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On pseudo valuation and pseudo almost valuation semidomains

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Abstract. In this paper, we characterize pseudo valuation semidomains and discuss some conditions which force a semidomain to be a pseudo valuation semidomain. Also, the notion of pseudo almost valuation semidomains is introduced and some results regarding pseudo almost valuation semidomains are investigated.

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Contents

1. Introduction

Several algebraists have done immense work in the field of valuation rings. Pseudo valuation domains are the rings which are closely related to valuation domains. In [12], Hedstrom introduced the notion of pseudo valuation domains and further studied by Anderson [2] and Badawi [3]. Semirings [8] are the generalization of elaborately studied algebraic structures such as rings, bounded distributive lattices and have significant applications in computer science, engineering and optimization theory (cf. [11, 13, 14]). The brief structure of semirings have been studied by various researchers (cf. [5–10, 16, 18]). The algebraic structure pseudo valuation semidomain is a generalization of the pseudo valuation domain.

In this paper, by a semiring S, we mean a nonempty set S on which operations of addition and multiplication have been defined such that the following conditions are satisfied:
(i) $(S,+)$ is a commutative monoid with identity element 0;
(ii) $(S,.)$ is a monoid with identity element 1;
(iii) Multiplication distributes over addition from either side;
(iv) $0s=0=s0$, for all $s\in S$;
(v) $1\neq 0$.

Moreover, a commutative semiring S is said to be a semidomain if S is multiplicatively cancellative semiring i.e., $xy=xz$ implies $y=z$ for all $x,y,z\in S$ with $x\neq 0$. Using techniques adapted from the ring theory, it is easy to show that a semidomain S can be embedded into a semifield, known as the semifield of fractions, denoted

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by $F(S)$ (see p. 22 in [7]). For instance, we denote every element of $F(S)$ by a/b, where $a \in S$ and $b \in S\backslash\{0\}$. Throughout this article, S represents a semidomain with semifield of fractions $F(S)$.

A semidomain S is a valuation semidomain if and only if its lattice of ideals is a chain [17, Theorem 2.4]. A proper ideal $\wp$ of a semiring S is said to be a prime ideal, if $xy \in \wp$ implies either $x \in \wp$ or $y \in \wp$. An ideal N of a semiring S is a maximal ideal if it is maximal among all proper ideals of S. If x and y are elements of a semiring S, then y divide x, written as $y|x$, if there exists $a \in S$ such that $x = ay$. This is equivalent to $(x) \subseteq (y)$ and for more on the ideals of semiring, one can refer to (cf. [15, 16, 19, 20]). A prime ideal $\wp$ of a semidomain S is called strongly prime if $x, y \in F(S)$ and $xy \in \wp$ infers that $x \in \wp$ or $y \in \wp$. If every prime ideal of a semidomain S is a strongly prime, then S is called a pseudo valuation semidomain. However, an oversemiring of a semidomain S is a semiring between S and $F(S)$. If I is an ideal of S, then we define the set $(I : I) = \{x \in F(S)|xI \subseteq I\}$. One can easily verify that $(I : I)$ is an oversemiring of S and I is an ideal of $(I : I)$.

The radical of an ideal I of a semiring S, denoted by $\sqrt{I}$, is the intersection of all prime ideals of S containing I. Furthermore, a semiring S is said to be quasi-local if it has only one maximal ideal. It was shown in Nasehpour [17, Proposition 3.5] that a semiring S is quasi-local if and only if $S\backslash U(S)$ is an ideal of S, where $U(S)$ is the set of all unit elements of S. In this paper, we prove some results of pseudo valuation semidomains and introduce a new closely related class of semidomains called pseudo almost valuation semidomains. The main objective of this paper is to obtain results analogous to Hedstrom's and Badawi's results.

The second section is devoted to the characterization of pseudo valuation semidomains. Furthermore, several comparable conditions which allow a semidomain to be a pseudo valuation semidomain are discussed. Also, the relationship between valuation, pseudo valuation and quasi-local semidomains are derived. In section 3, we introduce and characterize the notion of pseudo almost valuation and almost valuation semidomains. In addition, we study some properties of pseudo almost valuation semidomains by establishing the connection between pseudo almost valuation, almost valuation and quasi-local semidomains.

2. Properties of Pseudo valuation semidomains

In this section, we characterize the pseudo valuation semidomains. We start this section with the following examples of pseudo valuation semidomain.

Example 2.1. *Let $S = \{0, 1\}$, we define operations with the help of the following tables:*

$\oplus$	0	1
0	0	1
1	1	1

$\odot$	0	1
0	0	0
1	0	1

One can easily see that S is a semidomain with only one prime ideal $I = \{0\}$, which is a strongly prime ideal. Therefore, S is a pseudo valuation semidomain.

Example 2.2. *Consider $S = \{0, 1, 2, ..., p-1\}$, where p is a prime number with binary operations addition $\oplus$ and multiplication $\odot$ as follows:*

$$\left\{\begin{array}{c} a \oplus b = a + b \text{ if } a + b \leq p-1, \text{ otherwise, } a \oplus b \equiv a + b (mod p); \\ a \odot b = ab \text{ if } ab \leq p-1, \text{ otherwise, } a \odot b \equiv ab (mod p), \end{array}\right\}$$

for all $a, b \in S$. Clearly, S is a pseudo valuation semidomain.

The proofs of the next two lemmas are quite easy, so we omit the proofs.

Lemma 2.3. *Let S be a semidomain. Then the following statements hold:*
(i) If $\wp$ is a strongly prime ideal of S and I is an ideal of S, then $\wp$ and I are comparable.
(ii) If S is a pseudo valuation semidomain, then S is quasi-local.

Lemma 2.4. *If $\wp$ is a prime ideal of S, then $\wp$ is a strongly prime ideal if and only if for every $a \in F(S)\backslash S$, $a^{-1}\wp \subseteq \wp$.*

Theorem 2.5. *A semidomain S is a pseudo valuation semidomain if and only if for a maximal ideal N of S, $a^{-1}N \subseteq N$, for every $a \in F(S)\backslash S$.*

Proof. Let S be a pseudo valuation semidomain with a maximal ideal N. Then by definition of S, N is a strongly prime ideal and by the above lemma, $a^{-1}N \subseteq N$, for every $a \in F(S)\backslash S$.

Conversely, assume that N is a maximal ideal of S such that for every $a \in F(S)\backslash S$, $a^{-1}N \subseteq N$. First, we will show that S is quasi-local. For this, let $N^{'}$ be a maximal ideal of S such that $N \neq N^{'}$. By using Lemma 2.3 and Lemma 2.4, we get that N and $N^{'}$ are comparable, which is a contradiction. Thus S is quasi-local with maximal ideal N. Now, let $\wp$ be any prime ideal of S. To show S is a pseudo valuation semidomain, it suffices to prove that $\wp$ is a strongly prime ideal of S. Again by using Lemma 2.3, $\wp$ and N are comparable. Since N is a maximal ideal, so $\wp \subseteq N$. Further, by using the hypothesis, we have $a^{-1}\wp \subseteq N$, for every $a \in F(S)\backslash S$. For $p \in \wp$, we obtain $a^{-1}pa^{-1} \in N$ which infers that $(a^{-1}p)(a^{-1}p) \in \wp$. This leads to $a^{-1}p \in \wp$, since $\wp$ is a prime ideal. By Lemma 2.4, $\wp$ is a strongly prime ideal of S. Hence, S is a pseudo valuation semidomain. ■

In light of the proof of Theorem 2.5, we have the following corollary.

Corollary 2.6. *Let $\wp$ be a strongly prime ideal of S. If U is a prime ideal of S and $U \subseteq \wp$, then U is a strongly prime ideal of S.*

Proposition 2.7. *If N is a maximal ideal of S such that xN and N are comparable for each $x \in F(S)$, then S is quasi-local.*

Proof. Let N be a maximal ideal of S such that xN and N are comparable for each $x \in F(S)$. Suppose that $N^{'}$ is a maximal ideal of S with $N^{'} \neq N$. Choose $a \in N\backslash N^{'}$ and $b \in N^{'}\backslash N$. By the hypothesis, either $ab^{-1}N \subseteq N$ or $N \subseteq ab^{-1}N$. If $ab^{-1}N \subseteq N$, then $aN \subseteq bN \subseteq N^{'}$. As $N^{'}$ is a prime ideal and $a \in N$, so $a^2 \in aN \subseteq N^{'}$ implies $a \in N^{'}$, which is not possible. Now, if $N \subseteq ab^{-1}N$, then $a^{-1}bN \subseteq N$. As $a \in N$, so $b = a^{-1}ba \in N$, which is a contradiction. Hence S is quasi-local. ■

By using the above result, one can easily prove

Proposition 2.8. *If N is a maximal ideal of S, then the following statements are equivalent:*
(i) *aN and N are comparable, whenever $a \in F(S)$.*
(ii) *S is quasi-local and for each $a \in F(S)$, either $aN \subseteq S$ or $N \subseteq aS$.*

Lemma 2.9. *If N is a maximal ideal of S such that for each pair I_1, I_2 of ideals of S, either $I_1 \subseteq I_2$ or $I_2N \subseteq I_1$, then S is quasi-local.*

Proof. Let N be a maximal ideal of S such that for each pair I_1, I_2 of ideals of S, either $I_1 \subseteq I_2$ or $I_2N \subseteq I_1$. Assume that M is a maximal ideal of S with $M \neq N$. By using the given hypothesis, for $I_1 = M$ and $I_2 = N$, either $M \subseteq N$ or $N^2 \subseteq M$. Since M and N are two distinct maximal ideals of S, so $M \subseteq N$, a contradiction. But $N^2 \subseteq M$ implies $N \subseteq M$, which is again a contradiction. Hence, S is quasi-local. ■

Theorem 2.10. *Let N be a maximal ideal of S. Then S is a pseudo valuation semidomain if and only if for each pair I_1, I_2 of ideals of S, either $I_1 \subseteq I_2$ or $I_2N \subseteq I_1$.*

Proof. Let S be a pseudo valuation semidomain and I_1, I_2 be any pair of ideals of S. Since S is a pseudo valuation semidomain, then N is a strongly prime ideal. By Lemma 2.3, I_1 and I_2 are comparable to N. Further, let $I_1 \nsubseteq I_2$ and $a \in I_1\backslash I_2$. Then for each $b \in I_2$, we get $a/b \notin S$. By using Lemma 2.4, we have $(a/b)^{-1}N \subseteq N$, which infers that $bN \subseteq aN \subseteq I_1N$. Thus, $NI_2 \subseteq NI_1 \subseteq I_1$. Conversely, assume that for each pair I_1, I_2 of ideals of S, either $I_1 \subseteq I_2$ or $I_2N \subseteq I_1$. By the above lemma, S is quasi-local with maximal ideal N. We have to show that for every $x \in F(S)\backslash S$, $x^{-1}N \subseteq N$. Now, let $a, b \in S$ with $a/b \notin S$. Then $(a) \nsubseteq (b)$. As $(a) \nsubseteq (b)$, then by the hypothesis $N(b) \subseteq (a)$. Further, $Nb \subseteq (a)$ leads to $Nba^{-1} \subseteq S$. If $N(b/a) = S$, then $N = S(a/b)$ and $a/b \in N \subset S$, which is a contradiction. So $Nba^{-1} \subseteq N$. Therefore, $x^{-1}N \subseteq N$, whenever $x \in F(S)\backslash S$ and by Theorem 2.5, S is a pseudo valuation semidomain. ■

In the forthcoming result, we prove that a semidomain S is a pseudo valuation semidomain if and only if for each nonunit $a \in S$, we have $b^{-1}a \in S$ for all $b \in F(S)\backslash S$.

Theorem 2.11. *A semidomain S is a pseudo valuation semidomain if and only if for each nonunit $a \in S$, we have $b^{-1}a \in S$ for all $b \in F(S)\backslash S$.*

Proof. Let S be a pseudo valuation semidomain and a be a nonunit of S. By Lemma 2.3, the semidomain S is quasi-local. As a is a nonunit, so a belongs to the maximal ideal, say N of S. Further, by using Lemma 2.4, we conclude that $b^{-1}a \in N \subset S$, for each $b \in F(S)\backslash S$. Thus, $b^{-1}a \in S$ for each nonunit $a \in S$ and $b \in F(S)\backslash S$.

Conversely, assume that $b^{-1}a \in S$ for each nonunit $a \in S$ and $b \in F(S)\backslash S$. It suffices to prove that each prime ideal $\wp$ of S is a strongly prime ideal. Let $x, y \in F(S)$ such that $xy \in \wp$. If $x, y \in S$, then $x \in \wp$ or $y \in \wp$. Suppose that $x \in F(S)\backslash S$, then by the hypothesis $y = x^{-1}xy \in S$. Now, we have to show that y is a nonunit of S. If possible, let y be a unit of S. Then $x = xyy^{-1} \in \wp$, which is a contradiction. Thus, y is a nonunit and $y^2 = x^{-1}yxy \in \wp$. Since $\wp$ is a prime ideal of S, so $y \in \wp$. Therefore, $\wp$ is a strongly prime ideal and hence S is a pseudo valuation semidomain. ■

We need the following lemma for the proof of subsequent results.

Lemma 2.12. *[17] For a semidomain S, the following statements are equivalent:*
(i) *S is a valuation semiring.*
(ii) *For any element $a \in F(S)$, either $a \in S$ or $a^{-1} \in S$.*
(iii) *For any ideals I, J of S, either $I \subseteq J$ or $J \subseteq I$.*
(iv) *For any elements $a, b \in S$, either $(a) \subseteq (b)$ or $(b) \subseteq (a)$.*

Proposition 2.13. *If S is a pseudo valuation semidomain with a nonzero principal prime ideal, then S is a valuation semidomain.*

Proof. Let $\wp = (p)$ be a nonzero principal prime ideal generated by some prime p of S. Assume that $\wp$ is nonmaximal, then there exists a nonunit element $a \in S\backslash\wp$. By Lemma 2.3, we have $\wp \subset (a)$. In particular, $p \in (a)$, which is not true, as $\wp$ is prime and $a \notin \wp$, a is a nonunit element of S. This concludes that $\wp$ is a maximal ideal of S. Let a, b be nonunits in S and $ab^{-1} \in F(S)\backslash S$. By Lemma 2.4, we have $ba^{-1}p \in \wp$ and $ba^{-1} = s$, for some $s \in S$. Therefore, $b = as$ gives that $(b) \subseteq (a)$. Hence, by Lemma 2.12, S is a valuation semidomain. ■

By using Lemma 2.12, one can easily prove

Proposition 2.14. *Every valuation semidomain is a pseudo valuation semidomain.*

Now, we will close this section with

Theorem 2.15. *Let S be a quasi-local semidomain with a maximal ideal N. Then the following statements are equivalent:*
(i) *S is a pseudo valuation semidomain.*
(ii) *$(N : N)$ is a valuation semidomain with maximal ideal N.*

Proof. $(i) \Rightarrow (ii)$ Suppose that S is a pseudo valuation semidomain and $a \in F(S)\backslash S$. Then, by Lemma 2.4, we have $a^{-1}N \subseteq N$. So $a^{-1} \in (N : N)$ and Lemma 2.12 infers that $(N : N)$ is a valuation semidomain. Now, N is an ideal of $(N : N)$, so it is sufficient to prove that N is a maximal ideal of $(N : N)$. For this, let $a \in (N : N)$ be a nonunit element. Suppose that $a \notin N$. Then, $a \notin S$. Therefore, by Lemma 2.4, we have $a^{-1}N \subseteq N$. This implies $a^{-1} \in (N : N)$, a contradiction. Thus, if a is a nonunit of $(N : N)$, then $a \in N$. Hence, N is a maximal ideal of $(N : N)$.

$(ii) \Rightarrow (i)$ Suppose that $(N : N)$ is a valuation semidomain with a maximal ideal N. Let $\wp$ be any prime ideal of S and $p \in \wp, a \in (N : N)$. As S is quasi-local with maximal ideal N and $p \in \wp$, so $p \in N$. Thus,

$ap \in N$ and $(ap)(ap) \in \wp$, as $apa \in N$. From $(ap)(ap) \in \wp$, we get $ap \in \wp$. Hence, $\wp$ is an ideal of $(N : N)$. To show that $\wp$ is a prime ideal of $(N : N)$, let $ab \in \wp$ with $a, b \in (N : N)$. If $a, b \in S$, then either $a \in \wp$ or $b \in \wp$. Suppose that $a \notin S$, so $a \notin N$ and $a^{-1} \in (N : N)$. As $\wp$ is an ideal of $(N : N)$, we have $b = a^{-1}ab \in \wp$. Thus, $\wp$ is a prime ideal of $(N : N)$. By Proposition 2.14, $(N : N)$ is a pseudo valuation semidomain. So $\wp$ is a strongly prime ideal. Therefore, every prime ideal of S is strongly prime and hence S is a pseudo valuation semidomain. ■

3. Pseudo almost valuation semidomains

In this section, we study pseudo almost valuation semidomains. Let us recall that a prime ideal $\wp$ of an integral domain R with quotient field K is called a pseudo strongly prime ideal in sense of Badawi [4] if, whenever $a, b \in K$ and $ab\wp \subseteq \wp$, then there is a positive integer $n \geq 1$ such that either $a^n \in R$ or $b^n\wp \subseteq \wp$. If every prime ideal of R is a pseudo strongly prime ideal, then R is called a pseudo almost valuation domain. This motivated us to give the definition of a pseudo almost valuation semidomain and its several characterizations. Throughout this section, $\ell(A)$ =$\{a \in F(S) \mid a^n \notin S$ for every $n \geq 1\}$, where $A \subseteq S$. We begin with

Definition 3.1. *A prime ideal $\wp$ of S is said to be a pseudo strongly prime if, whenever $a, b \in F(S)$ and $ab\wp \subseteq \wp$, then there is a positive integer $n \geqslant 1$ such that either $a^n \in S$ or $b^n\wp \subseteq \wp$. If every prime ideal of S is a pseudo strongly prime ideal, then S is a pseudo almost valuation semidomain.*

Proposition 3.2. *Every pseudo valuation semidonain is a pseudo almost valuation semidomain.*

Proof. Assume that S is a pseudo valuation semidomain and $\wp$ is any prime ideal of S. To show that S is a pseudo almost valuation semidomain, it suffices to show that $\wp$ is a pseudo strongly prime ideal. Let $a, b \in F(S)$ such that $ab\wp \subseteq \wp$. Suppose that $a \in \ell(S)$. Then, by Lemma 2.4, we have $a^{-n}(a^nb^n\wp) \subseteq \wp$ for every $n \geq 1$, since $a^nb^n\wp \subseteq \wp$ and $a^n \in F(S)\backslash S$. This concludes that $b^n\wp \subseteq \wp$. Therefore, $\wp$ is a pseudo strongly prime ideal of S and hence S is a pseudo almost valuation semidomain. ■

Lemma 3.3. *If $\wp$ is a prime ideal of S, then following statements are equivalent:*
(i) $\wp$ is a pseudo strongly prime ideal.
(ii) For every $a \in \ell(S)$, there is an $n \geq 1$ such that $a^{-n}\wp \subseteq \wp$.

Proof. $(i) \Rightarrow (ii)$ Assume that $\wp$ is a pseudo strongly prime ideal of S and $a \in \ell(S)$. Let $p \in \wp$, then $p = aa^{-1}p \in \wp$ which infers that $a^{-1}p \in \wp$. As $a^{-1}p \in a^{-1}\wp$, so $a^{-1}\wp \subseteq \wp$. Thus, there is an $n \geq 1$ such that $a^{-n}\wp \subseteq \wp$, since $a \in \ell(S)$.

$(ii) \Rightarrow (i)$ Assume that for every $a \in \ell(S)$, there is an $n \geq 1$ such that $a^{-n}\wp \subseteq \wp$. To see that $\wp$ is a pseudo strongly prime ideal, let $ab\wp \subseteq \wp$ with $a, b \in F(S)$. Assume that $a \in \ell(S)$, then by the hypothesis, there is an $n \geq 1$ such that $a^{-n}\wp \subseteq \wp$. As $ab\wp \subseteq \wp$, so $a^nb^n\wp \subseteq \wp$. This implies that $b^n\wp = a^{-n}(a^nb^n\wp) \subseteq \wp$, since $a^{-n}\wp \subseteq \wp$. Therefore, $\wp$ is a pseudo strongly prime ideal. ■

Proposition 3.4. *If $\wp_1$ and $\wp_2$ are pseudo strongly prime ideals of S, then $\wp_1$ and $\wp_2$ are comparable.*

Proof. Suppose that $\wp_1$ and $\wp_2$ are pseudo strongly prime ideals of S. Our claim is that $\wp_1$ and $\wp_2$ are comparable. If not, then there exists $a \in \wp_1\backslash\wp_2$ and $b \in \wp_2\backslash\wp_1$. As $a \notin \wp_2$, so $a/b \in \ell(S)$. By using Lemma 3.3, there is an $n \geq 1$ such that $(a/b)^{-n}\wp_1 \subseteq \wp_1$. Therefore, $(b)^n = (a/b)^{-n}a^n \in \wp_1$. As $\wp_1$ is prime, so $(b)^n \in \wp_1$ concludes that $b \in \wp_1$, which is a contradiction. Hence, $\wp_1$ and $\wp_2$ are comparable. ■

Next corollary is an immediate consequence of Proposition 3.4.

Corollary 3.5. *If S is a pseudo almost valuation semidomain, then the prime ideals of S are linearly ordered. In particular, S is quasi-local.*

Theorem 3.6. *Let N be a maximal ideal of S. Then S is a pseudo almost valuation semidomain if and only if N is a pseudo strongly prime ideal.*

Proof. Let S be a pseudo almost valuation semidomain. Then, N is a pseudo strongly prime ideal, as N is a maximal ideal of S. Conversely, let N be a pseudo strongly prime ideal of S. We first observe that S is quasi-local. If not, then there exists a maximal ideal $N^{'}$ of S such that $N \neq N^{'}$. By using Proposition 3.4, we get that N and $N^{'}$ are comparable, which is a contradiction. Therefore, S is quasi-local with maximal ideal N. We must show that each prime ideal $\wp$ of S is a pseudo strongly prime ideal. By Proposition 3.4, we have $\wp \subseteq N$. Assume that $a \in \ell(S)$. Then, by Lemma 3.3, there exists an $n \geq 1$ such that $a^{-n}\wp \subseteq N$. Now, $(a^{-n}p)a^{-n} \in N$ for $p \in \wp$. So $(a^{-n}p)(a^{-n}p) \in \wp$ which leads to $a^{-n}\wp \subseteq \wp$, as $\wp$ is a prime ideal. Further, by Lemma 3.3, we conclude that $\wp$ is a pseudo strongly prime ideal of S. Hence, S is a pseudo almost valuation semidomain. ■

Corollary 3.7. *Let $\wp_1$ be a pseudo strongly prime ideal of S and $\wp_2$ be a prime ideal of S such that $\wp_2 \subseteq \wp_1$. Then $\wp_2$ is a pseudo strongly prime ideal of S.*

Recall from [1] that an integral domain R is an almost valuation domain if for each pair $a, b \in R\backslash\{0\}$, there exists an integer $n \geq 1$(depending on a, b) with $a^n|b^n$ or $b^n|a^n$. Influenced by this concept, we give the following definition

Definition 3.8. *A semidomain S is said to be an almost valuation semidomain if for each pair $a, b \in S\backslash\{0\}$, there exists an integer $n \geq 1$ (depending on a, b) with $a^n|b^n$ or $b^n|a^n$.*

The upcoming proposition follows immediately by definition of almost valuation semidomain.

Proposition 3.9. *For a semidomain S, the following statements are equivalent:*
(i) *S is an almost valuation semidomain.*
(ii) *For each $x \in F(S)\backslash\{0\}$, there exists an $n \geq 1$(depending on x) with x^n or $x^{-n} \in S$.*

Lemma 3.10. *Every valuation semidomain is an almost valuation semidomain.*

Proof. Let S be a valuation semidomain. Then, by using Lemma 2.12, we get that for any element $a \in F(S)$, either $a \in S$ or $a^{-1} \in S$. Further, consider $a \neq 0$, we have $a \in S$ or $a^{-1} \in S$. Therefore, for $n \geq 1$, we conclude that $a^n \in S$ or $a^{-n} \in S$. By Proposition 3.9, S is an almost valuation semidomain. ■

Remark 3.11. *If I is a proper ideal of S and $D = (I : I)$, then $ID = I$.*

Theorem 3.12. *Let S be a quasi-local semidomain with maximal ideal N. Then S is a pseudo almost valuation semidomain if and only if $D = (N : N)$ is almost valuation semidomain with maximal ideal $\sqrt{ND}$ (the radical of ND in D).*

Proof. Let S be a pseudo almost valuation semidomain and let, $a \in \ell(D)$. Obviously, $a \in \ell(S)$. By Lemma 3.3, we get that there is an $n \geq 1$ such that $a^{-n}N \subseteq N$, since N is a pseudo strongly prime ideal and $a \in \ell(S)$. Therefore, $a^{-n} \in D$. By Proposition 3.9 and Remark 3.11, D is almost valuation semidomain and $ND = N$. Now, we have to show that $\sqrt{ND}$ is a maximal ideal of D. Let a be a nonunit element of D. If $a \notin \sqrt{ND}$, then $a \in \ell(S)$, as $ND = N$ and a is a nonunit of D. Further, by using Lemma 3.3, we conclude that there is an $n \geq 1$ such that $a^{-n}N \subseteq N$ which infers that $a^{-n} \in D$. This implies that a is a unit in D, a contradiction. Therefore, $a \in \sqrt{ND}$ and hence $\sqrt{ND}$ is a maximal ideal of D.

Conversely, let $D = (N : N)$ be an almost valuation semidomain with maximal ideal $\sqrt{ND}$. Let $a \in \ell(S)$, then $a \notin \sqrt{ND}$. If $a^n \in D$ for some $n \geq 1$, then a^n is unit of D which implies $a^{-n}N \subseteq N$. If $a \in \ell(D)$, then there is an $m \geq 1$ such that $a^{-m} \in D$, as D is almost valuation semidomain. Therefore, there is an $m \geq 1$ such that $a^{-m}N \subseteq N$. So, N is a pseudo strongly prime ideal. By Theorem 3.6, S is a pseudo almost valuation semidomain. ■

Proposition 3.13. *If S is a semidomain and for every $a, b \in S$, there is an $n \geq 1$ such that either $a^n | b^n$ or $b^n | a^n c$ for every nonunit $c \in S$, then the prime ideals of S are linearly ordered.*

Proof. Assume that S is a semidomain and for every $a, b \in S$, there is an $n \geq 1$ such that either $a^n | b^n$ or $b^n | a^n c$ for every nonunit $c \in S$. If a is a nonunit, then there is an $n \geq 1$ such that either $a^n | b^n$ or $b^n | a^{n+1}$. Let $\wp_1$ and $\wp_2$ be distinct prime ideals of S and $a \in \wp_1 \backslash \wp_2$. Therefore, for every $b \in \wp_2$, there is an $n \geq 1$ such that $a^n | b^n$ which leads to $b \in \wp_1$. So $\wp_2 \subseteq \wp_1$. Hence, the prime ideals of S are linearly ordered. ■

Theorem 3.14. *If S is a semidomain, then the following statements are equivalent:*
(i) *S is a pseudo almost valuation semidomain.*
(ii) *For every $b \in \ell(S)$, there is an $n \geq 1$ such that $ab^{-n} \in S$ for every nonunit $a \in S$.*

Proof. $(i) \Rightarrow (ii)$ Let S be a pseudo almost valuation semidomain and a be a nonunit of S. By Corollary 3.5, S is quasi-local. So a belongs to the maximal ideal, say N of S. Moreover, by Lemma 3.3, we get that for every $b \in \ell(S)$, there is an $n \geq 1$ such that $b^{-n}a \in N \subset S$. Therefore, for every $b \in \ell(S)$, there is an $n \geq 1$ such that $ab^{-n} \in S$ for every nonunit $a \in S$.

$(ii) \Rightarrow (i)$ Assume that for every $b \in \ell(S)$, there is an $n \geq 1$ such that $ab^{-n} \in S$ for every nonunit $a \in S$. To show that S is a pseudo almost valuation semidomain, we need only show that each prime ideal $\wp$ of S is a pseudo strongly prime ideal. Let $xy\wp \subseteq \wp$ with $x, y \in F(S)$. Assume that $x \in \ell(S)$. By using the hypothesis, there is an $n \geq 1$ such that $(x^n y^n p)x^{-n} = y^n p \in S$ for all $p \in \wp$, as $x^n y^n \wp \subseteq \wp$. Now, it suffices to show that $y^n p$ is a nonunit of S. If possible, let $y^n p$ be a unit element for $p \in \wp$. Then, $x^n = x^n y^n p (y^n p)^{-1} \in S$, which is a contradiction. Thus, $y^n p$ is a nonunit of S and $(y^n p)x^{-n} \in S$ for $p \in \wp$. Further, $(y^n p)^2 = (y^n p x^{-n})(x^n y^n p) \in \wp$ infers that $y^n p \in \wp$, as $\wp$ is a prime ideal. So $y^n \wp \subseteq \wp$. Therefore, $\wp$ is a pseudo strongly prime ideal. Hence, S is a pseudo almost valuation semidomain. ■

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