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Hyers-Ulam-Rassias stability of nth order linear ordinary differential equations with initial conditions

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Abstract

In this paper, we investigate the stability of nth order linear ordinary differential non-homogeneous equation with initial conditions in the Hyers-Ulam-Rassias sense.

Keywords: Differential equation, Differential inequality, Hyers-Ulam-Rassias stability, Initial Value Problem, Integral Equation.

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1 Introduction

In 1940, S.M. Ulam while he was giving talk at Wisconsin University, he proposed the following problem: Under what conditions does there exist an additive mapping near an approximately additive mapping? for details see [18]. A year later, D.H. Hyers in [4] gave an answer to the problem of Ulam for additive functions defined on Banach spaces. Let E_1 and E_2 be two real Banach spaces and $f : X_1 \to X_2$ be a mapping. If there exist an $\epsilon \geq 0$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon$$

for all $x, y \in X_1$, then there exist a unique additive mapping $g : X_1 \to X_2$ with the property

$$\|f(x) - g(x)\| \leq \epsilon,$$

$\forall x \in X_1$. A generalized solution to Ulam's problem for approximately linear mappings was proved by Th.M. Rassias in 1978 [13]. He considered a mapping $f : E_1 \to E_2$ such that $t \to f(tx)$ is continuous in t for each fixed x. Assume that there exists $\theta \geq 0$ and $0 \geq p < 1$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \theta \left(\|x\|^p + \|y\|^p\right)$$

for any $x, y \in E_1$. After Hyers result, many mathematicians have extended Ulam's problem to other functional equations and generalized Hyers result in various directions, see ([2], [5]).

Soon-Mo Jung [17], investigated the Hyers-Ulam stability of a system of first order linear differential equations with constant coefficients. Miura et al [11], proved the Hyers-Ulam stability of the first-order linear differential equations of the form $y'(t) + g(t)y(t) = 0$, where $g(t)$ is a continuous function, while Jung [14], proved the Hyers-Ulam stability of differential equations of the form $\varphi(t)y'(t) = y(t)$. Furthermore, the result of Hyers-Ulam stability for first-order linear differential equations has been generalized in ([15], [16], [19]).

A. Javadian, E. Sorouri, G.H. Kim and M. Eshaghi Gordji [6], investigated generalized Hyers-Ulam stability

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of the second order linear differential equations of the form $y'' + P(x)y' + q(x)y = f(x)$ with some conditions. Maher Nazmi Qorawani [10], investigated Hyers-Ulam stability of second order linear differential equations of the form $z'' + p(x)z' + (q(x) - \alpha(x))\, z = 0$ and nonlinear differential equations of the form $z'' + p(x)z' + q(x)z = h(x)\,|z|^{\beta}\, e^{\left(\frac{\beta-1}{2}\right)\int p(x)dx} sgnz$ with initial conditions. Li and Yan [8], investigated the Hyers-Ulam Stability of nonhomogeneous second order Linear Differential Equations of the form $y'' + p(x)y' + q(x)y + r(x) = 0$ under some special conditions. Pasc Gavruta, Jung, Li [3], investigated the Hyers-Ulam stability for second order linear differential equations with boundary conditions of the form $y'' + \beta(x)y(x) = 0$. Jinghao Huang, Qusuay H. Alqifiary, and Yongjin Li [7], proved the generalized superstability of nth order linear differential equations with initial conditions of the form $y^{(n)}(x) + \beta(x)y(x) = 0$. Recently, M.I. Modebei, O.O. Olaiya, I. Otaide [12], investigated generalized Hyers-Ulam stability of second order linear ordinary differential equation $y'' + \beta(x)y = f(x)$ with initial condition.

In this paper, we investigate the Hyers-Ulam-Rassias Stability of nth order linear ordinary differential equations with initial condtions

$$y^{(n)} + \beta(x)y(x) = f(x)$$
$$y(a) = y'(a) = y''(a) = ... = y^{(n-1)}(a) = 0,$$

where $y \in C^n[a,b], \beta \in C[a,b]$ and $f : [a,b] \to \mathbb{R}$ continuous.

Let $(X, \|.\|)$ be a real or complex Banach space with $a, b \in \mathbb{R}$ where $-\infty < a < b < \infty$, ϵ be a positive real number. Let $y : (a,b) \to X$ be a continuouus function. We consider the following differential equation

$$y^{(n)}(t) = \sum_{k=0}^{n-1} P_k y^{(k)}(t), \quad t \in I \tag{1.1}$$

and the following differential inequality

$$\left| y^{(n)}(t) - \sum_{k=0}^{n-1} P_k y^{(k)}(t) \right| \leq \epsilon, \quad t \in I \tag{1.2}$$

and

$$\left| y^{(n)}(t) - \sum_{k=0}^{n-1} P_k y^{(k)}(t) \right| \leq \varphi(t), \quad t \in I \tag{1.3}$$

Definition 1.1. *The equation (1.1) is said to have the Hyers-Ulam stability for any $\epsilon > 0$, there exist a real number $K > 0$ such that for each approximate solution $y \in C^n(I, X)$ of (1.2) there exist a solution $y_0 \in C^n(I, X)$ of (1.1) with*

$$|y - y_0| \leq K\epsilon \quad \forall t \in I. \tag{1.4}$$

Definition 1.2. *The equation (1.1) is said to have the Hyers-Ulam-Rassias stability if there exist $\theta_{\varphi} \in C(\mathbb{R}_+, \mathbb{R}_+)$, such that for each approximate solution $y \in C^n(I, X)$ of (1.3) there exist a solution $y_0 \in C^n(I, X)$ of (1.1) with*

$$|y - y_0| \leq \theta_{\varphi}(t) \quad \forall t \in I. \tag{1.5}$$

Definition 1.3. *The equation $y^{(n)}(x) + \beta(x)y(x) = 0$ has the Hyers-Ulam stability with initial conditions $y(a) = y'(a) = ... = y^{(n-1)}(a) = 0$, if there exists a positive constant K with the following property: For every $\epsilon > 0, y \in C^n[a,b]$, if*

$$\left| y^{(n)}(x) + \beta(x)y(x) \right| \leq \epsilon, \tag{1.6}$$

and $y(a) = y'(a) = ... = y^{(n-1)}(a) = 0$, then there exists some $z \in C^n[a,b]$ satisfying $Z^{(n)} + \beta(x)z(x) = 0$ and $z(a) = z'(a) = ... = z^{(n-1)}(a) = 0$, such that

$$|y(x) - z(x)| \leq K\epsilon.$$

We need the following Lemma to prove our main results.

Lemma 1.1. ***(Generalized Replacement Lemma)*** *Suppose that $g : [a,b] \to \mathbb{R}$ is a continuous. Then*

$$\int_a^{s_{n-1}} \int_a^{s_{n-2}} ... \int_a^{s_2} \int_a^{s_1} \int_a^{x} g(s) ds ds_1 ds_2 ... ds_{n-1} = \int_a^x \frac{(x-s)^{n-1}}{(n-1)!} g(s) ds, \forall x \in [a,b]$$

The details of the proof we can see [1].

Theorem 1.1. *If* $\max|\beta(x)| < \dfrac{n!}{(b-a)^n}$ *Then*

$$y^{(n)}(x) + \beta(x)y(x) = 0 \tag{1.7}$$

has the Hyers-Ulam stability with initial conditions

$$y(a) = y'(a) = ... = y^{(n-1)}(a) = 0 \tag{1.8}$$

where $y \in C^n[a,b], \beta \in C[a,b]$ *and* $f : [a,b] \to \mathbb{R}$ *continuous.*

Proof. For every $\epsilon > 0$, By using the Taylor formula, we have

$$y(x) = y(a) + y'(x-a) + ... + \frac{y^{(n)}(\xi)}{n!}(x-a)^n.$$

Thus

$$\begin{aligned} |y(x)| &= \left|\frac{y^{(n)}(\xi)}{n!}(x-a)^n\right| \\ &\le \max\left|y^{(n)}(x)\right| \frac{(b-a)^n}{n!} \qquad \forall x \in [a,b], \end{aligned}$$

then

$$\max|y(x)| \le \frac{(b-a)^n}{n!}\left[\max\left|y^{(n)}(x) - \beta(x)y(x) + \beta(x)y(x)\right|\right]$$

Now using (1.7), we obtain

$$\begin{aligned} \max|y(x)| &\le \frac{(b-a)^n}{n!}\left[\max\left|y^{(n)}(x) - \beta(x)y(x)\right| + \max|\beta(x)|\max|y(x)|\right] \\ &\le \frac{(b-a)^n}{n!}\epsilon + \frac{(b-a)^n}{n!}\max|\beta(x)|\max|y(x)|. \end{aligned}$$

Let $\eta = \left((b-a)^n \max|\beta(x)|\right)/n!, \quad K = (b-a)^n/\left(n!(1-\eta)\right)$. It is easy to see that $z_0(x) = 0$ is a solution of $y^{(n)}(x) - \beta(x)y = 0$ with initial conditions (1.8).

$$|y - z_0| \le K\epsilon.$$

Hence (1.7) has the Hyers-Ulam-Rassias stability with initial conditions (1.8). □

2 Main Result

In this section, we shall prove the Generalized Hyers-Ulam-Rassias Stability of the IVP

$$y^{(n)} + \beta(x)y(x) = f(x) \tag{2.9}$$

$$y(a) = y'(a) = y''(a) = ... = y^{(n-1)}(a) = 0, \tag{2.10}$$

where $y \in C^n[a,b], \beta \in C[a,b]$ and $f : [a,b] \to \mathbb{R}$ continuous.

Theorem 2.2. *Suppose* $|\beta(x)| < M$ *where* $M = \dfrac{n!}{(b-a)^n}$, $\varphi : [a,b] \to [0,\infty)$ *in an increasing function. The equation (2.9) has the Hyers-Ulam-Rassias stability if for* $\theta_\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$ *and for each approximate solution* $y \in C^n[a,b]$ *of (2.9) satisfying*

$$\left|y^{(n)} - \beta(x)y(x) - f(x)\right| \le \varphi(x) \tag{2.11}$$

there exist a solution $z_0 \in C^n[a,b]$ *of (2.9) with condition (2.10) such that*

$$|y(x) - z_0(x)| \le \theta_\varphi(x). \tag{2.12}$$

Proof. From (2.11) we have that

$$-\varphi(x) \leq y^{(n)} - \beta(x)y(x) - f(x) \leq \varphi(x).$$

Integrating from a to x, and applying condition (2.10) we have

$$-\int_a^x \varphi(s)ds \leq y^{(n-1)}(x) - \int_a^x \beta(s)y(s)ds - \int_a^x f(s)ds \leq \int_a^x \varphi(s)ds.$$

On further integration and also applying condition (2.10) we have

$$\begin{aligned}-\int_a^{s_1}\int_a^x \varphi(s)dsds_1 \leq y^{(n-2)}(x) - &\int_a^{s_1}\int_a^x \beta(s)y(s)dsds_1 \\ &- \int_a^{s_1}\int_a^x f(s)dsds_1 \leq \int_a^{s_1}\int_a^x \varphi(s)dsds_1.\end{aligned}$$

Continuing the process finally we can get,

$$\begin{aligned}-\int_a^{s_{n-1}}\int_a^{s_{n-2}}\ldots\int_a^{s_2}\int_a^{s_1}\int_a^x &\varphi(s)dsds_1\ldots ds_{n-1} \\ &\leq y(x) - \int_a^{s_{n-1}}\int_a^{s_{n-2}}\ldots\int_a^{s_2}\int_a^{s_1}\int_a^x \beta(s)y(s)dsds_1\ldots ds_{n-1} \\ &- \int_a^{s_{n-1}}\int_a^{s_{n-2}}\ldots\int_a^{s_2}\int_a^{s_1}\int_a^x f(s)dsds_1\ldots ds_{n-1} \\ &\leq \int_a^{s_{n-1}}\int_a^{s_{n-2}}\ldots\int_a^{s_2}\int_a^{s_1}\int_a^x \varphi(s)dsds_1\ldots ds_{n-1}.\end{aligned}$$

Now applying Lemma (1.1), we obtain

$$\begin{aligned}-\int_a^x \frac{(x-s)^{n-1}}{(n-1)!}\varphi(s)ds \leq y(x) - \int_a^x \frac{(x-s)^{n-1}}{(n-1)!}\beta(s)y(s)ds - &\int_a^x \frac{(x-s)^{n-1}}{(n-1)!}f(s)ds \\ &\leq \int_a^x \frac{(x-s)^{n-1}}{(n-1)!}\varphi(s)ds.\end{aligned}$$

Hence we have

$$\left|y(x) - \int_a^x \frac{(x-s)^{n-1}}{(n-1)!}\left(\beta(s)y(s)ds + f(s)ds\right)\right| \leq \int_a^x \frac{(x-s)^{n-1}}{(n-1)!}\varphi(s)ds. \tag{2.13}$$

If we choose $z_0(x)$ such that it solves equation (2.9) with (2.10) such that

$$z_0(x) = \int_a^x \frac{(x-s)^{n-1}}{(n-1)!}\left(\beta(s)z_0(s)ds + f(s)ds\right),$$

thus we estimate

$$\begin{aligned}|y(x) - z_0(x)| &\leq \left|y(x) - \int_a^x \frac{(x-s)^{n-1}}{(n-1)!}\left(\beta(s)y(s)ds + f(s)ds\right)\right| \\ &+ \int_a^x \left|\frac{(x-s)^{n-1}}{(n-1)!}\left(\beta(s)y(s)ds + f(s)ds\right) - \int_a^x \frac{(x-s)^{n-1}}{(n-1)!}\left(\beta(s)z_0(s)ds + f(s)\right)\right| ds\end{aligned}$$

$$|y(x) - z_0(x)| \leq \int_a^x \frac{(x-s)^{n-1}}{(n-1)!}\varphi(s)ds + \int_a^x \left|\frac{(x-s)^{n-1}}{(n-1)!}\beta(s)[y(s) - z_0(s)]\right| ds.$$

Now applying (2.13) and Theorem 1.1, we get

$$|y(x) - z_0(x)| \leq \int_a^x \frac{(x-s)^{n-1}}{(n-1)!}\varphi(s)ds + |\beta(s)|\int_a^x \frac{(x-s)^{n-1}}{(n-1)!}|y(s) - z_0(s)|\, ds$$

$$|y(x) - z_0(x)| \leq \int_a^x \frac{(x-s)^{n-1}}{(n-1)!}\varphi(s)ds + M\int_a^x \frac{(x-s)^{n-1}}{(n-1)!}|y(s) - z_0(s)|\, ds.$$

Applying Gronwall's inequality, we have

$$|y(x) - z_0(x)| \leq \int_a^x \frac{(x-s)^{n-1}}{(n-1)!} \varphi(s) ds \exp\left\{ M \int_a^x \frac{(x-s)^{n-1}}{(n-1)!} ds \right\}$$
$$|y(x) - z_0(x)| \leq \int_a^x \frac{(x-s)^{n-1}}{(n-1)!} \varphi(s) ds \exp\left\{ M \left[\frac{(x-a)^n}{n!} \right] \right\}$$
$$|y(x) - z_0(x)| \leq c \int_a^x \frac{(x-s)^{n-1}}{(n-1)!} \varphi(s) ds$$

with

$$c = \exp\left\{ \left[\frac{x-a}{b-a} \right]^n \right\}$$

and the proof is completed.

Remark: Note that as $x \to b$, then the above system considered is Hyers-Ulam stable. □

Conclusion

We obtained the Hyers-Ulam-Rassias stability of nth order linear ordinary differential nonhomogeneous equation with initial conditions. Hyers-Ulam-Rassias stability guarantees that there is a close exact solution of the system.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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