Malaya J. Mat. 4(2)(2016) 243–254

Malaya Journal of Matematik

www.malayajournal.org

MJM
an international journal of mathematical sciences with computer applications...

ISSN : 2319-3786

Results on Weighted Stepanov-like Pseudo Periodic Functions and Applications

Shan Zheng*

Department of Mathematics, Lanzhou Jiaotong University, Lanzhou 730070, China.

Abstract

In this article, we first established some new results on composition of weighted Stepanov-like pseudo periodic function of class r under a uniform continuity condition with respect to L^p norm. And then, we proved the existence and uniqueness of weighted pseudo periodic solutions to a semi-linear functional differential equation with finite delay under Stepanov-like nonlinear term.

Keywords: Weighted Stepanov-like pseudo periodic function of class r, L^p norm, differential equation with delay.

2010 MSC: 34G20.

1 Introduction

The periodic, almost periodic and almost automorphic solutions to differential equations can profoundly characterize the asymptotic behavior of the corresponding dynamic systems, which have gained great attention by many scholars [10, 17, 18]. Particularly, the concept of weighted pseudo periodicity was introduced in [3]; Xia presented the notion of weighted Stepanov-like pseudo periodicity in [20]. In order to investigate differential equations with delay, Xia in [21] further studied new types of functions so called weighted pseudo periodic of class r, weighted Stepanov-like pseudo periodic of class r, respectively. For more details on this topic, we refer to [1, 2, 4, 5, 7, 11–15, 19, 25, 26] and references therein.

The main purpose of present paper is to make a further investigation on the composition results for weighted Stepanov-like pseudo periodic function of class r. Considering the space of weighted Stepanov-like pseudo periodic function of class r with an integral norm coming from L^p norm, we first prove a new composition theorem for weighted Stepanov-like pseudo periodic function of class r under a uniform continuity condition with respect to the L^p norm suggested by [16]. And then, we apply the obtained results to prove the existence and uniqueness of weighted pseudo periodic solution to the following semi-linear delay differential equation with a weighted Stepanov-like pseudo periodic nonlinear term

$$u'(t) = Au(t) + f(t, u_t), \quad t \in \mathbb{R}, \tag{1.1}$$

where A is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$, $u_t \in \mathfrak{B}$ is defined by $u_t(\theta) = u(t+\theta)$ for $\theta \in [-r, 0]$, r is a no-negative constant, f and $\mathfrak{B}$ are specified in the later.

The rest of this paper is organized as follows. In Section 2, we recall some basic definitions, lemmas, and preliminary results which will be used throughout this paper. In Section 3, we establish some new results on composition of weighted Stepanov-like pseudo periodic function of class r under a L^p norm uniform continuity condition. In Section 4, we prove the existence of pseudo periodic mild solutions to the existence and uniqueness of weighted pseudo periodic solutions to the equation (1.1) under Stepanov-like nonlinear term. An example is also given to illustrate the main results.

*Corresponding author.
E-mail address: 1318434313@qq.com

2 Preliminaries

Let $(\mathbb{X}, \|\cdot\|)$ and $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ be two Banach spaces and $\mathbb{N}$, $\mathbb{R}$ stand for sets of natural numbers and real numbers, respectively. To facilitate the discussion below, we further introduce the following notations:

· $BC(\mathbb{R}, \mathbb{X})$ (respectively, $BC(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$): The Banach spaces of bounded continuous function from $\mathbb{R}$ to $\mathbb{X}$ (respectively, from $\mathbb{R} \times \mathbb{Y}$ to $\mathbb{X}$) with the sup norm.

· $L^p(\mathbb{R}, \mathbb{X})$: The space of all classes of equivalence (with respect to the equality almost everywhere on $\mathbb{R}$) of measurable function $f : \mathbb{R} \to \mathbb{X}$ such that $\|f\| \in L^p(\mathbb{R}, \mathbb{R})$.

· $L^p_{loc}(\mathbb{R}, \mathbb{X})$: The space of all classes of equivalence of measurable function $f : \mathbb{R} \to \mathbb{X}$ such that the restriction of f to every bounded subinterval of $\mathbb{R}$ is in $L^p(\mathbb{R}, \mathbb{X})$.

· $\mathfrak{B}$: The space $C([-r, 0], \mathbb{X})$ endowed with the sup norm $\|\psi\|_{\mathfrak{B}}$ on $[-r, 0]$.

Definition 2.1 ([21]). A function $f \in C(\mathbb{R}, \mathbb{X})$ is said to be anti-periodic if there exists a $\omega \in \mathbb{R} \setminus \{0\}$ with the property that $f(t + \omega) = -f(t)$ for all $t \in \mathbb{R}$. The least positive ω with this property is called the anti-periodic of f. The collection of such functions is denoted by $P_{\omega ap}(\mathbb{R}, \mathbb{X})$.

Definition 2.2 ([21]). A function $f \in C(\mathbb{R}, \mathbb{X})$ is called to be periodic if there exists a $\omega \in \mathbb{R} \setminus \{0\}$ such that $f(t + \omega) = f(t)$ for all $t \in \mathbb{R}$. The least positive ω with this property is called the periodic of f. The collection of those ω periodic functions is denoted by $P_\omega(\mathbb{R}, \mathbb{X})$.

Let $\mathbb{U}$ denote the set of all functions $\rho : \mathbb{R} \to (0, \infty)$, which are locally integrable over $\mathbb{R}$ such that $\rho > 0$ almost everywhere. For a given $T > 0$ and for each $\rho \in \mathbb{U}$, we set $\mu(T, \rho) := \int_{-T}^{T} \rho(t)dt$. Thus the spaces of weights $\mathbb{U}_\infty$ and $\mathbb{U}_B$ are defined by

$$\mathbb{U}_\infty := \left\{ \rho \in \mathbb{U} : \lim_{T\to\infty} \mu(T, \rho) = \infty \right\},$$

$$\mathbb{U}_B := \left\{ \rho \in \mathbb{U}_\infty : \rho \text{ is bounded and } \inf_{t\in\mathbb{R}} \rho(t) > 0 \right\}.$$

For a given $\rho_1, \rho_2 \in \mathbb{U}_\infty$, we define respectively

$$V(T, f, r, \rho_1, \rho_2) = \frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \left(\sup_{\theta\in[t-r,t]} \|f(\theta)\| \right) \rho_2 dt;$$

$$WPP_0(\mathbb{R}, \mathbb{X}, \rho_1, \rho_2) := \left\{ f \in BC(\mathbb{R}, \mathbb{X}) : \lim_{T\to\infty} \frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \|f(t)\| \rho_2(t) dt = 0 \right\};$$

$$WPP_0(\mathbb{R}, \mathbb{X}, , r, \rho_1, \rho_2) := \left\{ f \in BC(\mathbb{R}, \mathbb{X}) : \lim_{T\to\infty} \frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \left(\sup_{\theta\in[t-r,t]} \|f(\theta)\| \right) \cdot \rho_2(t) dt = 0 \right\};$$

$$WPP_0(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, r, \rho_1, \rho_2) := \left\{ f \in BC(\mathbb{R}, \mathbb{X}) : \lim_{T\to\infty} \frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \left(\sup_{\theta\in[t-r,t]} \|f(\theta)\| \right) \cdot \rho_2(t) dt = 0 \text{ and uniformly in } u \in \mathbb{Y} \right\}$$

$$= \left\{ f \in BC(\mathbb{R}, \mathbb{X}) : \lim_{T\to\infty} V(T, f, r, \rho_1, \rho_2) = 0 \text{ uniformly in } u \in \mathbb{Y} \right\}.$$

Definition 2.3 ([21]). Let $\rho_1, \rho_2 \in \mathbb{U}_\infty$. A function $f \in C(\mathbb{R}, \mathbb{X})$ is called weighted pseudo anti-periodic for $\omega \in \mathbb{R} \setminus \{0\}$ if it can be decomposed as $f = g + \varphi$, where $g \in P_{\omega ap}(\mathbb{R}, \mathbb{X})$ and $\varphi \in WPP_0(\mathbb{R}, \mathbb{X}, \rho_1, \rho_2)$. Denote by $WPP_{\omega ap}(\mathbb{R}, \mathbb{X}, \rho_1, \rho_2)$ the set of such function.

Definition 2.4 ([21]). Let $\rho_1, \rho_2 \in \mathbb{U}_\infty$. A function $f \in C(\mathbb{R}, \mathbb{X})$ is called weighted pseudo periodic for $\omega \in \mathbb{R} \setminus \{0\}$ if it can be decomposed as $f = g + \varphi$, where $g \in P_\omega(\mathbb{R}, \mathbb{X})$ and $\varphi \in WPP_0(\mathbb{R}, \mathbb{X}, \rho_1, \rho_2)$. Denote by $WPP_\omega(\mathbb{R}, \mathbb{X}, \rho_1, \rho_2)$ the set of such function.

Definition 2.5 ([21]). Let $\rho_1, \rho_2 \in \mathbb{U}_\infty$. A function $f \in C(\mathbb{R}, \mathbb{X})$ is called weighted pseudo anti-periodic of class r for $\omega \in \mathbb{R} \setminus \{0\}$ if it can be decomposed as $f = g + \varphi$, where $g \in P_{\omega ap}(\mathbb{R}, \mathbb{X})$ and $\varphi \in WPP_0(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2)$. Denote by $WPP_{\omega ap}(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2)$ the set of such function.

Definition 2.6 ([21]). Let $\rho_1, \rho_2 \in \mathbb{U}_\infty$. A function $f \in C(\mathbb{R}, \mathbb{X})$ is called weighted pseudo periodic of class r for $\omega \in \mathbb{R} \setminus \{0\}$ if it can be decomposed as $f = g + \varphi$, where $g \in P_\omega(\mathbb{R}, \mathbb{X})$ and $\varphi \in WPP_0(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2)$. Denote by $WPP_\omega(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2)$ the set of such function.

Remark 2.1 ([21]). If $r = 0$, then the weighted pseudo anti-periodic function of class r reduces to the weighted pseudo anti-periodic function, the weighted pseudo periodic function of class r reduces to the weighted pseudo periodic function. That is, $WPP_{\omega ap}(\mathbb{R}, \mathbb{X}, 0, \rho_1, \rho_2) = WPP_{\omega ap}(\mathbb{R}, \mathbb{X}, \rho_1, \rho_2)$, and $WPP_\omega(\mathbb{R}, \mathbb{X}, 0, \rho_1, \rho_2) = WPP_\omega(\mathbb{R}, \mathbb{X}, \rho_1, \rho_2)$.

Let $\|\cdot\|_p$ denote the norm of space $L^p(0,1;\mathbb{X})$ for $p \in [1, \infty)$, we give the following definitions.

Definition 2.7 ([6, 17]). Let $p \in [1, \infty)$. The space $BS^p(\mathbb{R}, \mathbb{X})$ of all Stepanov-like bounded functions, with the exponent p, consists of all measurable functions $f : \mathbb{R} \to \mathbb{X}$ such that $f^b \in L^\infty(\mathbb{R}, L^p(0,1;\mathbb{X}))$, where f^b is the Bochner transform of f defined by $f^b(t,s) := f(t+s), t \in \mathbb{R}, s \in [0,1]$. This is a Banach space with the norm

$$\|f\|_{S^p} = \|f^b\|_{L^\infty(\mathbb{R}, L^p)} = \sup_{t \in \mathbb{R}} \left(\int_t^{t+1} \|f(\tau)\|^p d\tau \right)^{1/p} = \sup_{t \in \mathbb{R}} \|f(t+\cdot)\|_p.$$

For $\rho_1, \rho_2 \in U_\infty$, we list the following weighted ergodic space in $BS^p(\mathbb{R}, \mathbb{X})$:

$$R(T, f, r, \rho_1, \rho_2) = \frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \rho_2(t) \left(\sup_{\theta \in [t-r, t]} \left(\int_\theta^{\theta+1} \|f_2(s)\|^p \cdot ds \right)^{\frac{1}{p}} \right) dt,$$

$$\begin{aligned} S^pWPP_0(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2) := \Bigg\{ f \in BS^p(\mathbb{R}, \mathbb{X}) : \lim_{T \to \infty} \frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \rho_2(t) \\ \left(\sup_{\theta \in [t-r, t]} \left(\int_\theta^{\theta+1} \|f(s)\|^p ds \right)^{\frac{1}{p}} \right) dt = 0 \Bigg\} \\ = \left\{ f \in BS^p(\mathbb{R}, \mathbb{X}) : \lim_{T \to \infty} R(T, f, r, \rho_1, \rho_2) = 0 \right\}, \end{aligned}$$

$$\begin{aligned} S^pWPP_0(\mathbb{R} \times \mathbb{Y}, \mathbb{X}, r, \rho_1, \rho_2) := \Big\{ f \in BS^p(\mathbb{R}, \mathbb{X}) : \lim_{T \to \infty} R(T, f(\cdot, u), r, \rho_1, \rho_2) \\ = 0 \text{ uniformly in } u \in \mathbb{Y} \Big\}. \end{aligned}$$

Definition 2.8 ([21]). Let $\rho_1, \rho_2 \in \mathbb{U}_\infty$. A function $f \in BS^p(\mathbb{X})$ is said to be Stepanov-like weighted pseudo anti-periodic of class r (or S^p-weighted pseudo anti-periodic of class r) if there exist $\phi \in S^pWPP_0(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2)$ such that $g = f - \phi$ satisfied $g(t+\omega) + g(t) = 0$ a.e $t \in \mathbb{R}$. Denote by $S^pWPP_{\omega ap}(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2)$ the collection of such function.

Definition 2.9 ([21]). Let $\rho_1, \rho_2 \in \mathbb{U}_\infty$. A function $f \in BS^p(\mathbb{X})$ is said to be Stepanov-like weighted pseudo periodic of class r (or S^p-weighted pseudo periodic of class r) if there exist $\phi \in S^pWPP_0(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2)$ such that $g = f - \phi$ satisfied $g(t+\omega) - g(t) = 0$ a.e $t \in \mathbb{R}$. Denote by $S^pWPP_\omega(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2)$ the collection of such function.

3 Results on composition theorem

The main aim of this section is to establish some new results on composition of weighted Stepanov-like pseudo periodic function of class r. We first list the following "uniform continuity condition" with respect to the L^p norm for a function h: $\mathbb{R} \times \mathbb{X} \to \mathbb{X}$ with $h(\cdot, u) \in L^p_{Loc}(\mathbb{R}, \mathbb{X})$ for each $u \in \mathbb{X}$, which was initially adopted in [16]:

(A1) For any $\varepsilon > 0$, there exists $\sigma > 0$ such that $x, y \in L^p(0,1,\mathbb{X})$ and $\|x-y\|_p < \sigma$ imply that

$$\|h(t+\cdot, x(\cdot)) - h(t+\cdot, y(\cdot))\|_p < \varepsilon, \quad t \in \mathbb{R}.$$

Definition 3.10 ([21]). Let $\rho_1, \rho_2 \in U_\infty$. The function ρ_1 is said to be equivalent to ρ_2(i.e. $\rho_1 \sim \rho_2$) if $\frac{\rho_1}{\rho_2} \in U_B$.

From arguments in [21], we know that the notation "$\sim$" is a binary equivalence relation on U_∞. For a given weight $\rho \in U_\infty$, its equivalence class can be denoted by $cl(\rho) = \{\varrho \in U_\infty : \rho \sim \varrho\}$. It is clear that $U_\infty = \bigcup_{\rho \in U_\infty} cl(\rho)$.

Let $\rho \in U_\infty$, $\tau \in \mathbb{R}$ be given, and define ρ^τ by $\rho^\tau(t) = \rho(t+\tau)$ for $t \in \mathbb{R}$. Denote [22]

$$U_T = \{\rho \in U_\infty : \rho \sim \rho^\tau \text{ for each } \tau \in \mathbb{R}\}.$$

In view of [21], we know the conclusion that for $\rho_1, \rho_2 \in U_T$ and $\inf_{T>0} \frac{\mu(T,\rho_1)}{\mu(T,\rho_2)} = \delta_0 > 0$, the space $(WPP_\omega(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2), \|\cdot\|_\infty)$ is a Banach space with sup norm.

Using similar ideas as in [8, 9, 23, 24], one can easily show the following result.

Lemma 3.1. If $\rho_1, \rho_2 \in U_T$ and $\inf_{T>0} \frac{\mu(T,\rho_1)}{\mu(T,\rho_2)} = \delta_0 > 0$, then the decomposition of weighted Stepanov-like pseudo periodic function of class r is unique.

Lemma 3.2. Let $f \in BS^p(\mathbb{R}, \mathbb{X})$, $\rho_1, \rho_2 \in U_\infty$, $\sup_{T>0} \frac{\mu(T,\rho_2)}{\mu(T,\rho_1)} < \infty$, then $f \in S^pWPP_0(\mathbb{R}, L^p(0,1,\mathbb{X}), r, \rho_1, \rho_2)$ if and only if for every $\varepsilon > 0$,

$$\lim_{T\to\infty} \frac{1}{\mu(T,\rho_1)} \int_{M(T,\varepsilon,f)} \rho_2(t) \cdot dt = 0,$$

where $M(T,\varepsilon,f) = \left\{ t \in [-T,T] : \sup_{\theta\in[t-r,t]} \left(\int_\theta^{\theta+1} \|f(s)\|^p ds \right)^{\frac{1}{p}} \geq \varepsilon \right\}$.

Proof. Sufficiency: From the statement of lemma it is clear that $\|f\|_{S^p} < \infty$ and for any $\varepsilon > 0$, there exist $T_0 > 0$ such that $T > T_0$,

$$\frac{1}{\mu(T,\rho_1)} \int_{M(T,\varepsilon,f)} \rho_2(t) dt < \frac{\varepsilon}{\mathscr{M}+1}, \quad \mathscr{M} = \|f\|_{S^p}.$$

Thus

$$\begin{aligned}
&\frac{1}{\mu(T,\rho_1)} \int_{-T}^{T} \left(\sup_{\theta\in[t-r,t]} \left(\int_\theta^{\theta+1} \|f(s)\|^p ds \right)^{\frac{1}{p}} \rho_2(t) \right) dt \\
&:= \frac{1}{\mu(T,\rho_1)} \int_{M(T,\varepsilon,f)} \left(\sup_{\theta\in[t-r,t]} \left(\int_\theta^{\theta+1} \|f(s)\|^p ds \right)^{\frac{1}{p}} \rho_2(t) \right) dt \\
&+ \frac{1}{\mu(T,\rho_1)} \int_{[-T,T]\setminus M(T,\varepsilon,f)} \left(\sup_{\theta\in[t-r,t]} \left(\int_\theta^{\theta+1} \|f(s)\|^p ds \right)^{\frac{1}{p}} \rho_2(t) \right) dt \\
&\leq \frac{\mathscr{M}}{\mu(T,\rho_1)} \int_{M(T,\varepsilon,f)} \rho_2(t) dt + \frac{\varepsilon}{\mu(T,\rho_1)} \int_{-T}^{T} \rho_2(t) dt \\
&\leq \frac{\mathscr{M}\varepsilon}{\mathscr{M}+1} + \frac{\varepsilon}{\mu(T,\rho_1)} \int_{-T}^{T} \rho_2(t) dt \\
&\leq \varepsilon + \sup_{T>0} \frac{\mu(T,\rho_2)}{\mu(T,\rho_1)} \varepsilon.
\end{aligned}$$

So

$$\lim_{T\to\infty} \frac{1}{\mu(T,\rho_1)} \int_{-T}^{T} \left(\sup_{\theta\in[t-r,t]} \left(\int_\theta^{\theta+1} \|f(s)\|^p ds \right)^{\frac{1}{p}} \rho_2(t) \right) dt = 0,$$

that is $f \in S^pWPP_0(\mathbb{R}, L^p(0,1,\mathbb{X}), r, \rho_1, \rho_2)$.

Necessity: Suppose on the contrary that exist $\varepsilon_0 > 0$, such that

$$\frac{1}{\mu(T,\rho_1)} \int_{M(T,\varepsilon_0,f)} \rho_2(t) dt$$

does not converge to 0 as $T \to \infty$, then there exist $\delta > 0$ such that for each n,

$$\frac{1}{\mu(T,\rho_1)} \int_{M(T_n,\varepsilon_0,f)} \rho_2(t)dt \geq \delta, \text{ for some } T_n \geq n.$$

Then

$$\begin{aligned}
&\frac{1}{\mu(T_n,\rho_1)} \int_{-T_n}^{T_n} \left(\sup_{\theta\in[t-r,t]} \left(\int_{\theta}^{\theta+1} \|f(s)\|^p ds \right)^{\frac{1}{p}} \rho_2(t) \right) dt \\
&\geq \frac{1}{\mu(T_n,\rho_1)} \int_{M(T_n,\varepsilon_0,f)} \left(\sup_{\theta\in[t-r,t]} \left(\int_{\theta}^{\theta+1} \|f(s)\|^p ds \right)^{\frac{1}{p}} \rho_2(t) \right) dt \\
&\geq \frac{\varepsilon_0}{\mu(T_n,\rho_1)} \int_{M(T_n,\varepsilon_0,f)} \rho_2(t)dt \\
&\geq \varepsilon_0\delta,
\end{aligned}$$

which contradicts the fact that $f \in S^pWPP_0(\mathbb{R}, L^p(0,1,\mathbb{X}), r, \rho_1, \rho_2)$, and the proof is complete. □

Lemma 3.3 ([16]). Let h be the function in (A1), and x: $\mathbb{R} \to \mathbb{X}$ with $\overline{x(\mathbb{R})}$ compact. For $\varepsilon > 0$, there exist a finite set $\{x_k\}_{k=1}^m \subset \overline{x(\mathbb{R})}$ such that

$$\|h(t+\cdot, x(t+\cdot))\|_p < \varepsilon + m \sup_{1\leq k\leq m} \|h(t+\cdot, x_k\|_p, \quad t \in \mathbb{R}.$$

Next, we establish main composition results for weighted Stepanov-like pseudo periodic function.

Theorem 3.1. Assume that $\rho_1, \rho_2 \in U_\infty, r \geq 0$, $f = g + \phi \in S^pWPP_\omega(\mathbb{R} \times \mathbb{X}, \mathbb{X}, r, \rho_1, \rho_2)$, $h = h_1 + h_2 \in S^pWPP_\omega(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2)$ with $\overline{h_1(\mathbb{R})}$ compact, $g(t+\omega) - g(t) = 0$, $h_1(t+\omega) - h_1(t) = 0$. Assume g satisfies (A1), ϕ satisfies (A1) and $\{f(\cdot,z) : z \in \mathbb{K}\}$ is bounded in $S^pWPP_\omega(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2)$ for any bounded $\mathbb{K} \subset \mathbb{X}$, then $f(\cdot, h(\cdot)) \in S^pWPP_\omega(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2)$.

Proof. Let $H_1(t) = g\,(t, h_1(t))\,, H_2(t) = f\,(t, h(t)) - f\,(t, h_1(t))\,, H_3(t) = \phi\,(t, h_1(t))\,, t \in \mathbb{R}$. Then

$$f(t,h(t)) = g\,(t,h_1(t)) + f\,(t,h(t)) - f\,(t,h_1(t)) + \phi\,(t,h_1(t)) = H_1(t) + H_2(t) + H_3(t).$$

Since

$$g(t+\omega, h_1(t+\omega)) = g(t, h_1(t+\omega)) = g(t, h_1(t)),$$

we have

$$H_1(t+\omega) - H_1(t) = 0.$$

Thus we need only to prove $H_2, H_3 \in S^pWPP_0(\mathbb{R}, L^p(0,1,\mathbb{X}), r, \rho_1, \rho_2)$.

It is easy to see that $H_2 \in BS^p(\mathbb{R}, \mathbb{X})$ since $\{f(\cdot,z) : z \in \mathbb{K}\}$ is bounded in $S^pWPP_\omega(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2)$ for any bounded $\mathbb{K} \subset \mathbb{X}$. Noticing that f satisfies (A1) since g and ϕ satisfy (A1), for $\varepsilon > 0$, let $\sigma > 0$ be given by (A1), then

$$\|H_2(t+\cdot)\|_p = \|f(t+\cdot, h(\cdot)) - f(t+\cdot, h_1(\cdot))\|_p < \varepsilon, \text{ for } \|h_2(t+\cdot)\| < \sigma.$$

This implies that $M_{T,\varepsilon}(H_2) \subset M_{T,\sigma}(h_2)$ by the notation defined in Lemma 3.2. Meanwhile since $h_2 \in S^pWPP_0(\mathbb{R}, L^p(0,1,\mathbb{X}), r, \rho_1, \rho_2)$, $\sup_{T>0} \frac{\mu(T,\rho_2)}{\mu(T,\rho_1)} < \infty$ by Lemma 3.2,

$$\lim_{T\to\infty} \frac{1}{\mu(T,\rho_1)} \int_{M(T,\sigma,h_2)} \rho_2(t) \cdot dt = 0.$$

Thus

$$\lim_{T\to\infty} \frac{1}{\mu(T,\rho_1)} \int_{M(T,\varepsilon,H_2)} \rho_2(t) \cdot dt = 0,$$

which shows that $H_2 \in S^pWPP_0(\mathbb{R}, L^p(0,1,\mathbb{X}), r, \rho_1, \rho_2)$.

For $\varepsilon > 0$, let σ be given by (A1) with ϕ, in view of Lemma 3.3, there is a finite set $\{x_k\}_{k=1}^m \subset \overline{h_1(\mathbb{R})}$ such that for $t \in \mathbb{R}$,

$$\|\phi(t+\cdot, h_1(t+\cdot))\|_p < \varepsilon + m \sup_{1\leq k\leq m} \|\phi(t+\cdot, x_k\|_p, \quad t \in \mathbb{R},$$

so

$$\|\phi(t,h_1(t)\|_p < \varepsilon + m \sup_{1\le k\le m} \|\phi(t,x_k\|_p, \quad t\in\mathbb{R}.$$

Since $\phi(\cdot,x)\in S^pWPP_0(\mathbb{R},L^p(0,1,\mathbb{X}),r,\rho_1,\rho_2)$, for each $x\in\mathbb{X}$, there is $T>T_0$, $1\le K\le m$,

$$\begin{aligned}R\left(T,\phi(\cdot,x_k),r,\rho_1,\rho_2\right) &= \frac{1}{\mu(T,\rho_1)}\int_{-T}^{T}\left(\sup_{\theta\in[t-r,t]}\left(\int_\theta^{\theta+1}\|\phi(s,x_k)\|^p ds\right)^{\frac{1}{p}}\right)\rho_2(t)\cdot dt\\ &< \frac{\varepsilon}{m}.\end{aligned}$$

Then for $T>T_0$,

$$\begin{aligned}R\left(T,H_3,r,\rho_1,\rho_2\right) :=& \frac{1}{\mu(T,\rho_1)}\int_{-T}^{T}\left(\sup_{\theta\in[t-r,t]}\left(\int_\theta^{\theta+1}\|\phi(s,h_1(\theta))\|^p ds\right)^{\frac{1}{p}}\right)\rho_2(t)\cdot dt\\ &\le \varepsilon + m\sup_{1\le k\le m}\frac{1}{\mu(T,\rho_1)}\int_{-T}^{T}\left(\sup_{\theta\in[t-r,t]}\left(\int_\theta^{\theta+1}\|\phi(s,x_k)\|^p ds\right)^{\frac{1}{p}}\right)\\ &\rho_2(t)\cdot dt\\ &= \varepsilon + mR\left(T,\phi(\cdot,x_k),r,\rho_1,\rho_2\right)\\ &= \varepsilon + m\cdot\frac{\varepsilon}{m}\\ &= 2\varepsilon.\end{aligned}$$

This yields that

$$\lim_{T\to\infty} R(T,H_3,r,\rho_1,\rho_2) = 0.$$

That is $H_3\in S^pWPP_0(\mathbb{R},L^P(0,1,\mathbb{X}),r,\rho_1,\rho_2)$. The proof is complete. □

According to Theorem 3.1, we can obtain the following corollaries.

Corollary 3.1. Assume that $\rho_1,\rho_2\in U_\infty, r\ge 0$, $f=g+\phi\in S^pWPP_{\omega ap}(\mathbb{R}\times\mathbb{X},\mathbb{X},r,\rho_1,\rho_2)$, $h=h_1+h_2\in S^pWPP_{\omega ap}(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2)$ with $\overline{h_1(\mathbb{R})}$ compact. $g(t+\omega)+g(t)=0$, $h_1(t+\omega)+h_1(t)=0$. Assume g satisfies (A1), ϕ satisfies (A1) and $\{f(\cdot,z): z\in\mathbb{K}\}$ is bounded in $S^pWPP_{\omega ap}(\mathbb{R}\times\mathbb{X},\mathbb{X},r,\rho_1,\rho_2)$ for any bounded $\mathbb{K}\subset\mathbb{X}$, then $f(\cdot,h(\cdot))\in S^pWPP_{\omega ap}(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2)$.

Corollary 3.2. Assume that $\rho_1,\rho_2\in U_\infty, r\ge 0$, $f=g+\phi\in S^pWPP_{\omega}(\mathbb{R}\times\mathbb{X},\mathbb{X},r,\rho_1,\rho_2)$, $h\in WPP_\omega(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2)$. Assume g satisfies (A1), ϕ satisfies (A1) and $\{f(\cdot,z): z\in\mathbb{K}\}$ is bounded in $S^pWPP_{\omega}(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2)$ for any bounded $\mathbb{K}\subset\mathbb{X}$, then $f(\cdot,h(\cdot))\in S^pWPP_{\omega}(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2)$.

Corollary 3.3. Assume that $\rho_1,\rho_2\in U_\infty, r\ge 0$, $f=g+\phi\in S^pWPP_{\omega ap}(\mathbb{R}\times\mathbb{X},\mathbb{X},r,\rho_1,\rho_2)$, $h\in WPP_{\omega ap}(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2)$. Assume g satisfies (A1), ϕ satisfies (A1) and $\{f(\cdot,z): z\in\mathbb{K}\}$ is bounded in $S^pWPP_{\omega ap}(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2)$ for any bounded $\mathbb{K}\subset\mathbb{X}$, then $f(\cdot,h(\cdot))\in S^pWPP_{\omega ap}(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2)$.

4 Weighted pseudo periodic mild solution

In this section, we deal with weighted pseudo periodic mild solutions to the problem (1.1). We list the following basic assumptions:

(A2) $\rho_1,\rho_2\in U_T$, $\inf_{T>0}\frac{\mu(T,\rho_2)}{\mu(T,\rho_1)}>0$ and $\sup_{T>0}\frac{\mu(T,\rho_2)}{\mu(T,\rho_1)}<\infty$.

(A3) The operator $T(t)$ generated by A is exponentially stable, that is, there exist constants M, $c>0$ such that $\|T(t)\|\le Me^{-ct}$ for $t\ge 0$.

(A4) $f\in S^pWPP_\omega(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2)$, and there exist a positive constant L_f such that for $\psi_i\in\mathfrak{B}$, $i=1,2$, $\|f(t,\psi_1)-f(t,\psi_2)\|_p\le L_f\|\psi_1-\psi_2\|_{\mathfrak{B}}$.

Let $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Denote

$$\alpha_0 = M\left(\frac{e^{qc-1}}{qc}\right)^{\frac{1}{q}}, \quad \alpha = \alpha_0 \sum_{k=1}^{\infty} e^{-ck}.$$

Under the condition (A3), we give the following definition.

Definition 4.11. A function u: $\mathbb{R} \to \mathbb{X}$ is said to be a mild solution to the problem (1.1) if

$$u(t) = \int_{-\infty}^{t} T(t-s) f(s, u_s) ds,$$

for all $t \in \mathbb{R}$.

Lemma 4.4. Let $\rho_1, \rho_2 \in U_T$, $u \in S^p WPP_\omega(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2)$, then u_t belongs to $S^p WPP_\omega(\mathbb{R}, \mathfrak{B}, r, \rho_1, \rho_2)$.

Proof. Suppose that $u = \alpha + \beta$, where $\alpha \in P_\omega(\mathbb{R}, \mathbb{X})$ and $\beta \in S^p WPP_0(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2)$, then $u_t = \alpha_t + \beta_t$ and $\alpha_t \in P_\omega(\mathbb{R}, \mathfrak{B}, r, \rho_1, \rho_2)$. On the other hand, for $T > 0$, we see that

$$\begin{aligned}
&\frac{1}{\mu(T,\rho_1)} \int_{-T}^{T} \left(\sup_{\theta \in [t-r,t]} \left(\int_{\theta}^{\theta+1} \sup_{\tau \in [-r,0]} \|\beta(s+\tau)\|^p ds \right)^{\frac{1}{p}} \right) \rho_2(t) dt \\
&\leq \frac{1}{\mu(T,\rho_1)} \int_{-T}^{T} \left(\sup_{\theta \in [t-2r,t]} \left(\int_{\theta}^{\theta+1} \|\beta(s)\|^p ds \right)^{\frac{1}{p}} \right) \rho_2(t) dt \\
&\leq \frac{1}{\mu(T,\rho_1)} \int_{-T}^{T} \left(\sup_{\theta \in [t-2r,t-r]} \left(\int_{\theta}^{\theta+1} \|\beta(s)\|^p ds \right)^{\frac{1}{p}} \right. \\
&\left. + \sup_{\theta \in [t-r,t]} \left(\int_{\theta}^{\theta+1} \|\beta(s)\|^p ds \right)^{\frac{1}{p}} \right) \rho_2(t) dt \\
&\leq \frac{1}{\mu(T,\rho_1)} \int_{-T-r}^{T-r} \left(\sup_{\theta \in [t-r,t]} \left(\int_{\theta}^{\theta+1} \|\beta(s)\|^p ds \right)^{\frac{1}{p}} \right) \rho_2(t+\tau) dt \\
&+ \frac{1}{\mu(T,\rho_1)} \int_{-T}^{T} \left(\sup_{\theta \in [t-r,t]} \left(\int_{\theta}^{\theta+1} \|\beta(s)\|^p ds \right)^{\frac{1}{p}} \right) \rho_2(t) dt \\
&\leq \frac{\mu(T+r,\rho_1)}{\mu(T,\rho_1)} \cdot \frac{1}{\mu(T+r,\rho_1)} \\
&\int_{-T-r}^{T-r} \left(\sup_{\theta \in [t-r,t]} \left(\int_{\theta}^{\theta+1} \|\beta(s)\|^p ds \right)^{\frac{1}{p}} \right) \rho_2(t) \frac{\rho_2(t+r)}{\rho_2(t)} dt \\
&+ \frac{1}{\mu(T,\rho_1)} \int_{-T}^{T} \left(\sup_{\theta \in [t-r,t]} \left(\int_{\theta}^{\theta+1} \|\beta(s)\|^p ds \right)^{\frac{1}{p}} \right) \rho_2(t) dt.
\end{aligned}$$

The fact $\rho_1, \rho_2 \in U_T$ implies that there exists $\eta > 0$ such that

$$\frac{\rho_1(t+r)}{\rho_1(t)} \leq \eta, \quad \frac{\rho_1(t-r)}{\rho_1(t)} \leq \eta, \quad \frac{\rho_2(t-r)}{\rho_2(t)} \leq \eta.$$

For $T > r$,

$$\begin{aligned}
\mu(T+r,\rho_1) &= \int_{-T-r}^{T-r} \rho_1(t) dt + \int_{T-r}^{T+r} \rho_1(t) dt \\
&\leq \int_{-T-r}^{T-r} \rho_1(t) dt + \int_{-T+r}^{T+r} \rho_1(t) dt \\
&= \int_{-T}^{T} \rho_1(t-r) dt + \int_{-T}^{T} \rho_1(t+r) dt \\
&\leq 2\eta \mu(T,\rho_1),
\end{aligned}$$

then

$$\begin{aligned}
&\frac{1}{\mu(T,\rho_1)}\int_{-T}^{T}\left(\sup_{\theta\in[t-r,t]}\left(\int_{\theta}^{\theta+1}\sup_{\tau\in[-r,0]}\|\beta(s+\tau)\|^p ds\right)^{\frac{1}{p}}\right)\rho_2(t)dt\\
&\le \frac{2\eta^2}{\mu(T+r,\rho_1)}\int_{-T-r}^{T+r}\left(\sup_{\theta\in[t-r,t]}\left(\int_{\theta}^{\theta+1}\|\beta(s)\|^p ds\right)^{\frac{1}{p}}\right)\rho_2(t)dt\\
&+\frac{1}{\mu(T,\rho_1)}\int_{-T}^{T}\left(\sup_{\theta\in[t-r,t]}\left(\int_{\theta}^{\theta+1}\|\beta(s)\|^p ds\right)^{\frac{1}{p}}\right)\rho_2(t)dt.
\end{aligned}$$

Note that $\beta \in S^pWPP_0(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2)$, $\rho_1,\rho_2 \in U_T$, then $\beta_t \in S^pWPP_0(\mathbb{R},\mathfrak{B},r,\rho_1,\rho_2)$. Therefore $u_t \in S^pWPP_\omega(\mathbb{R},\mathfrak{B},r,\rho_1,\rho_2)$. □

From the proof of Lemma 4.4, we can easily deduce the following corollary.

Corollary 4.4. ([21, Lemma 2.14]) Let $\rho_1,\rho_2 \in U_T, u \in WPP_\omega(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2)$, then u_t belong to $WPP_\omega(\mathbb{R},\mathfrak{B},r,\rho_1,\rho_2)$.

Lemma 4.5 ([21])**.** Let $\phi_n \to \phi$ uniformly on $\mathbb{R}$ where each $\phi_n \in WPP_0(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2),\rho_1,\rho_2 \in U_\infty$, if $\sup_{T>0}\frac{\mu(T,\rho_1)}{\mu(T,\rho_2)} < \infty$, then $\phi \in WPP_0(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2)$.

Lemma 4.6. Assume that (A2)–(A3) hold, if $\phi \in S^pWPP_\omega(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2)$, then

$$(\Gamma\phi)(t) = \int_{-\infty}^{t} T(t-s)\phi(s)ds \in WPP_\omega(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2).$$

Proof. By $\phi \in S^pWPP_\omega(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2)$, we let $\phi(s) = \phi_1(s)+\phi_2(s)$, where $\phi_2 \in S^pWPP_0(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2)$ and $\phi_1(t+\omega)-\phi_1(t) = 0$ a.e. $t \in \mathbb{R}$, then

$$(\Gamma\phi)(t) = \int_{-\infty}^{t} T(t-s)\phi_1(s)ds + \int_{-\infty}^{t} T(t-s)\phi_2(s)ds = (\Gamma_1\phi_1)(t) + (\Gamma_2\phi_2)(t).$$

First, we show that $\Gamma_2\phi_2 \in WPP_0(\mathbb{R},\mathbb{X},r,\rho_1,\rho_2)$. Consider the integrals

$$X_n(t) = \int_{t-n}^{t-n+1} T(t-s)\phi_2(s)ds.$$

Fix $n \in \mathbb{N}$ and $t \in \mathbb{R}$, we have

$$\begin{aligned}
\|X_n(t+h)-X_n(t)\| &\le \int_{n-1}^{n}\|T(s)(\phi_2(t+h-s)-\phi_2(t-s))\|ds\\
&\le M\int_{t-n}^{t-n+1}\|\phi_2(s+h)-\phi_2(s)\|ds\\
&\le M\left(\int_{t-n}^{t-n+1}\|\phi_2(s+h)-\phi_2(s)\|^p ds\right)^{\frac{1}{p}}.
\end{aligned}$$

In view of $\phi_2 \in L^p_{loc}(\mathbb{R},\mathbb{X})$, we get

$$\lim_{h\to 0}\int_{t-n}^{t-n+1}\|\phi_2(s+h)-\phi_2(s)\|^p ds = 0,$$

which yields $\lim_{h\to 0}\|X_n(t+h)-X_n(t)\|$. This means that $X_n(t)$ is continuous.

By Hölder's inequality, one has

$$\begin{aligned}
\|X_n(t)\| &\le \int_{n-1}^{n} \|T(s)\phi_2(t-s)\|ds \\
&\le \int_{n-1}^{n} Me^{-cs}\|\phi_2(t-s)\|ds \\
&\le Me^{-c(n-1)} \int_{n-1}^{n} \|\phi_2(t-s)\|ds \\
&\le Me^{-c(n-1)} \int_{t-n}^{t-n+1} \|\phi_2(s)\|ds \\
&\le Me^{-c(n-1)} \left(\int_{t-n}^{t-n+1} \|\phi_2(s)\|^p ds\right)^{\frac{1}{p}} \\
&\le Me^{-c(n-1)}\|\phi_2\|_{S^p}.
\end{aligned}$$

Since

$$\sum_{n=1}^{\infty} Me^{-c(n-1)}\|\phi_2\|_{S^p} \le \frac{M}{1-e^{-\delta}}\|\phi_2\|_{S^p} < +\infty,$$

it follows that $\sum_{n=1}^{\infty} X_n(t)$ converges uniformly on $\mathbb{R}$. Let $X(t) = \sum_{n=1}^{\infty} X_n(t)$ for $t \in \mathbb{R}$. Then

$$X(t) = (\Gamma_2\phi_2)(t) = \int_{-\infty}^{t} T(t-s)\phi_2(s)ds,\ t \in \mathbb{R}\,.$$

It is obvious that $X(t) \in BC(\mathbb{R}, \mathbb{X})$. So, we only need to show that

$$\lim_{T\to\infty} \frac{1}{\mu(T,\rho_1)} \int_{-T}^{T} \rho_2(t) \left(\sup_{\theta\in[t-r,t)} \|X(\theta)\|\right) dt = 0. \tag{4.2}$$

In fact, by Hölder's inequality,

$$\begin{aligned}
\|X_n(t)\| &\le \int_{n-1}^{n} Me^{-cs}\|\phi_2(t-s)\|ds \\
&\le \widetilde{M} \int_{t-n}^{t-n+1} \|\phi_2(s)\|ds \\
&\le \widetilde{M} \left(\int_{t-n}^{t-n+1} \|\phi_2(s)\|^p ds\right)^{\frac{1}{p}},
\end{aligned}$$

for some constant $\widetilde{M} > 0$, then

$$\begin{aligned}
&\frac{1}{\mu(T,\rho_1)} \int_{-T}^{T} \rho_2(t) \left(\sup_{\theta\in[t-r,t)} \|X_n(\theta)\|\right) dt \\
&\le \frac{\widetilde{M}}{\mu(T,\rho_1)} \int_{-T}^{T} \rho_2(t) \left(\sup_{\theta\in[t-r,t)} \left(\int_{\theta-n}^{\theta-n+1} \|\phi_2(s)\|^p ds\right)^{\frac{1}{p}}\right) dt,
\end{aligned}$$

and hence $X_n \in WPP_0(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2)$ since $\phi_2 \in S^pWPP_0(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2)$. By Lemma 4.5, the equation (4.2) holds, whence $\Gamma_2\phi_2 \in WPP_0(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2)$.

From $\phi_1(t+\omega) - \phi_1(t) = 0$ a.e. $t \in \mathbb{R}$, one has

$$(\Gamma_1\phi_1)(t+\omega) = \int_{-\infty}^{t+\omega} T(t+\omega-s)\phi_1(s)ds = (\Gamma_1\phi_1)(t),\ \text{a.e. } t \in \mathbb{R}\,.$$

Hence $\Gamma\phi \in WPP_\omega(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2)$. This completes of the proof. □

Theorem 4.2. Let conditions (A2)–(A4) hold, then the problem (1.1) has a unique weighted pseudo periodic mild solution if $\alpha L_f < 1$.

Proof. Define $\mathcal{F} : WPP_\omega(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2) \to WPP_\omega(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2)$ as $(\mathcal{F}u)(t) = \int_{-\infty}^{t} T(t-s)f(s, u_s)ds$.

If $u \in WPP_\omega(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2)$, by Corollary 4.4 and Corollary 3.2, $f(s, u_s) \in S^p WPP_\omega\,(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2)$. Owing to Lemma 4.6, it is not difficult to see that $\mathcal{F}(WPP_\omega(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2)) \subseteq WPP_\omega(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2)$.

For any $u, v \in WPP_\omega(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2)$, we have

$$\begin{aligned}
\|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)\| &= \left\| \int_{-\infty}^{t} T(t-s)(f(s,u_s) - f(s,v_s))ds \right\| \\
&= \left\| \int_{0}^{\infty} T(s)\left(f(t-s, u_{t-s}) - f(t-s, v_{t-s})\right) ds \right\| \\
&\leq M \sum_{k=1}^{\infty} \left(\int_{k-1}^{k} e^{-cqs} ds \right)^{\frac{1}{q}} \left(\int_{k-1}^{k} \|f(s,u_s) - f(s,v_s)\|^p ds \right)^{\frac{1}{p}} \\
&= \alpha_0 \sum_{k=1}^{\infty} e^{-ck} \|f(t+k-2+\cdot, u_{t+k-2+\cdot}) - f(t+k-2+\cdot, \\
&\quad v_{t+k-2+\cdot})\|_p \\
&= \alpha \|f(t+k-2+\cdot, u_{t+k-2+\cdot}) - f(t+k-2+\cdot, v_{t+k-2+\cdot})\|_p \\
&\leq \alpha L_f \|u_{t+k-2+\cdot} - v_{t+k-2+\cdot}\|_{\mathfrak{B}} \\
&\leq \alpha L_f \|u(t+k-2+\cdot) - v(t+k-2+\cdot)\| \\
&= \alpha L_f \|u - v\|,
\end{aligned}$$

then $\mathcal{F}$ is a contraction since $\alpha L_f < 1$. By the Banach contraction mapping principle, $\mathcal{F}$ has a unique fixed point in $WPP_\omega\,(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2)$, which is the unique WPP_ω mild solution to the problem (1.1). $\square$

Corollary 4.5. Assume that conditions (A2), (A3) and the following condition (A4′) are satisfied:

(A4′) $f \in S^p WPP_{\omega ap}(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2)$, and there exist a positive constant L_f such that for $\psi_i \in \mathfrak{B}$, $i = 1, 2$, $\|f(t, \psi_1) - f(t, \psi_2)\|_p \leq L_f \|\psi_1 - \psi_2\|_{\mathfrak{B}}$.

Then the problem (1.1) admits a unique weighted pseudo anti-periodic mild solution provided that $\alpha L_f < 1$.

Example 4.1. Consider the partial differential equation which was inspired by [21]

$$\begin{gathered}
\frac{\partial}{\partial t} u(t, \xi) = \frac{\partial^2}{\partial \xi^2} u(t, \xi) + a_0(t)u(t, \xi) + \int_{-r}^{0} a_1(s)u(t+s, \xi), \quad (t, \xi) \in \mathbb{R}, \\
u(t, 0) = u(t, \pi) = 0,
\end{gathered} \tag{4.3}$$

where $a_0 \in WPP_\omega(\mathbb{R}, \mathbb{X}, r, \rho_1, \rho_2)$, $\rho_1 = e^t$, $\rho_2 = 1 + t^2$.

Let $X = \left(L^2([0, \pi], \mathbb{R}), \|\cdot\|_{L^2}\right)$ and define the operator A on $\mathbb{X}$. By $Au = u''$ with

$$D(A) = \left\{ u \in \mathbb{X} : u'' \in \mathbb{X}, u(0) = u(\pi) = 0 \right\}.$$

It is well-known that A is the infinitesimal generator of C_0-semigroup $(T(t))_{t>0}$ on $\mathbb{X}$ such that $\|T(t)\| \leq e^{-t}$ for every $t \geq 0$.

Define the function $f : \mathbb{R} \times \mathfrak{B} \to \mathbb{X}$ by

$$f(t, \psi)(\xi) = a_0(t)\psi(0, \xi) + \int_{-r}^{0} a_1(s)\psi(s, \xi)ds,$$

then the equation (4.3) can be rewritten as an abstract system in the form (1.1), where $u(t) = u(t, \cdot)$. Moreover, we can show

$$\|f(t, \cdot)\| \leq \|a_0\| + \sqrt{r\left(\int_{-r}^{0} a^2(s)ds\right)}, \quad t \in \mathbb{R}.$$

In view of Theorem 4.2, the equation (4.3) has a unique weighted pseudo periodic mild solution whenever $\|a_0\| + \sqrt{r\left(\int_{-r}^{0} a^2(s)ds\right)} < 1$.

References

[1] S. Abbas, D. Bahuguna; Almost periodic solutions of neutral functional differential equations, Comput. Math. Appl. **55** (2008), 2593-2601.

[2] S. Abbas; Pseudo almost automorphic solutions of fractional order neutral differential equation, Semigroup Forum **81** (2010), 393-404.

[3] N. S. Al-lslam, S. M. Alsulami, T. Diagana; Extience of weighted pseudo anti-periodic solutions to some non-autonomous differential equations, Appl. Math. Comput. **218** (2012), 6536-6548.

[4] J. Cao, Q. Yang, Z. Huang; Existence of anti-periodic mild solutions for a class of semilinear fractional differential equations, Commun. Nonlinear Sci. Numer. Simulat. **17** (2012), 277-283.

[5] J. Cao, Z. Huang, C. Zeng; Weighted pseudo almost automorphic classical solutions and optimal mild solutions for fractional differential equations and application in fractional reactionCdiffusion equations, J. Math. Chem. **52** (2014), 1984-2012.

[6] T. Diagana, G. M. N'Guérékata; Stepanov-like almost automorphic functions and applications to some semilinear equations, Appl. Anal. **86** (2007), 723-733.

[7] T. Diagana, E. M. Hernández; Extience and uniqueness of pseudo almost periodic solutions to some abstract partial neutral functional-differential equations and applications, J. Math. Anal. Appl. **327** (2007), 776-791.

[8] T. Diagana; Double-weighted pseudo almost periodic functions, Afr. Diaspora J. Math. **12** (2011), 121-136.

[9] T. Diagana; Existence of doubly-weighted pseudo almost periodic solutions to nonautonomous differential equation, Electron. J. Diff. Equ. **28** (2011), 1-15.

[10] T. Diagana, Almost Automorphic Type and Almost Periodic Type Functions in Abstract Spaces, Springer, New York, 2013.

[11] H. S. Ding, J. Liang, T. J. Xiao; Some properties Stepanov-like almost automorphic functions and applications to abstract evolution equations, Appl. Anal. **88** (2009), 1079-1091.

[12] H. S. Ding, J. Liang, T. J. Xiao; Almost automorphic solutions to nonautonomous semilinear evolution equations in Banach space, Appl. Anal. **73** (2010), 1426-1438.

[13] K. Ezzinbi, H. Toure, I. Zabsonre; Pseudo almost periodic solutions of class r for neutral partial functional differential equations, Afrika Mat. **24** (2013), 691-704.

[14] Z. B. Fan, J. Liang, T. J. Xiao; On Stepanov-like (pseudo) almost automorphic functions, Nonlinear Anal. **74** (2011), 2853-2861.

[15] Z. B. Fan, J. Liang, T. J. Xiao; Compostion of Stepanov-like pseudo almost automorphic functions and applications to nonautomous evolution equations, Nonlinear Anal. **13** (2012), 131-140.

[16] C. H. Hua, H. X. Li; Composition of S^p-weighted pseudo almost automorphic function and applications, Electron. J. Diff. Equ. **236** (2014), 1-15.

[17] G. M. N'Guérékata; Almost Automorphic and AlmostPeriodic Functions in Abstract Space, Kluwer Academic, NewYork, 2001.

[18] G. M. N'Guérékata; Topics in Almost Automorphy, Springer, New York, 2005.

[19] R. Sakthivel, P. Revathi, S. Marshal Anthoni; Existence of pseudo almost automorphic mild solutions to stochastic fractional differential equations, Nonlinear Anal. **75** (2012), 3339-3347.

[20] Z. N. Xia; Weighted Stepanov-like pseudo periodicity and applications, Abstr. Appl. Anal. Volume (2014), Arcicle ID 980869, 14 pages.

[21] Z. N. Xia; Pseudo almost automorphic solutions of some nonlinear integro-differential equations, Electron. J. Diff. Equ. **191** (2014), 1-16.

[22] L. Zhang, H. Li; Weighted pseudo almost periodic solutions for some abstract differential equations with uniform continuity, Bull. Aust. Math. Soc.**82** (2010), 424-436.

[23] R. Zhang, Y. K. Chang, G. M. N'Guérékata; New composition theorems of Stepanov-like almost automorphic functions and applications to nonautonomous evolution equations, Nonlinear Anal. RWA. **13** (2012), 2866-2879.

[24] R. Zhang, Y. K. Chang, G. M. N'Guérékata; Weighted pseudo almost automorphic solutions for non-autonomous neutral functional sifferential equations with infinite delay(in chineses), Sci. Math. **43** (2013), 273-292.

[25] H. Zhou, Z. Zhou, Q. Wang; Positive almost periodic solution for a class of LasotaCWazewska model with infinite delays, Appl. Math. Comput. **218** (2011), 4501-4506.

[26] H. Zhou, Z. Zhou, Z. Qiao; Mean-square almost periodic solution for impulsive stochastic Nicholsons blowflies model with delays, Appl. Math. Comput. **219** (2013), 5943-5948.

Received: February 22, 2016; *Accepted*: March 1, 2016

UNIVERSITY PRESS

Website: http://www.malayajournal.org/