

# Harvesting Model for Fishery Resource with Reserve Area and Modified Effort Function

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## Abstract

The aim of this paper is to study the dynamics of fishery resource system in an aquatic region consisting of two zones. One zone is free for fishing and other is restricted for any kind of fishing. In the proposed harvesting model, a modified effort function  $E$  is considered, which depends on the density effect of fish population. The criteria for local stability, global stability and instability are established for the proposed system. The theoretical results obtained are illustrated with numerical simulations in the last section.

*Keywords:* Fishery resource, Fishery effort, Stability, Harvesting.

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## 1 Introduction

Renewable resources like fishery, forestry and oil exploration are important sources of food and materials which play an important role for the survival and growth of biological population. The continuous and unplanned harvesting/exploration of these resources may lead to the extinction of the resources and it will further effect the survival of biological population. Therefore, its conservation and management are important fields to be analyzed through research. During the last few decades, there has been a considerable interest in modeling the dynamics of fishery resource systems [1, 2, 5–10].

Chaudhuri [5] in 1979 explained the problem of combined harvesting of two competing fish species and showed that the open-access fishery may possess a bionomic equilibrium which drives one species to extinction. Biological and economic interpretations of the results associated with the optimal equilibrium solution are discussed. Kitabatake [8] in 1982 studied a dynamical model for fishery resources with predator-prey relationships based on observational data for Lake Kasumigaura, Japan. He showed that the extensive use of diesel-powered trawling, which enables the large-scale catch of prey species in comparison with the traditional method of sailing trawling, may lead to the extinction of predator as well as prey species.

Mesterton-Gibbons [9] in 1996 described a technique to get the optimal harvesting policy for a Lotka-Volterra ecosystem of two interdependent populations, when the harvest rate is proportional to harvesting effort. The author explained that if two species coexist in the absence of harvesting, one species may be driven to extinction, if it is more catchable than the other. Fan and Wang [7] in 1998 examined the exploitation of single population modeled by time-dependent Logistic equation with periodic coefficients. Pradhan and Chaudhuri [10] in 1999 explained the dynamic reaction model of a fishery consisting of two competing species, each of which obeys the logistic law of growth. The authors use capital theoretic approach to formulate the dynamical system consisting of the growth equations of the two-species and fishing effort. Then they studied the existence of its steady states and their stability using eigen value analysis. Dubey et. al. [6] in 2002 explained that both the equilibrium density of fish population as well as the maximum sustainable yield

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increase as resource biomass density increase. The authors use Pontryagin's maximum Principle to discuss the optimal harvesting policy. Dubey et. al. [1] in 2003 proposed and analysed a mathematical model to study the dynamics of fishery resource system in an aquatic environment consisting of two zones: a free fishing zone and a reserve zone where fishing is not allowed. Here the authors obtained biological and bionomic equilibria of the system and showed that, even if, fishery is exploited continuously in the unreserved zone, fish population can be maintained at an appropriate equilibrium level in the habitat. Dubey and Patra [2] in 2013 proposed a dynamical model and analyzed the effect of the population on the resource biomass by taking into account the crowding effect. An appropriate Hamiltonian function is constructed for the discussion of optimal harvesting of resource which is utilized by the population using Pontryagin's Maximum Principal. The aim of this paper is to study the dynamics of fishery resource system in an aquatic region consisting of two zones. One zone is free for fishing and other is restricted for any kind of fishing. In the proposed harvesting model, a modified effort function  $E$  is considered which depends on the density effect of fish population. The criteria for local stability, global stability and instability are established for the proposed system. The theoretical results obtained are illustrated with the help of numerical simulations in the last section.

## 2 The Model

Consider a fishery resource system consisting of two zones: a free fishery zone and a reserve zone where fishing is not allowed. Let  $x(t)$  be the biomass density of fish population inside the unreserved zone and  $y(t)$  be the biomass density of same fish population inside the reserved zone at time  $t$ . If the fish population of reserved and unreserved zones are allowed to migrate within the zones, Dubey et. al. [1] in 2003 proposed and analyzed a mathematical model to study the dynamics of fishery resource system which is governed by following autonomous system of differential equations:

$$\begin{aligned}\frac{dx}{dt} &= rx\left(1 - \frac{x}{K}\right) - \sigma_1x + \sigma_2y - qEx, \\ \frac{dy}{dt} &= sy\left(1 - \frac{y}{L}\right) + \sigma_1x - \sigma_2y,\end{aligned}\quad (2.1)$$

where  $x(0) > 0, y(0) > 0$ . In this model  $r$  and  $s$  are the intrinsic growth rates of fish subpopulation inside the unreserved and reserved areas respectively;  $K$  and  $L$  are the carrying capacities of fish species in the unreserved and reserved areas respectively;  $\sigma_1$  is migration rate of fishes from unreserved area to reserved area and  $\sigma_2$  is migration rate of fishes from reserved area to unreserved area;  $q$  is catchability coefficient of fish species and  $E$  is the total effort applied for harvesting the fish species in the unreserved zone.

In this case the fishing effort  $E$  is simply taken as a function of  $t$  i.e.  $E = E(t)$ , which do not address the inverse effect of fish abundance on fishing effort. That is, it do not address the fact that higher the density of fishes, lesser the amount of effort needed to catch unit harvest. In order to overcome this deficiency, Idels and Wang [11] proposed a modified effort function which is a function of  $t$  as well as  $x$  and is given by

$$E(t, x) = \alpha(t) - \beta(t) \frac{1}{x} \frac{dx}{dt}, \quad (2.2)$$

where  $\alpha \geq 0, \beta \geq 0$  are continuous functions of  $t$ . Incorporating (2.2), we get the following modified version of the model (2.1) as:

$$\begin{aligned}\frac{dx}{dt} &= \frac{r}{1 - q\beta} x\left(1 - \frac{x}{K}\right) - \frac{\sigma_1x}{1 - q\beta} + \frac{\sigma_2y}{1 - q\beta} - \frac{q\alpha x}{1 - q\beta}, \\ \frac{dy}{dt} &= sy\left(1 - \frac{y}{L}\right) + \sigma_1x - \sigma_2y.\end{aligned}\quad (2.3)$$

The parameters  $r, s, q, \sigma_1, \sigma_2, K$  and  $L$  are all assumed to be positive constants. It is to be noted that, if there is no migration of fish population from reserved area to unreserved area (i.e.  $\sigma_2 = 0$ ) and  $\frac{1}{1 - q\beta}(r - \sigma_1 - q\alpha) < 0$ , then  $\dot{x} < 0$ . Similarly, if there is no migration of fish population from unreserved area to reserved area (i.e.  $\sigma_1 = 0$ ) and  $s - \sigma_2 < 0$ , then  $\dot{y} < 0$ . Hence throughout our analysis we assume that

$$1 - q\beta > 0, r - \sigma_1 - q\alpha > 0 \text{ and } s - \sigma_2 > 0. \quad (2.4)$$

**Lemma 2.1.** All the solutions of system (2.3) which initiate in  $\mathfrak{R}_2^+$  are uniformly bounded.

**Proof:** We define a function

$$\omega(t) = x(t) + y(t). \tag{2.5}$$

The time derivative of (2.5) along the solution of system (2.3) is

$$\begin{aligned} \frac{d\omega}{dt} + \eta\omega &= \frac{dx}{dt} + \frac{dy}{dt} + \eta x + \eta y \\ &\leq \frac{K}{4r(1-q\beta)}(r + (1-q\beta)\eta - q\alpha - q\sigma_1\beta)^2 + \frac{L}{4s}(s + \eta + \frac{q\beta\sigma_2}{1-q\beta})^2 \\ &= \mu. \end{aligned}$$

Applying a theory of differential inequality (Birkhoff and Rota, [3] in 1982), we obtain

$$0 < \omega(x(t), y(t)) \leq \frac{\mu}{\eta}(1 - e^{-\eta t}) + \omega(x(0), y(0))e^{-\eta t},$$

and for  $t \rightarrow \infty, 0 < \omega \leq \frac{\mu}{\eta}$ . This proves the lemma.

### 3 Dynamical Behavior of the System

The dynamical behavior of a system is studied at equilibrium points and equilibrium points of model (2.3) are obtained by solving  $\dot{x} = \dot{y} = 0$ . There are two feasible equilibrium points for the system (2.3), namely (i)  $E_0 = (0, 0)$ , which is trivial equilibrium point and (ii)  $E^* = (x^*, y^*)$ , which is endemic equilibrium state, where  $x^*, y^*$  are positive solutions of following algebraic equations:

$$\sigma_2 y = q\alpha x + \sigma_1 x - rx + \frac{r}{K}x^2, \tag{3.6}$$

$$\sigma_1 x = \sigma_2 y - sy(1 - \frac{y}{L}). \tag{3.7}$$

Substituting the value of  $x$  from equation (3.7) into equation (3.6), we get a cubic equation in  $y$  as  $ay^3 + by^2 + cy + d = 0$ , where

$$\begin{aligned} a &= \frac{rs^2}{KL^2\sigma_1^2}, \\ b &= -2\frac{rs(s - \sigma_2)}{KL\sigma_1^2}, \\ c &= \frac{r}{K\sigma_1}(s - \sigma_2)^2 - \frac{s}{\sigma_1 L}(r - \sigma_1 - q\alpha), \\ d &= \frac{1}{\sigma_1}(r - \sigma_1 - q\alpha)(s - \sigma_2) - \sigma_2. \end{aligned}$$

The above equation has a unique positive solution  $y = y^*$ , if the following inequalities hold:

$$\frac{r}{K\sigma_1}(s - \sigma_2)^2 < \frac{s}{\sigma_1 L}(r - \sigma_1 - q\alpha), \tag{3.8}$$

$$(r - \sigma_1 - q\alpha)(s - \sigma_2) < \sigma_1\sigma_2. \tag{3.9}$$

After getting the value of  $y^*$ , the value of  $x^*$  can be easily computed from (3.7). It may be noted that for  $x^*$  to be positive, we must have

$$\sigma_2 + \frac{sy^*}{L} > s.$$

We now investigate the dynamical behaviour of system (2.3) at equilibrium points. The general variational matrix corresponding to the system (2.3) is given by

$$W = \begin{bmatrix} \frac{r}{1-q\beta} - \frac{2rx}{(1-q\beta)k} - \frac{\sigma_1}{1-q\beta} - \frac{q\alpha}{1-q\beta} & \frac{\sigma_2}{1-q\beta} \\ \sigma_1 & s - \frac{2sy}{L} - \sigma_2 \end{bmatrix}.$$

At  $E_0(0,0)$ , keeping in view equation (2.4), we note that all the eigen values of variational matrix are positive. Therefore, the trivial equilibrium  $E_0$  is unstable. At  $E^*(x^*, y^*)$ , using the Routh-Hurwitz criteria, it is easy to check that all eigenvalues of the variational matrix corresponding to  $E^*$  have negative real parts, and hence  $E^*$  is locally asymptotic stable in  $XY$  plane. This implies that we can find a small circle with center  $E^*$  such that any solution  $(x(t), y(t))$  of system (2.3), which is inside the circle at some time  $t = t_0$ , will remain inside the circle for all  $t \geq t_0$  and will tend to  $(x^*, y^*)$  as  $t \rightarrow \infty$ .

**Theorem 3.1.** The nontrivial equilibrium  $E^*$  is globally asymptotically stable with respect to all solutions initiating in the interior of the positive quadrant.

**Proof:** Consider the following positive definite function about  $E^*$ :

$$V = (x - x^* - x^* \ln \frac{x}{x^*}) + \frac{y^* \sigma_2}{(1 - q\beta)x^* \sigma_1} (y - y^* - y^* \ln \frac{y}{y^*}). \tag{3.10}$$

Differentiating  $V$  with respect to time  $t$  along the solutions of model (2.3) and after some algebraic manipulation, we get

$$\frac{dV}{dt} = -\frac{r}{(1 - q\beta)K} (x - x^*)^2 - \frac{s}{L(1 - q\beta)x^* \sigma_1} (y - y^*)^2 - \frac{\sigma_2}{(1 - q\beta)xx^*y} (x^*y - y^*x)^2 < 0. \tag{3.11}$$

This shows that  $\frac{dV}{dt}$  is negative definite and hence by Liapunov’s theorem on stability [14], it follows that the positive equilibrium  $E^*$  is globally asymptotically stable with respect to all solutions initiating in the interior of the positive quadrant.

The above theorem implies that in an aquatic environment consisting of two zones, if one zone is reserved where fishing is not allowed and fish population are harvested only outside the reserved zone, then in both the reserved and unreserved zones fish species settle down to their respective equilibrium levels, whose magnitude depends upon the intrinsic growth rates of fish species, their migration coefficients and carrying capacities. This implies that fish populations may be sustained at an appropriate equilibrium level even after continuous harvesting of fish populations in unreserved zone.

**Theorem 3.2.** System (2.3) cannot have any limit cycle in the interior of positive quadrant.

**Proof:** Let

$$\begin{aligned} H(x, y) &= \frac{1}{xy}, \\ h_1(x, y) &= \frac{rx}{1 - q\beta} (1 - \frac{x}{K}) - \frac{\sigma_1 x}{1 - q\beta} + \frac{\sigma_2 y}{1 - q\beta} - \frac{q\alpha x}{1 - q\beta}, \\ h_2(x, y) &= sy(1 - \frac{y}{L}) + \sigma_1 x - \sigma_2 y. \end{aligned}$$

Clearly  $H(x, y) > 0$  in the interior of the positive quadrant of  $XY$  plane. Then we have,

$$\delta(x, y) = \frac{\partial}{\partial x}(Hh_1) + \frac{\partial}{\partial y}(Hh_2) = -\frac{1}{(1 - q\beta)y} \left[ \frac{r}{K} + \frac{\sigma_2 y}{x^2} \right] - \frac{1}{x} \left[ \frac{s}{L} + \frac{\sigma_2 x}{y^2} \right] < 0.$$

This shows that  $\delta(x, y)$  does not change sign and is not identically zero in the positive quadrant of  $XY$ -plane. Then by Bendixson-Dulac criterion, the system (2.3) has no closed trajectory and hence there is no periodic solution in the interior of the positive quadrant of  $XY$ -plane.

### 4 Numerical Simulation

In order to investigate the dynamics of the model (2.3) with the help of computer simulations, we choose the following set of values of parameters (other set of parameter may also exist):

$$r = 0.9, K = 100, \sigma_1 = 0.3, q = 0.01, \beta = 1, \sigma_2 = 0.4, \alpha = 50, s = 0.5, L = 100 \tag{4.12}$$

with initial conditions  $x(0) = 50$  and  $y(0) = 50$ . For this set of parameters, the conditions 3.8–3.9 for the existence of the interior equilibrium are satisfied. This shows that the endemic equilibrium point  $E^*(x^*, y^*)$  exists and is given by

$$\begin{aligned} x^* &= 62.3954 \\ y^* &= 71.9978. \end{aligned}$$



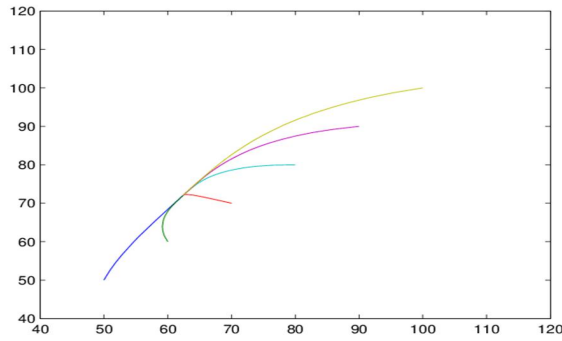
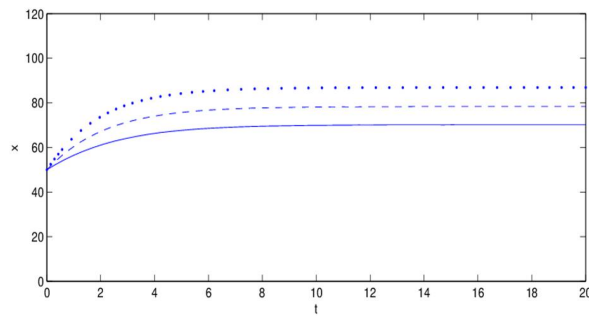


Figure 3: Global stability of  $E^*(x^*, y^*)$ .



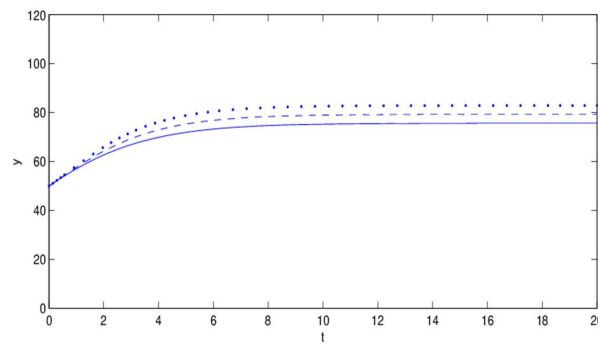
- $\alpha = 40$
- $\alpha = 30$
- $\alpha = 20$

Figure 4: Plot of  $x$  verses time  $t$  for different values of  $\alpha$ , the other values of parameters being same as given in (4.12).

are important parameters governing the dynamics of the system. Therefore, we have plotted the behavior of  $x$  and  $y$  with respect to time for different values of  $\alpha$  in Figure 4 and Figure 5. From Figure 4, we note that as the value of  $\alpha$  decreases, the value of  $x$  increases. Similarly, the figure 5 implies that the value of  $y$  increases as the value of  $\alpha$  decreases.

### 5 Conclusion

In this paper, a mathematical model representing harvesting of fishery resources with reserve area and modified effort function has been proposed. It has been assumed that the aquatic ecosystem consists of two zones: one free fishing zone and other reserved zone where fishing is strictly prohibited. It has been assumed that fish populations are growing logistically inside and outside the reserved zone and they migrate from reserved zone to unreserved zone and vice versa. Using stability theory of ordinary differential equation, it has been proved that the interior equilibrium exists under certain condition and it is globally asymptotically stable. It has been shown that the system under consideration does not have any limit cycle. It has been further found that if a reserved zone is created in an open-access fishery region where fishing is not allowed and harvesting of fish population is permitted only outside the reserved zone, the fish populations settle down at the respective equilibrium levels inside as well as outside the reserved zone.



$$\alpha = 40$$

$$\alpha = 30$$

$$\alpha = 20$$

Figure 5: Plot of  $y$  verses time  $t$  for different values of  $\alpha$ , the other values of parameters being same as given in (4.12).

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