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Energy decay of solutions for the wave equation with a time varying delay term in the weakly nonlinear internal feedbacks

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Abstract

We consider the nonlinear wave equation in a bounded domain with a time varying delay term in the weakly nonlinear internal feedback

$$\left(|u_t|^{\gamma-2}u_t\right)_t - \Delta_x u - \int_0^t g(t-s)\Delta u(s)ds + \mu_1\psi(u_t(x,t)) + \mu_2\psi(u_t(x,t-\tau(t))) = 0,$$

we study the asymptotic behavior of solutions in using the Lyapunov functional , we extend and improve the previous result due to [30] ,

Keywords: Energy decay; viscoelastic term ; time varying delay term.

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1 Introduction

In this paper we investigate the decay properties of solutions for the initial boundary value problem of a nonlinear wave equation of the form

$$\begin{cases} \left(|u_t|^{\gamma-2}u_t\right)_t - \Delta_x u - \int_0^t g(t-s)\Delta u(s)ds + \mu_1\psi(u_t(x,t)) + \mu_2\psi(u_t(x,t-\tau(t))) = 0, & \text{in } \Omega\times]0,+\infty[, \\ u(x,t) = 0, & \text{on } \Gamma\times]0,+\infty[, \\ u(x,0) = u_0(x), \;\; u_t(x,0) = u_1(x) & \text{in } \Omega, \\ u_t(x,t-\tau(0)) = f_0(x,t-\tau(0)), & \text{in } \Omega\times]0,\tau(0)[, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}^*$, with a smooth boundary $\partial\Omega = \Gamma$, $\tau(t) > 0$ is a time varying delay, μ_1 and μ_2 are positive real numbers, and the initial data (u_0, u_1, f_0) belong to a suitable function space. In absence of delay ($\mu_2 = 0$), the problem of existence and energy decay have been extensively studied by several authors (see **[3]**, **[5]**, **[6]**, **[9]**, **[12]**, **[13]**, **[17]**, **[23]**) and many energy estimates have been derived for arbitrary growing feedbacks (polynomial, exponential or logarithmic decay). The decay rate of the energy (when t goes to infinity) depends on the function σ and on the function H which represents the growth at the origin of ψ.

Time delay is the property of a physical system by which the response to an applied force is delayed in its effect (see **[25]**). Whenever material, information or energy is physically transmitted from one place to another, there is a delay associated with the transmission. Time delays so often arise in many physical, chemical, biological and economical phenomena. In recent years, the control of PDEs with time delay effects has become an active area of research, see for example **[1]**, **[26]**, **[28]** and the references therein. In **[7]**, the authors showed that a small delay in a boundary control could turn such well-behave hyperbolic system into a wild one and

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therefore, delay becomes a source of instability. However, sometimes it also can improve the performance of the systems (see **[26]**).
To stabilize a hyperbolic system involving input delay terms, additional control terms will be necessary (see **[18]**, **[19]**, **[27]**). For instance in **[18]** the authors studied the wave equation with linear internal damping term with constant delay (ψ linear, $\tau(t) = const$ in the problem (1.1)). They determined suitable relations between μ_1 and μ_2, for which the stability or alternatively instability takes place. More precisely, they showed that the energy is exponentially stable if $\mu_2 < \mu_1$ and they also found a sequence of delays for which the corresponding solution of (1.1) will be instable if $\mu_2 \geq \mu_1$. The main approach used in **[18]**, is an observability inequality obtained with a Carleman estimate. The same results were obtained if both the damping and the delay are acting in the boundary. We also recall the result by Xu, Yung and Li **[27]**, where the authors proved a result similar to the one in **[18]** for the one-space dimension by adopting the spectral analysis approach.
The case of time varying delay in the wave equation has been studied recently by Nicaise, Valein and Fridman **[22]** in one-space dimension and in the linear case (ψ linear in problem (1.1) and proved an exponential stability result under the condition

$$\mu_2 < \sqrt{1-d}\mu_1,$$

where the constant d satisfies

$$\tau'(t) \leq d < 1, \quad \forall t > 0.$$

In **[21]** Nicaise, Pignotti and Valein extended the above result to higher-space dimension and established an exponential decay.
Our purpose in this paper is to give an energy decay estimate of the solution to problem (1.1) for a weakly nonlinear damping and in the presence of a time varying delay term.
In this article, we use some technique from (see **[3]**), **[30]**) and **[31]**).to give energy decay estimates of solutions to the problem (1.1) for a nonlinear damping and a time varying delay term. To prove decay estimates, we use a suitable energy and Lyapunov functionals and some properties of convex functions. These arguments of convexity were introduced and developed by Lasiecka et al. **[4]**, and **[13]**, and used by Liu and Zuazua **[15]**, Eller et al **[8]**.

2 Preliminaries and main results

In order to state and prove our results, we need some assumptions, as well as, some lemmas.
First assume the following hypotheses
(H1) $g : R^+ \to R^+$ is a bounded C^1 function satisfying

$$g(0) > 0, \qquad 1 - \int_0^\infty g(s)ds = l < 1,$$

and there exists a non-increasing differentiable function : $\xi : R_+ \to R_+$ such that $g'(t) \leq -\xi(t)g(t)$.
(H2) $\psi : \mathbb{R} \to \mathbb{R}$ is a non-decreasing function of the class $C(\mathbb{R})$ such that there exist ϵ_1, c_1, $c_2 > 0$ and a convex and increasing function $H : \mathbb{R}_+ \to \mathbb{R}_+$ of the class $C^1(\mathbb{R}_+) \cap C^2(]0, \infty[)$ satisfying $H(0) = 0$, and H linear on $[0, \epsilon_1]$ or ($H'(0) = 0$ and $H'' > 0$ on $]0, \epsilon_1]$), such that

$$c_1|s| \leq |\psi(s)| \leq c_2|s| \quad \text{if } |s| \geq \epsilon_1, \tag{2.2}$$

$$s^2 + \psi^2(s) \leq H^{-1}(s\psi(s)) \quad \text{if } |s| \leq \epsilon_1. \tag{2.3}$$

$\psi : \mathbb{R} \to \mathbb{R}$ is an odd non-decreasing function of the class $C^1(\mathbb{R})$ such that there exist $c_3, \alpha_1, \alpha_2 > 0$

$$|\psi'(s)| \leq c_3 \tag{2.4}$$

$$\alpha_1\, s\psi(s) \leq G(s) \leq \alpha_2\, s\psi(s), \tag{2.5}$$

where

$$G(s) = \int_0^s \psi(r)\, dr,$$

with l satisfying

$$\gamma - 1 \le \frac{n+2}{n-2}, \quad if\ n > 2,$$
$$\gamma - 1 < \infty, \qquad if\ n \le 2.$$

(H3) τ is a function such that

$$\tau \in W^{2,\infty}([0,T]), \forall T > 0, \tag{2.6}$$

$$0 < \tau_0 \le \tau(t) \le \tau_1, \quad \forall t > 0, \tag{2.7}$$

$$\tau'(t) \le d < 1, \quad \forall t > 0,, \tag{2.8}$$

where τ_0 and τ_1 are two positive constants.
(H4) The weight of dissipation and the delay satisfy:

$$\mu_2 < \frac{\alpha_1(1-d)}{\alpha_2(1-\alpha_1 d)}\mu_1. \tag{2.9}$$

We now state some Lemmas needed later.

Lemma 2.1 (Sobolev-Poincaré's inequality)**.** *Let q be a number with $2 \le q < +\infty$ $(n = 1,2)$ or $2 \le q \le 2n/(n-2)$ $(n \ge 3)$. Then there exists a constant $c_* = c_*(\Omega, q)$ such that*

$$\|u\|_q \le c_* \|\nabla u\|_2 \quad for \quad u \in H_0^1(\Omega).$$

Lemma 2.2. [30]. *For any $g \in C^1(R_+)$ and $\varphi \in H^1(0,T)$, we have*

$$\int_0^t \int_\Omega g(t-s)\varphi(x,s)\varphi_t(x,t)dxds = -\frac{1}{2}\frac{d}{dt}\left((go\varphi)(t) + \int_0^t g(s)ds\|\varphi\|_2^2\right) - g(t)\|\varphi\|_2^2 + (g'o\varphi)(t),$$

where

$$(go\varphi)(t) = \int_0^t g(t-s)\int_\Omega |\varphi(x,s) - \varphi(x,t)|^2 dxds,$$

and

$$\|\varphi\|_2^2 = \int_\Omega |\varphi(x,s)|^2 dx.$$

Lemma 2.3. [30]. *For $u \in H_0^1(\Omega)$, we have*

$$\int_\Omega \left(\int_0^t g(t-s)(u(x,t) - u(x,s))ds\right)^2 dx \le (1-l)c_s^2(go\nabla u)(t), \tag{2.10}$$

where

$$(go\nabla u)(t) = \int_0^t g(t-s)\int_\Omega |u(x,s) - u(x,t)|^2 dxds,$$

and c_s^2 is the poincaré constant and l is given in $(H1)$.

We introduce, as in **[18]**, the new variable

$$z(x,\rho,t) = u_t(x, t-\tau(t)\rho), \quad x \in \Omega,\ \rho \in (0,1), \quad t > 0. \tag{2.11}$$

Then, we have

$$\tau(t)z_t(x,\rho,t) + (1-\tau'(t)\rho)z_\rho(x,\rho,t) = 0, \text{ in } \Omega \times (0,1) \times (0,+\infty). \tag{2.12}$$

Therefore, problem (1.1) is equivalent to:

$$\begin{cases} \left(|u_t|^{\gamma-2}u_t\right)_t - \Delta_x u - \int_0^t g(t-s)\Delta u(s)ds + \mu_1\psi(u_t(x,t)) + \mu_2\psi(u_t(x,t-\tau(t))) = 0, & in\ \Omega\times]0,+\infty[, \\ \tau(t)z_t(x,\rho,t) + (1-\tau'(t)\rho)z_\rho(x,\rho,t) = 0, & in\ \Omega\times]0,1[\times]0,+\infty[, \\ u(x,t) = 0, & on\ \partial\Omega\times]0,+\infty[\\ z(x,0,t) = u_t(x,t), & on\ \Omega\times]0,+\infty[, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & in\ \Omega, \\ z(x,\rho,0) = f_0(x,-\rho\tau(0)), & in\ \Omega\times]0,1[. \end{cases} \tag{2.13}$$

where ξ satisfies

$$\frac{\mu_2(1-\alpha_1)}{\alpha_1(1-d)} < \xi < \frac{\mu_1 - \alpha_2\mu_2}{\alpha_2}. \tag{2.14}$$

We define the energy associated to the solution of the problem (2.13) by:

$$E(t) = \frac{\gamma-1}{\gamma}\|u_t(t)\|_\gamma^\gamma + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2 + \xi(t)\tau(t)\int_\Omega\int_0^1 G(z(x,\rho,t))\,d\rho\,dx, \tag{2.15}$$

Lemma 2.4. *Let (u,z) be a solution of the problem (2.13). Then, the energy functional defined by (2.15) satisfies*

$$\begin{aligned} E'(t) \le &-(\mu_1 - \xi(t)\alpha_2 - \mu_2\alpha_2)\int_\Omega u_t\psi(u_t)\,dx \\ &-\big(\xi(t)(1-\tau'(t))\alpha_1 - \mu_2(1-\alpha_1)\big)\int_\Omega z(x,1,t)\psi(z(x,1,t))\,dx \\ &+\frac{1}{2}(g'o\nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2 \\ &\le 0. \end{aligned} \tag{2.16}$$

Proof. Multiplying the first equation in (2.13) by u_t, integrating over Ω and using integration by parts, we get

$$\frac{1}{2}\frac{d}{dt}\left(\|u_t\|_\gamma^\gamma + \|\nabla u\|_2^2\right) + \mu_1\int_\Omega u_t\psi(u_t)\,dx + \mu_2\int_\Omega \psi(z(x,1,t))u_t(x,t)\,dx = 0. \tag{2.17}$$

We multiply the second equation in (2.13) by $\xi(t)\psi(z)$ and integrate over $\Omega\times(0,1)$, to obtain:

$$\xi(t)\tau(t)\int_\Omega\int_0^1 z_t\psi(z(x,\rho,t))\,d\rho\,dx = -\xi(t)\int_\Omega\int_0^1(1-\tau'(t)\rho)\frac{\partial}{\partial\rho}G(z(x,\rho,t))\,d\rho\,dx. \tag{2.18}$$

Consequently,

$$\begin{aligned} &\frac{d}{dt}\left(\xi(t)\tau(t)\int_\Omega\int_0^1 G(z(x,\rho,t))\,d\rho\,dx\right) \\ &= -\xi(t)\int_0^1\int_\Omega\frac{\partial}{\partial\rho}((1-\tau'(t)\rho)G(z(x,\rho,t)))\,d\rho\,dx + \xi_t(t)\tau(t)\int_0^1\int_\Omega G(z(x,\rho,t)))\,dxd\rho. \\ &= \xi(t)\int_\Omega(G(z(x,0,t)) - G(z(x,1,t)))\,dx + \xi(t)\tau'(t)\int_\Omega G(z(x,1,t)))\,dx \\ &+\xi_t(t)\tau(t)\int_0^1\int_\Omega G(z(x,\rho,t)))\,dxd\rho. \end{aligned} \tag{2.19}$$

From (2.17), (2.18), lemma 2.2 we get

$$\begin{aligned} E'(t) \le &-(\mu_1 - \xi(t)\alpha_2)\int_\Omega u_t\psi(u_t)\,dx - \xi(t)(1-\tau'(t))\int_\Omega G(z(x,1,t))\,dx \\ &-\mu_2\int_\Omega u_t(t)\psi(z(x,1,t))\,dx + \frac{1}{2}(g'o\nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2. \end{aligned} \tag{2.20}$$

Let us denote by G^* the conjugate function of the convex function G, i.e., $G^*(s) = \sup_{t\in\mathbb{R}_+}(st - G(t))$. Then G^* is the Legendre transform of G, which is given by (see Arnold **[2]**, p. 61-62)

$$G^*(s) = s(G_2')^{-1}(s) - G[(G')^{-1}(s)], \quad \forall s \ge 0 \tag{2.21}$$

and satisfies the following inequality

$$st \le G^*(s) + G(t), \quad \forall s,t \ge 0. \tag{2.22}$$

Then, from the definition of G_2, we get

$$G^*(s) = s\psi^{-1}(s) - G(\psi^{-1}(s)).$$

Hence

$$\begin{aligned} G^*(\psi(z(x,1,t))) &= z(x,1,t)\psi(z(x,1,t)) - G(z(x,1,t)) \\ &\le (1-\alpha_1)z(x,1,t)\psi(z(x,1,t)). \end{aligned} \tag{2.23}$$

Making use of (2.19) and (2.22), we have

$$\begin{aligned} E'(t) \leq &-(\mu_1 - \xi(t)\alpha_2)\int_\Omega u_t\psi(u_t)\,dx - \xi(t)(1-\tau'(t))\int_\Omega G(z(x,1,t))\,dx \\ &+\mu_2\int_\Omega \left(G(u_t) + G^*(\psi(z(x,1,t)))\right)\,dx \\ &+\frac{1}{2}(g'o\nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2. \end{aligned} \tag{2.24}$$

From (2.5) and (2.22), we obtain

$$\begin{aligned} E'(t) \leq &-(\mu_1 - \xi(t)\alpha_2 - \mu_2\alpha_2)\int_\Omega u_t\psi(u_t)\,dx \\ &-\left(\xi(t)(1-\tau'(t))\alpha_1 - \mu_2(1-\alpha_1)\right)\int_\Omega z(x,1,t)\psi(z(x,1,t))\,dx \\ &+\frac{1}{2}(g'o\nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2. \end{aligned}$$

Then, by using (2.8) and (2.14), our conclusion follows. □

3 Asymptotic Behavior

In this section we prove the energy decay result by constructing a suitable Lyapunov functionnal.
We denote by c various positive constants which may be different at different occurrences.
Now we define the following functional

$$L(t) = ME(t) + \epsilon\phi(t) + \epsilon\varphi(t) + \epsilon I(t), \tag{3.25}$$

where

$$\phi(t) = \int_\Omega u|u_t|^{\gamma-2}u_t dx, \tag{3.26}$$

$$\varphi(t) = -\int_\Omega |u_t|^{\gamma-2}u_t\int_0^t g(t-s)(u(t)-u(s))dsdx, \tag{3.27}$$

and

$$I(t) = \int_\Omega\int_0^1 e^{-2\rho\tau(t)}G(z(x,\rho,t))d\rho dx. \tag{3.28}$$

We need also the following lemma

Lemma 3.1. . *Let (u,z) be a solution of problem (2.13), then there exists two positive constants λ_1,λ_2 such that*

$$\lambda_1 E(t) \leq L(t) \leq \lambda_2 E(t), \quad t \geq 0, \tag{3.29}$$

for M sufficiently large .

Proof. Thank's to the Holder and Young's inequalities, lemma 2.1 , we have

$$\begin{aligned} \int_\Omega u|u_t|^{\gamma-2}u_t dx &\leq C_\epsilon\int_\Omega |u|^\gamma dx + \epsilon\int_\Omega |u_t|^l dx \\ &\leq C_\epsilon\|\nabla u\|_2^\gamma + \epsilon\|u_t\|_\gamma^\gamma \\ &\leq C_\epsilon E^{\frac{\gamma}{2}}(t) + c\epsilon E(t) \\ &\leq C_\epsilon E^{\frac{\gamma-2}{2}}(0)E(t) + c\epsilon E(t), \end{aligned} \tag{3.30}$$

$$\begin{aligned} \int_\Omega u|u_t|^{\gamma-2}u_t dx &\geq -C_\epsilon\int_\Omega |u|^\gamma dx - \epsilon\int_\Omega |u_t|^\gamma dx \\ &\geq -C_\epsilon\|\nabla u\|_2^\gamma - \epsilon\|u_t\|_\gamma^\gamma \\ &\geq -C_\epsilon E^{\frac{\gamma}{2}}(t) - c\epsilon E(t) \\ &\geq -C_\epsilon E^{\frac{\gamma-2}{2}}(0)E(t) - c\epsilon E(t), \end{aligned} \tag{3.31}$$

and

$$\begin{aligned}\varphi(t) &= \left|-\int_\Omega |u_t|^{\gamma-2}u_t \int_0^t g(t-s)(u(t)-u(s))dsdx\right| \\ &\leq \frac{1}{2}\|u_t\|_\gamma^\gamma + \frac{1}{2}\int_\Omega \left(\int_0^t g(t-s)(u(t)-u(s))ds\right)^2 dx \\ &\leq \frac{1}{2}\left(\|u_t\|_\gamma^\gamma + (1-l)c_s^2 \int_0^t g(t-s)\|\nabla u(t)-\nabla u(s)\|_2^2 ds\right) \\ &\leq \frac{1}{2}\left(\|u_t\|_\gamma^\gamma + (1-l)c_s^2 (go\nabla u)(t)\right),\end{aligned} \tag{3.32}$$

it follows from (3.28) that $\forall c > 0$

$$\begin{aligned}|I(t)| &= \left|\int_\Omega \int_0^1 e^{-2\rho\tau(t)} G(z(x,\rho,s))dkdx\right| \\ &\leq c\int_\Omega \int_0^1 G(z(x,\rho,s))d\rho dx.\end{aligned} \tag{3.33}$$

Hence, combining (3.30)-(3.33). This yields

$$\begin{aligned}|L(t)-ME(t)| &= \epsilon\phi(t) + \varphi(t) + \epsilon I(t) \\ &\leq C_\epsilon E^{\frac{\gamma-2}{2}}(0)E(t) + c\epsilon E(t)\epsilon\|u_t\|_\gamma^\gamma + \epsilon(1-l)c_s^2(go\nabla u)(t) \\ &+ c\int_\Omega \int_0^1 G(z(x,\rho,t))d\rho dx.\end{aligned} \tag{3.34}$$

Where

$$|L(t)-ME(t)| \leq c_5 E(t), \tag{3.35}$$

where $c_5 = max(c_1, c_2, c_3, c_4)$. Thus, from the definition of E(t) and selecting M sufficiently large,

$$\beta_2 E(t) \leq L(t) \leq \beta_1 E(t). \tag{3.36}$$

Where $\beta_1 = (M - \epsilon c_5)$, $\beta_2 = (M + \epsilon c_5)$. This completes the proof. □

Lemma 3.2. *Let (u,z) be the solution of (2.13). Then it holds*

$$\begin{aligned}\frac{d}{dt}\phi(t) &\leq \left(\frac{(1+l)(1-l)^2 + (\mu_1+\mu_2)\alpha c_s^2}{2} - 1\right)\|\nabla u\|_2^2 + \frac{(1-l)}{2}(go\nabla u)(t) \\ &+ \frac{\mu_2}{4\alpha}\|\psi(z(x,1,t))\|_2^2 + \|u_t\|_\gamma^\gamma + \frac{\mu_1}{4\alpha}\|\psi(u_t)\|_2^2.\end{aligned} \tag{3.37}$$

Proof. We take the derivative of $\phi(t)$. It follows from (3.26) that

$$\frac{d}{dt}\phi(t) = \int_\Omega (|u_t|^{\gamma-2}u_t)_t u dx + \|u_t\|_\gamma^\gamma, \tag{3.38}$$

using the problem (2.13), then we have

$$\begin{aligned}\frac{d}{dt}\phi(t) &= \|u_t\|_\gamma^\gamma - \|\nabla u\|_2^2 + \int_\Omega \int_0^t g(t-s)\nabla u(s)\nabla u(t)dsdx \\ &-\mu_2 \int_\Omega \psi(z(x,1,t))u(t)dx - \mu_1 \int_\Omega \psi(u_t)u(t)dx,\end{aligned} \tag{3.39}$$

we estimate the third term in the right hand side of (3.39) as follows

$$\begin{aligned}&\left|\int_\Omega \nabla u(t)\int_0^t g(t-s)\nabla u(s)dsdx\right| \\ &\leq \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int_\Omega \left(\int_0^t g(t-s)(|\nabla u(s)-\nabla u(t)| + |\nabla u(t)|)ds\right)^2 dx \\ &\leq \frac{1+(1+\lambda)(1-l)^2}{2}\|\nabla u\|_2^2 + \frac{(1+\frac{1}{\lambda})(1-l)}{2}(go\nabla u)(t),\end{aligned} \tag{3.40}$$

for the forth and fifth term in (3.39), Holder and Young's to get

$$\left|\int_\Omega \psi(u_t)u dx\right| \le \alpha c_s^2\|\nabla u\|_2^2 + \frac{1}{4\alpha}\|\psi(u_t)\|_2^2, \tag{3.41}$$

and

$$\left|\int_\Omega \psi(z(x,1,t))u dx\right| \le \alpha c_s^2\|\nabla u\|_2^2 + \frac{1}{4\alpha}\|\psi(z(x,1,t))\|_2^2. \tag{3.42}$$

Let $\lambda = \frac{l}{1-l}$ in (3.40) and using (3.41), (3.42), then (3.40) becomes

$$\begin{aligned} \frac{d}{dt}\phi(t) \le \epsilon\left(\frac{(1+l)(1-l)^2 + (\mu_1+\mu_2)\alpha c_s^2}{2} - 1\right)\|\nabla u\|_2^2 + \epsilon\frac{(1-l)}{2}(go\nabla u)(t) \\ + \epsilon\frac{\mu_2}{4\alpha}\|\psi(z(x,1,t))\|_2^2 + \|u_t\|_\gamma^\gamma + \frac{\epsilon\mu_1}{4\alpha}\|\psi(u_t)\|_2^2. \end{aligned} \tag{3.43}$$

This completes the proof. □

Lemma 3.3. *. Let (u,z) be the solution of (2.13). Then $\varphi(t)$ satisfies*

$$\begin{aligned} \varphi'(t) &\le \alpha\left(1 + 2(1-l)^2\right)\|\nabla u\|_2^2 - (g_0 - \alpha)\|u_t\|_\gamma^\gamma \\ &+ \mu_1\|\psi(u_t)\|_2^2 + \frac{g(0)c_s^2}{4\alpha}(-g'o\nabla u)(t) \\ &+ \frac{\mu_2}{4\alpha}(1-l)(2(\alpha+1) + c_s^2)(go\nabla u)(t) \\ &+ \frac{1}{4\alpha}c_s^2(1-l)^2\mu_2\int_\Omega \psi^2(z(x,1,t))dx. \end{aligned} \tag{3.44}$$

Proof. Now Taking the derivatives of $\varphi(t)$, using the problem (2.13), we obtain

$$\begin{aligned} \varphi'(t) &= -\int_\Omega (|u_t|^{\gamma-2}u_t)_t \int_0^t g(t-s)(u(t)-u(s))dsdx \\ &- \int_\Omega |u_t|^{\gamma-2}u_t \int_0^t g'(t-s)(u(t)-u(s))dsdx - \left(\int_0^t g(s)ds\right)\int_\Omega u_t^\gamma dx \\ &= \int_\Omega \nabla u(t)\left(\int_0^t g(t-s)(\nabla u(t) - \nabla u(s))ds\right)dx \\ &- \int_\Omega \left(\int_0^t g(t-s)\nabla u(s)ds\right)\left(\int_0^t g(t-s)(\nabla u(t) - \nabla u(s))ds\right)dx \\ &+ \int_\Omega \mu_1\psi(u(t))\int_0^t g(t-s)(u(t)-u(s))dsdx \\ &+ \int_\Omega \mu_2\psi(z(x,1,t))\int_0^t g(t-s)(u(t)-u(s))dsdx \\ &- \int_\Omega |u_t|^{\gamma-2}u_t \int_0^t g'(t-s)(u(t)-u(s))dsdx - \left(\int_0^t g(s)ds\right)\int_\Omega u_t^\gamma dx. \end{aligned} \tag{3.45}$$

Next we will estimate the right hand side of (3.45), using Holder, Young's inequalities and $(H1)$ to have

$$\begin{aligned} &\int_\Omega \nabla u\left(\int_0^t g(t-s)(\nabla u(t) - \nabla u(s))ds\right)dx \\ &\le \alpha\|\nabla u\|_2^2 + \frac{(1-l)}{4\alpha}(go\nabla u)(t), \end{aligned} \tag{3.46}$$

and

$$
\begin{aligned}
&-\int_\Omega \left(\int_0^t g(t-s)\nabla u_s ds\right)\left(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds\right)dx\\
&\le \alpha\int_\Omega \left(\int_0^t g(t-s)(|\nabla u(t)|_2^2 ds\right)^2 dx\\
&+\frac{1}{\alpha}\int_\Omega \left|\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|ds\right|^2 dx\\
&\le \alpha\int_\Omega \left(\int_0^t g(t-s)(|\nabla u(t)-\nabla u(s)|+|\nabla u(t)|)ds\right)^2 dx\\
&+\frac{1}{\alpha}\left(\int_0^t g(t-s)ds\right)\int_\Omega\int_0^t g(t-s)(|\nabla u(t)|-\nabla u(s)|^2 dsdx\\
&\le \alpha\left(\int_\Omega\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|ds\right)^2 dx\\
&+2\alpha(1-l)^2|\nabla u(t)|_2^2+\frac{1}{4\alpha}(1-l)(go\nabla u)(t)\\
&\le 2\alpha(1-l)^2\|\nabla u(t)\|_2^2+\left(2\alpha+\frac{1}{4\alpha}\right)(1-l)(go\nabla u)(t),
\end{aligned}
\tag{3.47}
$$

where g is positive, continuous and $g(0)>0$, for any t_0, we have

$$
\int_0^t g(s)ds \ge \int_0^{t_0} g(s)ds = g_0, \quad \forall t\ge t_0, \tag{3.48}
$$

then we use (3.48) to get

$$
\begin{aligned}
&\int_\Omega |u_t|^{\gamma-2}u_t\int_0^t g'(t-s)(u(t)-u(s))dsdx-\left(\int_0^t g(s)ds\right)\int_\Omega u_t^\gamma dx\\
&\le \alpha\|u_t\|_\gamma^\gamma+\frac{g(0)c_s^2}{4\alpha}(-g'o\nabla u)(t)-g_0\|u_t\|_\gamma^\gamma,
\end{aligned}
\tag{3.49}
$$

$$
\begin{aligned}
&\left|-\int_\Omega \mu_1\psi(u_t)\int_0^t g(t-s)(u(t)-u(s))dsdx\right|\\
&\le \mu_1\|\psi(u_t)\|_2^2+\frac{\mu_1(1-l)c_s^2}{4\alpha}(go\nabla u)(t),
\end{aligned}
\tag{3.50}
$$

and

$$
\begin{aligned}
&\left|-\int_\Omega \mu_2\psi(z(x,1,t))\int_0^t g(t-s)(u(t)-u(s))dsdx\right|\\
&\le \mu_2\int_\Omega \psi^2(z(x,1,t))dx+\frac{\mu_2(1-l)c_s^2}{4\alpha}(go\nabla u)(t).
\end{aligned}
\tag{3.51}
$$

A substitution of (3.49)-(3.51) into (3.47) yields

$$
\begin{aligned}
\varphi'(t) &\le \alpha\left(1+2(1-l)^2\right)\|\nabla u\|_2^2-(g_0-\alpha)\|u_t\|_\gamma^\gamma\\
&+\mu_1\|\psi(u_t)\|_2^2+\frac{g(0)c_s^2}{4\alpha}(-g'o\nabla u)(t)\\
&+\frac{\mu_2}{4\alpha}(1-l)(2(\alpha+1)+c_s^2)(go\nabla u)(t)\\
&+\frac{1}{4\alpha}c_s^2(1-l)^2\mu_2\int_\Omega \psi^2(z(x,1,t))dx.
\end{aligned}
\tag{3.52}
$$

□

Lemma 3.4. *. The functional defined by (3.28) can be estimated by*

$$
\frac{d}{dt}I(t)\le -2I(t)-\frac{c\xi(t)}{2\tau_1}\int_\Omega G(z_1(x,1,t))dx+\frac{\xi(t)}{\tau_0}\|\psi(u_t)\|_2^2 \tag{3.53}
$$

where τ_0, τ_2 are some positive constant.

Proof. Differentiating (3.28) with respect to t and using the second equation in (2.13), we have

$$\begin{aligned}
\frac{I(t)}{dt} &\leq \frac{d}{dt}\left[\xi(t)e^{-\rho\tau(t)}\int_\Omega\int_0^1 z^2(x,\rho,t))d\rho dx\right]\\
&= \left[\xi'(t)e^{-\tau(t)\rho}\int_\Omega\int_0^1 G(z(x,\rho,t))d\rho dx - \xi(t)\rho e^{-\tau(t)\rho}\tau'(t)\int_\Omega\int_0^1 G(z(x,\rho,t))d\rho dx\right]\\
&+ \frac{1}{\tau(t)}e^{-\tau(t)\rho}\tau(t)\xi(t)\int_\Omega\int_0^1 \frac{d}{dt}G(z(x,\rho,t))d\rho dx\\
&= \left[\xi'(t)e^{-\tau(t)\rho}\int_\Omega\int_0^1 G(z(x,\rho,t))d\rho dx - \xi(t)\rho e^{-\tau(t)\rho}\tau'(t)\int_\Omega\int_0^1 G(z(x,\rho,t))d\rho dx\right]\\
&+ \frac{1}{\tau(t)}e^{-\tau(t)\rho}\xi(t)\int_\Omega\int_0^1 \frac{\partial}{\partial\rho}(1-\tau'(t)\rho)G(z(x,\rho,t))d\rho dx\\
&\leq -\xi(t)\rho e^{-\tau(t)\rho}\tau'(t)\int_\Omega\int_0^1 G(z(x,\rho,t))d\rho dx + \xi(t)\frac{\beta}{\tau(t)}\int_\Omega G(z(x,1,t))dx\\
&+ \frac{1}{\tau(t)}\left[\xi(t)\int_\Omega [G(z(x,0,t))dx - G(z(x,1,t))]dx\right]\\
&\leq -2c\xi(t)I(t) - \frac{c\xi(t)}{2\tau_1}\int_\Omega G(z(x,1,t))dx + \frac{\xi(t)}{\tau_0}\|\psi(u_t)\|_2^2,
\end{aligned} \tag{3.54}$$

□

Theorem 3.1. . *Let $(H1)-(H4)$ hold. And $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$, $f_0 \in L^2(\Omega\times(0,1))$ be given. Then the solution of the problem (2.13) is global and bounded in time. Furthermore, we have the following decay estimates:*

$$E(t) \leq \omega_1 H_1^{-1}(\omega_2 t + \omega_3), \quad \forall t > 0,$$

where

$$H_1(t) = \int_t^1 \frac{1}{H_2(s)}ds. \tag{3.55}$$

Proof. . First, we prove $T = \infty$, it is sufficient to show that $l\|\nabla u\|_2^2$ is bounded independently of t. We have from (2.15)

$$E(0) \geq E(t) \geq l\|\nabla u\|_2^2.$$

Then the energy is uniformly bounded.
Hence we conclude from lemma 3.2, lemma 3.3 and 3.4 that

$$\begin{aligned}
\frac{dL(t)}{dt} &\leq \left\{\frac{M}{2} - \frac{\epsilon g(0)c_s^2}{4\alpha}\right\}(g' o\nabla u)(t) + \left\{\epsilon\left(\frac{\mu_2(1-l)}{4\alpha}\left(2(\alpha+1) + c_s^2 + \frac{1-l}{2}\right)\right)\right\}(go\nabla u)(t)\\
&- \left\{\epsilon\left((1-\alpha-(1-l)^2(2+(1+l))) - (\mu_1+\mu_2)\alpha c_s^2\right)\right\}\|\nabla u\|_2^2 - \epsilon(g_0-\alpha-1)\|u_t\|_\gamma^\gamma\\
&- Mc_1\int_\Omega u_t\psi(u_t)dx - Mc_2\int_\Omega z(x,1,t)\psi(z(x,1,t))dx + \frac{\epsilon\mu_1}{4\alpha}\|\psi(u_t)\|_2^2 + \frac{\xi(t)}{\tau_0}\|\|\psi(u_t)\|_2^2\\
&+ \epsilon\left\{\frac{\mu_2 c_s^2(1-l)^2}{4\alpha} + \frac{\mu_2}{4\alpha}\right\}\|\psi(z(x,1,t))\|_2^2 - 2\epsilon\int_\Omega\int_0^1 e^{-2\rho\tau(t)}G(z(x,\rho,t))d\rho dx\\
&- \epsilon\frac{e^{-2\tau(t)}}{\tau_1}\int_\Omega G(z(x,1,t))dx + \frac{\epsilon}{\tau_0}\int_\Omega G(z(x,0,t))dx - \frac{M}{2}g(t)\|\nabla u(t)\|_2^2.
\end{aligned} \tag{3.56}$$

Choosing carefully ϵ sufficiently small and M sufficiently large and put

$$\left\{\frac{M}{2} - \frac{\epsilon g(0)c_s^2}{4\alpha}\right\} = \eta_0 > 0,$$

$$\left\{\frac{\mu_2(1-l)}{2\alpha}\left(2(\alpha+1) + c_s^2 + \frac{1-l}{2}\right)\right\} = \eta_1 > 0,$$

$$\left\{(1-\alpha-(1-l)^2(2+(1+l))) - (\mu_1+\mu_2)\alpha c_s^2\right\} = \eta_2 > 0,$$

$$\{g_0 - \alpha - 1\} = \eta_3 > 0,$$

then (3.56) takes the form

$$\frac{dL(t)}{dt} \leq -\theta\epsilon E(t) + \epsilon\frac{\eta_1}{2}(go\nabla u)(t) + \epsilon c\|\psi(u_t)\|_2^2, \tag{3.57}$$

where θ is positive constant, setting

$$\lambda_1 = \frac{\theta\epsilon}{\beta_2},\ \lambda_2 = \frac{\eta_1\epsilon}{2},\ \lambda_3 = \epsilon c,$$

the last inequality becomes

$$\frac{dL(t)}{dt} \leq -\lambda_1 E(t) + \lambda_2(go\nabla u)(t) + \lambda_3\|\psi(u_t)\|_2^2, \tag{3.58}$$

multiplying (3.58) by $\xi(t)$ we get

$$\begin{aligned}\xi(t)\frac{dL(t)}{dt} &\leq -\lambda_1\xi(t)E(t) + \lambda_2\xi(t)(go\nabla u)(t) + \lambda_3\xi(t)\|\psi(u_t)\|_2^2\\ &\leq -\lambda_1\xi(t)E(t) - \lambda_2\xi(t)(g^{'}o\nabla u)(t) + \lambda_3\xi(t)\|\psi(u_t)\|_2^2\\ &\leq -\lambda_1\xi(t)E(t) - cE^{'}(t) + \lambda_3\xi(t)\|\psi(u_t)\|_2^2,\end{aligned} \tag{3.59}$$

we consider the following partition on Γ_1

$$\Omega_{11} = \{x \in \Omega; |u_t| \geq \epsilon^{'}\}, \quad \Omega_{12} = \{x \in \Omega; |u_t| \leq \epsilon^{'}\},$$

then it is clear that $F = L(t) + c\xi(t)E(t)$ is equivalent to E(t), then

$$F^{'}(t) \leq -\lambda_1\xi(t)E(t) + \lambda_3\xi(t)\|\psi(u_t)\|_2^2 \quad \forall t \geq t_0, \tag{3.60}$$

from (2.2) and (2.3), it follows that

$$\int_{\Omega_{12}} |\psi(u_t)|^2 dx \leq \mu_1 \int_{\Omega_{12}} u_t\|\psi(u_t)\|_2^2 dx \leq -\mu_1 E^{'}(t). \tag{3.61}$$

case 1: H is linear then, according to $(H1)$

$$c_1^{'}|s| \leq |\psi(s)| \leq c_2^{'}|s|, \quad \forall s,$$

and so

$$\psi^2(s) \leq c_2^{'}s\psi(s), \quad \forall s.$$

H is linear on $[0,\epsilon^{'}]$. In this case one can easily check that there exists $\mu_1^{'} > 0$, such that $|\psi(s)| \leq \mu_1^{'}|s|$ for all $|s| \leq \epsilon^{'}$, and thus

$$\int_{\Omega_{11}} \|\psi(u_t)\|^2 dx \leq \mu_1^{'} \int_{\Omega_{11}} u_t\psi(u_t)dx \leq -\mu_1^{'}E^{'}(t), \tag{3.62}$$

using (3.61), (3.62) and the fact that $\xi^{'}(t) \leq 0$, it is clearly that $\vartheta = L(t)\xi(t) + c(\mu_1 + \mu_1^{'})E$ equivalent to E(t) then, from (3.60) produces

$$E(t) \leq ce^{-c\int_0^t \xi(s)ds} = H_1^{-1}\left(\int_0^t \xi(s)ds\right). \tag{3.63}$$

case2 : $H^{'}(0) = 0$ and $H^{''} > 0$ on $[0,\epsilon^{'}]$ since H is convex and increasing H^{-1} is concave and increasing by Jensen's inequality

$$\begin{aligned}\int_{\Omega_{12}} |\psi(u_t)|^2 dx &\leq \int_{\Omega_{12}} H^{-1}\left(u_t\psi(u_t)dx\right)\\ &\leq |\Omega_{12}|H^{-1}\left(\frac{1}{|\Omega_{12}|}u_t\psi(u_t)dx\right)\\ &\leq cH^{-1}(-c^{'}E^{'}(t)),\end{aligned} \tag{3.64}$$

then using (2.1), (3.62) and (3.64) we get

$$\begin{aligned}\int_{\Omega}|\psi(u_t)|^2dx &= \int_{\Omega_{11}}|\psi(u_t)|^2dx + \int_{\Omega_{12}}|\psi(u_t)|^2dx\\ &\leq \int_{\Omega_{12}}H^{-1}u_t\psi(u_t)dx + \int_{\Omega_{12}}u_t\psi(u_t)dx\\ &\leq |\Omega_{12}|H^{-1}\left(\frac{1}{|\Omega_{12}|}u_t\psi(u_t)dx\right) + \int_{\Omega_{12}}u_t\psi(u_t)dx\\ &\leq cH^{-1}(-c^{'}E^{'}(t)) - c\xi(t)\mu_1^{'}E^{'}(t),\end{aligned} \tag{3.65}$$

it is clearly $F = L(t) + c\mu_1 E(t)$ equivalent to E(t) therefore (3.65) becomes

$$F^{'}(t) \leq \lambda_1\xi(t)E(t) + cH^{-1}(-c^{'}E^{'}(t)), \quad \forall t \geq t_0. \tag{3.66}$$

Let us denote by H^* the conjugate function of the convex function H, i.e.,

$$H^* = \sup_{t\in R_+}(st - H(t)). \tag{3.67}$$

Then H^* is the Legendre transform of H which satisfies the following inequality

$$st \leq H^* + H(t), \quad \forall s,t \geq 0, \tag{3.68}$$

and

$$H^* = s(H^{'})^{-1}(s) - H[(H^{'})^{-1}(s)], \forall s \geq 0, \tag{3.69}$$

the relation (3.69) and the fact that $H^{'}(0) = 0$ and $(H^{'})^{-1}$, H are increasing function yield

$$H^*(s) \leq s(H^{'})^{-1}(s), \quad \forall s \geq 0, \tag{3.70}$$

using the fact that $E^{'} \leq 0$, $H^{'} \geq 0$, $H^{''} \geq 0$ we derive $\epsilon_0 > 0$ small enough we find that the functional F_1 defined by

$$F_1(t) = H^{'}(\epsilon_0 E(t))F(t) + c_3E(t), \tag{3.71}$$

satisfies, for some $\nu_1, \nu_2 > 0$

$$\nu_1F_1(t) \leq E(t) \leq \nu_2F_1(t), \tag{3.72}$$

taking the derivative of (3.71)

$$\begin{aligned}F_1^{'}(t) &= \epsilon_0E^{'}(t)H^{''}(\epsilon_0E(t))(H^{'}(\epsilon_0E(t))F(t) + c_3E(t)) + H^{'}(\epsilon_0E(t))(L^{'}(t) + c\mu_1E^{'}(t)) + c_3E^{'}(t)\\ &\leq -\lambda_1\xi(t)E(t)H^{'}(\epsilon_0E(t)) + \hat{c}_3H^{'}(\epsilon_0E(t))H^{-1}(-c^{'}E^{'}(t)) + \hat{c}_3c^{'}E^{'}(t)\\ &\leq -\lambda_1\xi(t)E(t)H^{'}(\epsilon_0E(t)) + \hat{c}_3H^*(H^{'}(\epsilon_0E(t))) - \hat{c}_3\xi(t)E^{'}(t) + c_3E^{'}(t)\\ &\leq -\lambda_1\xi(t)E(t)H^{'}(\epsilon_0E(t)) + \epsilon_0\hat{c}_3\xi(t)E(t)(H^{'}(\epsilon_0E(t))) - \hat{c}_3\xi(t)E^{'}(t) + c_3E^{'}(t)\\ &\leq -c\xi(t)H_2E(t),\end{aligned} \tag{3.73}$$

where $H_2(t) = tH^{'}(\epsilon_0t)$ we can observe from lemma 3.1 that L(t) is equivalent to E(t). So, $F_1(t)$ is also equivalent to E(t). By the fact that H_2 is increasing we obtain

$$F_1^{'}(t) \leq -\hat{c}\xi(t)H_2F_1(t), \forall t \geq 0. \tag{3.74}$$

Noting that $H_1^{'} = \frac{-1}{H_2}$, we infer from (3.74)

$$[F_1(t)H_1(F_1(t))]^{'} \geq \hat{c}\xi(t), \forall t \geq 0. \tag{3.75}$$

A simple integration over (0,t) yields

$$H_1(F_1(t)) \geq \hat{c}\int_0^t \xi(s)ds + H_1(F_1(0)), \tag{3.76}$$

exploiting the fact that H_1^{-1} is decreasing , we infer

$$F_1(t) \leq H_1^{-1}\left(\hat{c}\int_0^t \xi(s)ds + H_1(F_1(0))\right), \tag{3.77}$$

the equivalence of L, F_1 and E, yields the estimate

$$E(t) \leq H_1^{-1}\left(\hat{c}\int_0^t \xi(s)ds + H_1(F_1(0))\right). \tag{3.78}$$

Which completes the proof. □

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