1}}[0,t_1]} \\
\leq \quad & 4C_1^2 Tr(Q)A_\alpha^2\left\|\varphi_R\right\|_{L^{\frac{1}{2q_1-1}}[0,b]}\left(\frac{1-q_1}{q-q_1-\alpha q}\right)^{2-2q_1}(t_2-t_1)^{2(q-q_1-\alpha q)} \\
\longrightarrow \quad & 0 \quad \text{as} \quad t_2-t_1 \longrightarrow 0.
\end{aligned}$$

For $\epsilon > 0$ small enough, by Lemma 2.3 (5) and (6), the assumption (H2) and Hölder inequality, we know that

$$\begin{aligned}
I_4 &\leq \sup_{s\in[0,t_1-\epsilon]} \left\|\mathcal{T}_q(t_2-s)-\mathcal{T}_q(t_1-s)\right\|_\alpha^2 \cdot 8\left\|L^{-1}\right\|^2 Tr\,(Q)\int_0^{t_1-\epsilon}(t_1-s)^{2(q-1)}\\
&\quad\times E\left\|\sigma_1(s,x(s),Hx\,(s))\right\|^2 ds\\
&\quad+8\left\|L^{-1}\right\|^2 Tr\,(Q)\int_{t_1-\epsilon}^{t_1}(t_1-s)^{2(q-1)}\left\|T^\alpha\left(\mathcal{T}_q(t_2-s)-\mathcal{T}_q(t_1-s)\right)\right\|^2\\
&\quad\times E\left\|\sigma_1(s,x(s),Hx\,(s))\right\|^2 ds\\
&\leq \sup_{s\in[0,t_1-\epsilon]} \left\|\mathcal{T}_q(t_2-s)-\mathcal{T}_q(t_1-s)\right\|_\alpha^2 \cdot 8C_1^2 Tr\,(Q)\int_0^{t_1-\epsilon}(t_1-s)^{2(q-1)}\varphi_R\,(s)\,ds\\
&\quad+16C_1^2 Tr\,(Q)\int_{t_1-\epsilon}^{t_1}(t_1-s)^{2(q-1)}\left((t_2-s)^{-2\alpha q}-(t_1-s)^{-2\alpha q}\right)\varphi_R\,(s)\,ds\\
&\leq \sup_{s\in[0,t_1-\epsilon]} \left\|\mathcal{T}_q(t_2-s)-\mathcal{T}_q(t_1-s)\right\|_\alpha^2 \cdot 8C_1^2 Tr\,(Q)\left\|\varphi_R\right\|_{L^{\frac{1}{2q_1-1}}[0,b]}\\
&\quad\times\left(\frac{1-q_1}{q-q_1}\right)^{2-2q_1}\left(t_1^{\frac{q-q_1}{1-q_1}}-\epsilon^{\frac{q-q_1}{1-q_1}}\right)^{2-2q_1}\\
&\quad+32C_1^2 Tr\,(Q)\,A_\alpha^2\left\|\varphi_R\right\|_{L^{\frac{1}{2q_1-1}}[0,b]}\left(\frac{1-q_1}{q-q_1-\alpha q}\right)^{2-2q_1}\epsilon^{2(q-q_1-\alpha q)}\\
&\longrightarrow 0 \text{ as } t_2-t_1\longrightarrow 0 \text{ and } \epsilon\longrightarrow 0.
\end{aligned}$$

As a result, $E\left\|(\Psi_n x)(t_2)-(\Psi_n x)(t_1)\right\|_\alpha^2$ tends to zero independently of $x\in B_R$ as $t_2-t_1\to 0$, which means that $\Psi_n : B_R\to B_R$ is equicontinuous. Hence by the Arzela-Ascoli theorem one has that $\Psi_n : B_R\to B_R$ is a compact operator. Therefore, by Schauder fixed point theorem we obtain that for each $n\in\mathbb{N}$, Ψ_n has at least one fixed point $x_n\in B_R$ which is in turn a α-mild solution of the problem (3.2)-(3.3). Furthermore, for any $t\in J$, $x_n(t)$ is given by

$$\begin{aligned}
x_n\,(t) &= \mathcal{S}_q(t)S\,(\delta_n)\,M^{-1}\left[x_0+\tfrac{1}{\Gamma(1-q)}\int_0^t(t-s)^{-q}\sigma_2(s,x_n(s))dw_2(s)\right]\\
&+ \int_0^t(t-s)^{q-1}L^{-1}\mathcal{T}_q(t-s)\sigma_1(s,x_n(s),Hx_n\,(s))dw_1(s).
\end{aligned} \tag{3.12}$$

Finally, we show that the set $\{x_n : n\in\mathbb{N}\}\subset B_R$ is precompact in $\mathcal{H}_2$. Denote by

$$x_n^1\,(t) = \mathcal{S}_q(t)S\,(\delta_n)\,M^{-1}\left[x_0+\tfrac{1}{\Gamma(1-q)}\int_0^t(t-s)^{-q}\sigma_2(s,x_n(s))dw_2(s)\right],\quad t\in J$$

and

$$x_n^2\,(t) = \int_0^t(t-s)^{q-1}L^{-1}\mathcal{T}_q(t-s)\sigma_1(s,x_n(s),Hx_n\,(s))dw_1(s),\quad t\in J.$$

Therefore, it is sufficient to show that the sets $\{x_n^1 : n\in\mathbb{N}\}$ and $\{x_n^2 : n\in\mathbb{N}\}$ are precompact in $\mathcal{H}_2$.

Let $\xi\in(0,\delta)$ be fixed, where δ is the constant in (H4). By taking the method similar to the proof of the compactness of the operator Ψ_n, we can see that the sets $\{x_n^1 : n\in\mathbb{N}\}\big|_{[\xi,b]}$ and $\{x_n^2 : n\in\mathbb{N}\}$ are precompact in $C([\xi,b],L^2(\Gamma,X_\alpha))$ and $C([0,b],L^2(\Gamma,X_\alpha))$, respectively. In particular, the set $\{x_n^2 : n\in\mathbb{N}\}\big|_{[\xi,b]}$ is precompact in $C([\xi,b],L^2(\Gamma,X_\alpha))$. Therefore, we have proved that the set $\{x_n : n\in\mathbb{N}\}\big|_{[\xi,b]}$ is precompact in $C([\xi,b],L^2(\Gamma,X_\alpha))$.

Without loss of generality, we let

$$x_n\longrightarrow x \text{ in } C([\xi,b],L^2(\Gamma,X_\alpha)) \text{ as } n\longrightarrow\infty.$$

Denote by

$$x_n^\delta\,(t)=\begin{cases} x_n\,(\delta)\,,\ t\in[0,\delta]\,,\\ x_n\,(t)\,,\ t\in[\delta,b]\,.\end{cases},\quad x^\delta\,(t)=\begin{cases} x\,(\delta)\,,\ t\in[0,\delta]\,,\\ x\,(t)\,,\ t\in[\delta,b]\,.\end{cases}$$

Then we have

$$x_n^\delta\to x^\delta \text{ in } C([0,b],L^2(\Gamma,X_\alpha)) \text{ as } n\longrightarrow\infty.$$

Therefore, to prove that the set $\{x_n : n \in \mathbb{N}\}$ is precompact in $C([0,b], L^2(\Gamma, X_\alpha))$, we only need to prove that the set $\{x_n^1 : n \in \mathbb{N}\}\big|_{[0,\xi]}$ is precompact in $C([0,\xi], L^2(\Gamma, X_\alpha))$. By the strong continuity of the semigroup $S(t)(t \geq 0)$ and the assumptions (H3) and (H4), we have

$$\begin{aligned}
& E\left\| S(\delta_n) M^{-1}\left(x_0 + \frac{1}{\Gamma(1-q)}\int_0^t (t-s)^{-q}\sigma_2(s, x_n(s))dw_2(s)\right)\right. \\
& \left. -M^{-1}\left(x_0 + \frac{1}{\Gamma(1-q)}\int_0^t (t-s)^{-q}\sigma_2(s, x(s))dw_2(s)\right)\right\|_\alpha^2 \\
\leq\ & 2\left\|M^{-1}\right\|^2\left(\frac{1}{\Gamma(1-q)}\right)^2 \\
& \times E\left\|S(\delta_n)\int_0^t (t-s)^{-q}T^\alpha\left(\sigma_2(s, x_n^\delta(s)) - -\sigma_2(s, x^\delta(s)\right)dw_2(s)\right\|^2 \\
& +2\left\|M^{-1}\right\|^2\left(\frac{1}{\Gamma(1-q)}\right)^2 \times E\left\|S(\delta_n)\int_0^t (t-s)^{-q}T^\alpha\sigma_2(s, x^\delta(s))dw_2(s)\right. \\
& \left. -\int_0^t (t-s)^{-q}T^\alpha\sigma_2(s, x^\delta(s))dw_2(s)\right\|^2 \\
\leq\ & 2C_2^2\left(\frac{1}{\Gamma(1-q)}\right)^2 K_0^2 E\left\|\int_0^t (t-s)^{-q}\sigma_2(s, x_n^\delta(s))dw_2(s)\right. \\
& \left. -\int_0^t (t-s)^{-q}\sigma_2(s, x^\delta(s), H_2x^\delta(s))dw_2(s)\right\|_\alpha^2 \\
& +2C_2^2\left(\frac{1}{\Gamma(1-q)}\right)^2 \times E\left\|(S(\delta_n) - I)\int_0^t (t-s)^{-q}T^\alpha\sigma_2(s, x^\delta(s))dw_2(s)\right\|^2 \\
\longrightarrow\ & 0 \text{ as } n \longrightarrow \infty.
\end{aligned}$$

which means that the set $\left\{S(\delta_n) M^{-1}\left(x_0 + \frac{1}{\Gamma(1-q)}\int_0^t (t-s)^{-q}\sigma_2(s, x_n(s))dw_2(s)\right) : n \in \mathbb{N}\right\}$ is precompact in X_α. By the continuity of the operator $\mathcal{S}_q(t)(t \geq 0)$, we know that the set $\left\{\mathcal{S}_q(t)S(\delta_n) M^{-1}\left(x_0 + \frac{1}{\Gamma(1-q)}\int_0^t (t-s)^{-q}\sigma_2(s, x_n(s))dw_2(s)\right) : n \in \mathbb{N}\right\}$ is precompact in X_α for $t \in [0,\xi]$. By Lemma 2.2 (2) and the assumption (H3), we know that for every $n \in \mathbb{N}$ and $t_1, t_2 \in [0,\xi]$ with $t_1 < t_2$,

$$\begin{aligned}
& E\left\|\mathcal{S}_q(t_2)S(\delta_n) M^{-1}\left(x_0 + \frac{1}{\Gamma(1-q)}\int_0^{t_2} (t_2-s)^{-q}\sigma_2(s, x_n(s))dw_2(s)\right)\right. \\
& \left. -\mathcal{S}_q(t_1)S(\delta_n) M^{-1}\left(x_0 + \frac{1}{\Gamma(1-q)}\int_0^{t_1} (t_1-s)^{-q}\sigma_2(s, x_n(s))dw_2(s)\right)\right\|_\alpha^2 \\
\leq\ & 3E\left\|\left(\mathcal{S}_q(t_2) - \mathcal{S}_q(t_1)\right) S(\delta_n) M^{-1}\left[\frac{1}{\Gamma(1-q)}\int_0^{t_1} (t_1-s)^{-q}T^\alpha\sigma_2(s, x_n(s))dw_2(s)\right]\right\|_\alpha^2 \\
& +3E\left\|\mathcal{S}_q(t_2)S(\delta_n) M^{-1}\left[\frac{1}{\Gamma(1-q)}\int_0^{t_1}\left((t_2-s)^{-q} - (t_1-s)^{-q}\right)T^\alpha\sigma_2(s, x_n(s))dw_2(s)\right]\right\|_\alpha^2 \\
& +3E\left\|\mathcal{S}_q(t_2)S(\delta_n) M^{-1}\left[\frac{1}{\Gamma(1-q)}\int_{t_1}^{t_2} (t_2-s)^{-q}T^\alpha\sigma_2(s, x_n(s))dw_2(s)\right]\right\|_\alpha^2 \\
\longrightarrow\ & 0 \text{ as } t_2 - t_1 \longrightarrow 0.
\end{aligned}$$

This means that the set $\{x_n^1(t) : n \in \mathbb{N}\}$ is equicontinuous for $t \in [0,\xi]$. Therefore, applying Arzela-Ascoli theorem again one obtains that the set $\{x_n^1 : n \in \mathbb{N}\}\big|_{[0,\xi]}$ is precompact in $C([0,\xi], L^2(\Gamma, X_\alpha))$.

Therefore,we have proved that the set $\{x_n : n \in \mathbb{N}\}$ is precompact in $C([0,b], L^2(\Gamma, X_\alpha))$.

Hence, without losing the generality, we may suppose that

$$x_n \to x^* \text{ in } C([0,b], L^2(\Gamma, X_\alpha)) \text{ as } n \longrightarrow \infty.$$

Taking limits in (3.12) one has

$$\begin{aligned}
x^*(t) &= \mathcal{S}_q(t)S(\delta_n) M^{-1}\left[x_0 + \tfrac{1}{\Gamma(1-q)}\textstyle\int_0^t (t-s)^{-q}\sigma_2(s, x^*(s))dw_2(s)\right] \\
&+ \textstyle\int_0^t (t-s)^{q-1} L^{-1}\mathcal{T}_q(t-s)\sigma_1(s, x^*(s), Hx^*(s))dw_1(s).
\end{aligned}$$

for all $t \in J$, which means that $x^* \in C([0,b], L^2(\Gamma, X_\alpha))$ is a α-mild solution of the problem (1.1)-(1.2) and the proof of Theorem 3.1 is completed.

4 An example

n this section, we present an example, which do not aim at generality but indicate how our abstract result can be applied to concrete problem. Let $N \geq 1$ be an integer, $U \subset \mathbb{R}^N$ be a bounded domain, whose boundary $\partial\Omega$ is an $(N-1)$-dimensional $C^{2+\mu}$-manifold for some $0 < \mu < 1$. We consider the nonlocal problem of Sobolev type fractional stochastic parabolic partial differential equation of the form

$$ {}^C D_t^{\frac{2}{3}} \left[x(z,t) - x_{zz}(z,t) \right] - \frac{\partial^2}{\partial z^2} x(z,t) = \frac{\sin\left(z, t, x(z,t), \int_0^t K(t,s)\, x(z,s)\, ds \right) d\hat{w}_1(t)}{e^t dt}, \; z \in U, \; t \in J, \tag{4.1} $$

$$ x(z,0) = \frac{\partial^2}{\partial z^2} \left[x_0(z) + \frac{1}{\Gamma\left(\frac{1}{3}\right)} \sum_{k=1}^{m} c_k \int_0^t (t-s)^{-\frac{2}{3}} x(z,t_k)\, \hat{w}_2(s) \right], \; z \in U \tag{4.2} $$

where ${}^C D_t^q$ is the Caputo fractional derivative of order $q \in (0,1)$, $0 < t_1 < ... < t_m < b$ and c_k are positive constants, $k = 1, ..., m$; the functions $x(t)(z) = x(z,t)$, $\sigma_1(t, x(t), Hx(t))(z) = \frac{\sin\left(z,t,x(z,t),\int_0^t K(t,s)x(z,s)ds\right)}{e^t}$ and $\sigma_2(t, x(t))(z) = \sum_{k=1}^m c_k x(z, t_k)$; $\hat{w}_1(t)$ and $\hat{w}_2(t)$ are two sided and standard one dimensional Brownian motions defined on the filtered probability space (Ω, Γ, P), $J = [0,b]$, $K \in C(\Delta, \mathbb{R}^+)$, $\Delta = \left\{ (t,s) \in \mathbb{R}^2 : 0 \leq s \leq t \leq b \right\}$.

Let $X = L^2(U)$, define the operators $L : D(L) \subset X \to X$, $A : D(A) = H^2(U) \cap H_0^1(U) \subset X \to X$ and $M : D(M) \subset X \to X$ by $Lx = x - x''$, $Ax = -x''$ and $M^{-1}x = x''$ where the domains $D(L)$, $D(A)$ and $D(M)$ are given by

$$ \{x \in X : x, x' \text{ are absolutely continuous, } x'' \in X, x|_{\partial U} = 0\}. $$

It is easy to see that L^{-1} is compact, bounded with $\|L^{-1}\| \leq 1$ and $T = AL^{-1}$ generates the above strongly continuous semigroup $S(t)$ on $L^2(U)$ with $\|S(t)\| \leq e^{-t} \leq 1$. Therefore, with the above choices, the system (4.1)-(4.2) can be written as an abstract formulation of (1.1)-(1.2).

From the definitions of σ_1 and σ_2, it is easy to verify that $\sigma_1 : X_0 \times X_0 \to L_2^0$ and $\sigma_2 : X_0 \to L_2^0$ whenever $x \in C(J, L^2(\Gamma, X_0))$. Moreover, we see that the assumptions (H1)–(H4) and the condition (3.1) hold with

$$ q_1 = \frac{1}{2}, \quad \varphi_\tau(t) = \frac{|U|}{e^t}, \quad \Phi(\tau) = m\,|U| \sum_{k=1}^{m} c_k^2 \tau^{\frac{2}{3}}, \quad \rho_1 = \rho_2 = 0, \quad \delta = t_1. $$

Therefore, by Theorem 3.1, we have the following result.

Theorem 4.1 *The nonlocal problem of Sobolev type fractional stochastic parabolic partial differential equation has at least one* 0*-mild solution.*

References

[1] Agarwal, R. P., Benchohra, M., & Hamani, S. "A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions." Acta Applicandae Mathematicae 109.3 (2010): 973-1033.

[2] Agarwal, R. P., Lakshmikantham, V., & Nieto, J. J. "On the concept of solution for fractional differential equations with uncertainty." Nonlinear Analysis: Theory, Methods & Applications 72.6 (2010): 2859-2862.

[3] Baleanu, D., Diethelm, K., Scalas, E., & Trujillo, J. J. Fractional Calculus Models and Numerical Methods. 2012.

[4] Byszewski, L, and Lakshmikantham, V. "Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space." Applicable Analysis 40.1 (1991): 11-19.

[5] Byszewski, L. "Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem." Journal of Mathematical Analysis and Applications 162.2 (1991): 494-505.

[6] Byszewski, L. ”Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem.” Journal of Mathematical Analysis and Applications 162.2 (1991): 494-505.

[7] Chang, Y. K., Zhao, Z. H., N’Guérékata, G. M., & Ma, R. ”Stepanov-like almost automorphy for stochastic processes and applications to stochastic differential equations.” Nonlinear Analysis: Real World Applications 12.2 (2011): 1130-1139.

[8] Cui, J., Yan, L., & Wu, X. ”Nonlocal Cauchy problem for some stochastic integro-differential equations in Hilbert spaces.” Journal of the Korean Statistical Society 41.3 (2012): 279-290.

[9] Cui, J., & Yan, L. ”Existence result for fractional neutral stochastic integro-differential equations with infinite delay.” Journal of Physics A: Mathematical and Theoretical 44.33 (2011): 335201.

[10] Debbouche, A., & El-Borai, M. M. ”Weak almost periodic and optimal mild solutions of fractional evolution equations.” Electronic Journal of Differential Equations 2009.46 (2009): 1-8.

[11] Debbouche, A., & Nieto, J. J.”Sobolev type fractional abstract evolution equations with nonlocal conditions and optimal multi-controls.” Applied Mathematics and Computation 245 (2014): 74-85.

[12] Debbouche, A., & Torres, D. F. ”Sobolev Type Fractional Dynamic Equations and Optimal Multi-Integral Controls with Fractional Nonlocal Conditions.” arXiv preprint arXiv:1409.6028 (2014).

[13] Debbouche, A., Baleanu, D., & Agarwal, R. P. ”Nonlocal nonlinear integrodifferential equations of fractional orders.” Boundary Value Problems 2012.1 (2012): 1-10.

[14] Da Prato, G., & Zabczyk, J. Stochastic equations in infinite dimensions. Vol. 152. Cambridge university press, 2014.

[15] El-Borai, M. M. ”Some probability densities and fundamental solutions of fractional evolution equations.” Chaos, Solitons & Fractals 14.3 (2002): 433-440.

[16] El-Borai, M. M. ”On some stochastic fractional integro-differential equations.” Advances in Dynamical Systems and Applications 1.1 (2006): 49-57.

[17] Ezzinbi, K., Fu, X., & Hilal, K. ”Existence and regularity in the 3b1-norm for some neutral partial differential equations with nonlocal conditions.” Nonlinear Analysis: Theory, Methods & Applications 67.5 (2007): 1613-1622.

[18] Grecksch, W., & Tudor, C. ”Stochastic evolution equations (a Hilbert space approach).” Mathematical Research (1995).

[19] Ichikawa, A. ”Stability of semilinear stochastic evolution equations.” Journal of Mathematical Analysis and Applications 90.1 (1982): 12-44.

[20] Kerboua, M., Debbouche, A., & Baleanu, D. ”Approximate Controllability of Sobolev Type Nonlocal Fractional Stochastic Dynamic Systems in Hilbert Spaces.” Abstract and Applied Analysis. Vol. 2013 (2013), Article ID 262191, 10 pages.

[21] Liu, H., & Chang, J. C. ”Existence for a class of partial differential equations with nonlocal conditions.” Nonlinear Analysis: Theory, Methods & Applications 70.9 (2009): 3076-3083.

[22] Liang, J., Liu, J., & Xiao, T. J. ”Nonlocal Cauchy problems governed by compact operator families.” Nonlinear Analysis: Theory, Methods & Applications 57.2 (2004): 183-189.

[23] Mainardi, F. (Ed.). Fractals and fractional calculus in continuum mechanics. No. 378. Springer Verlag, 1997.

[24] Malinowska, A. B., & Torres, D. F. M. Fractional Calculus of Variations. Imperial College Press, Singapore, 2012.

[25] Miller, K. S., & Ross, B. ”An introduction to the fractional calculus and fractional differential equations.” (1993).

[26] Pazy, A. Semigroups of linear operators and applications to partial differential equations. 1983.

[27] Podlubny, I. Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. Vol. 198. Academic press, 1998.

[28] Ren, Y., & Sakthivel, R. ”Existence, uniqueness, and stability of mild solutions for second-order neutral stochastic evolution equations with infinite delay and Poisson jumps.” Journal of Mathematical Physics 53.7 (2012): 073517.

[29] Sakthivel, R., & Luo, J. ”Asymptotic stability of impulsive stochastic partial differential equations with infinite delays.” Journal of Mathematical Analysis and Applications 356.1 (2009): 1-6.

[30] Sakthivel, R., & Ren, Y. ”Exponential stability of second-order stochastic evolution equations with Poisson jumps.” Communications in Nonlinear Science and Numerical Simulation 17.12 (2012): 4517-4523.

[31] Sakthivel, R., Revathi, P., & Mahmudov, N. I. ”Asymptotic stability of fractional stochastic neutral differential equations with infinite delays.” Abstract and Applied Analysis. Vol. 2013. Hindawi Publishing Corporation, 2013.

[32] Sakthivel, R., Revathi, P., & Ren, Y. ”Existence of solutions for nonlinear fractional stochastic differential equations.” Nonlinear Analysis: Theory, Methods & Applications 81 (2013): 70-86.

[33] Wang, J., & Zhou, Y. ”A class of fractional evolution equations and optimal controls.” Nonlinear Analysis: Real World Applications 12.1 (2011): 262-272.

[34] Wang, R. N., Xiao, T. J., & Liang, J. ”A note on the fractional Cauchy problems with nonlocal initial conditions.” Applied Mathematics Letters 24.8 (2011): 1435-1442.

[35] Yan, Z., & Yan, X. ”Existence of solutions for a impulsive nonlocal stochastic functional integrodifferential inclusion in Hilbert spaces.” Zeitschrift für angewandte Mathematik und Physik 64.3 (2013): 573-590.

[36] Yan, Z., & Yan, X. ”Existence of solutions for impulsive partial stochastic neutral integrodifferential equations with state-dependent delay.” Collectanea Mathematica 64.2 (2013): 235-250.

Received: February 20, 2015; *Accepted*: August 20, 2015

UNIVERSITY PRESS

Website: http://www.malayajournal.org/