

On Separation Axioms in Ideal Topological Spaces

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Abstract

Separation axioms in ideal topological spaces are discussed in the literature. In this paper we define the separation axioms in ideal topological spaces in a new way which is more natural than the previous versions and discuss some properties. Also we discuss the relationship of our definition with other definitions and prove some results in the context of separation axioms in ideal topological space. We show a property that holds in ideal topological theory which does not hold in the classical theory of topology; and also we show a property that holds in the classical theory that does not hold in the ideal topological theory.

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1 Introduction

An ideal on a set X is a nonempty collection \mathcal{I} of subsets of X which is closed under finite union such that if A is in \mathcal{I} , then all subsets of A are also in \mathcal{I} . In 1944 Vaidyanathaswamy[12] introduced the concept of ideals in topological spaces. Later the concepts were further studied and discussed by Kuratowski[8], Noiri[3, 5] and many others[2, 6, 9]. If \mathcal{I} is an ideal on a topological space (X, \mathcal{T}) , then we can construct a topology on X , called $*$ -topology, denoted by \mathcal{T}^* . The triplet $(X, \mathcal{T}, \mathcal{I})$ is called an ideal topological space.

In 1995, Dontchev[2] introduced the notion of I -Hausdorff space and Abd El-Monsef[1] developed the notion of quasi- I -Hausdorff space in 2000. Later Nasef[9] has improved the concepts of I -Hausdorff space and quasi- I -Hausdorff space. Further, Hatir and Noiri[4] introduced semi- I -Hausdorff space which is weaker than Hausdorff space.

In the above stated, and in many other works, these concepts were developed using notions like \mathcal{I} -open, semi- \mathcal{I} -open, quasi \mathcal{I} -open and so on. But a theory highlighting the topology \mathcal{T}^* induced by an ideal \mathcal{I} was developed in [11]. In[11], several ideals on the same topological space (X, \mathcal{T}) were considered and the relationship among the topologies generated by these ideals were discussed.

In this paper we define a concept called \mathcal{S} -Hausdorffness, in the context of ideal topological spaces slightly different from the definition available in the literature[2]. Also we define regular, normal spaces in the context of ideal topological spaces and prove certain results similar to results available in classical theory. We also prove that the intersection of two \mathcal{S} -Hausdorff topologies is an \mathcal{S} -Hausdorff topology, in contrast to the classical result which states that the intersection of two Hausdorff topologies need not be a Hausdorff topology. Further we show that the product of two \mathcal{S} -Hausdorff spaces need not be an \mathcal{S} -Hausdorff space, in contrast to the classical result which states that the product of two Hausdorff spaces is a Hausdorff space.

In Section 2 we recall some definitions and results from the literature and prove certain results which we need in the sequel; in Section 3 we define and discuss Hausdorffness in ideal topological spaces; in Section 4

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we define and discuss regularity and normality in the context of ideal topological spaces and finally we give some concluding remarks.

2 Preliminary Definitions and Results

Let us start with the definition of an ideal in a topological space.

Definition 2.1. [12] Let X be any set. An ideal on X is a nonempty collection \mathcal{I} of subsets of X satisfying the following.

- i. If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.
- ii. If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.

If (X, \mathcal{T}) is a topological space and \mathcal{I} is an ideal on X , then the triplet $(X, \mathcal{T}, \mathcal{I})$ is called an ideal topological space or ideal space.

Throughout this paper X , \mathcal{T} and \mathcal{I} will denote, respectively a nonempty set, a topology on X and an ideal on X . If (X, \mathcal{T}) is a topological space and $x \in X$, $\mathcal{T}(x)$ denote the collection $\{U \in \mathcal{T} / x \in U\}$ of all open sets in (X, \mathcal{T}) containing x . We denote the complement of A in X by A^c . If X is any set, by $\mathcal{P}(X)$ we denote the collection of all subsets of X and call it as the power set of X . The definitions and results which are not stated explicitly are as in [10].

A closure operator on a set X is a function from $\mathcal{P}(X)$ to $\mathcal{P}(X)$, taking A to \bar{A} , satisfying the following conditions: $\bar{\emptyset} = \emptyset$, $A \subseteq \bar{A}$, $\overline{\bar{A}} = \bar{A}$ for all A , and for any A and B , $\overline{A \cup B} = \bar{A} \cup \bar{B}$. The above four conditions are called Kuratowski closure axioms [7]. If “—” is a closure operator on a set X , \mathcal{F} is the family of all subsets A of X for which $\bar{A} = A$, and if \mathcal{T} is the family of complements of members of \mathcal{F} , then \mathcal{T} is a topology on X and \bar{A} is the \mathcal{T} -closure of A for each subset A of X . This topology is called the topology generated by the closure operator “—”.

Definition 2.2. [8] For any subset A of X , define

$$A_{(X, \mathcal{T})}^* = \{x \in X / U \cap A \notin \mathcal{I} \text{ for every } U \in \mathcal{T}(x)\}.$$

Let $\bar{A} = A \cup A_{(X, \mathcal{T})}^*$. Then “—” is a Kuratowski closure operator which gives a topology on X , called the topology generated by \mathcal{I} , and is denoted by $\mathcal{T}_{\mathcal{I}}$. This topology is also called $*$ -topology or ideal topology.

Let (X, \mathcal{T}) be a topological space. Then the following results hold trivially.

- i. If $\mathcal{I} = \{\emptyset\}$, then $\mathcal{T}_{\mathcal{I}} = \mathcal{T}$.
- ii. If $A \in \mathcal{I}$, then $A^* = \emptyset$ and A is closed in $(X, \mathcal{T}_{\mathcal{I}})$.
- iii. If A is closed in $\mathcal{T}_{\mathcal{I}}$, then $A_{(X, \mathcal{T})}^* \subseteq A$.
- iv. If $\mathcal{I}_1 \subseteq \mathcal{I}_2$ are two ideals on X then $\mathcal{T}_{\mathcal{I}_1} \subseteq \mathcal{T}_{\mathcal{I}_2}$.

Now we prove a result which we use in the sequel.

Theorem 2.1. Let (X, \mathcal{T}) be a topological space. Let \mathcal{I}_1 and \mathcal{I}_2 be two ideals on X . Then $\mathcal{T}_{\mathcal{I}_1 \cap \mathcal{I}_2} = \mathcal{T}_{\mathcal{I}_1} \cap \mathcal{T}_{\mathcal{I}_2}$.

Proof. Since the intersection of two ideals is an ideal, $\mathcal{T}_{\mathcal{I}_1 \cap \mathcal{I}_2}$ is meaningful. As $\mathcal{I}_1 \cap \mathcal{I}_2 \subseteq \mathcal{I}_1$ and $\mathcal{I}_1 \cap \mathcal{I}_2 \subseteq \mathcal{I}_2$, we have $\mathcal{T}_{\mathcal{I}_1 \cap \mathcal{I}_2} \subseteq \mathcal{T}_{\mathcal{I}_1}$ and $\mathcal{T}_{\mathcal{I}_1 \cap \mathcal{I}_2} \subseteq \mathcal{T}_{\mathcal{I}_2}$. Therefore $\mathcal{T}_{\mathcal{I}_1 \cap \mathcal{I}_2} \subseteq \mathcal{T}_{\mathcal{I}_1} \cap \mathcal{T}_{\mathcal{I}_2}$.

Conversely, let us assume that $V \in \mathcal{T}_{\mathcal{I}_1} \cap \mathcal{T}_{\mathcal{I}_2}$ and let $A = V^c$. Then A is closed in $\mathcal{T}_{\mathcal{I}_1} \cap \mathcal{T}_{\mathcal{I}_2}$ and hence A is closed in $\mathcal{T}_{\mathcal{I}_1}$ and A is closed in $\mathcal{T}_{\mathcal{I}_2}$. This implies that $A_{(X, \mathcal{T})}^* \subseteq A$ and $A_{(X, \mathcal{T})}^* \subseteq A$. We aim to prove that A is closed in $\mathcal{T}_{\mathcal{I}_1 \cap \mathcal{I}_2}$. It is enough to show that $A_{(\mathcal{I}_1 \cap \mathcal{I}_2, \mathcal{T})}^* \subseteq A$. Assume that $x \notin A$. This implies that $x \notin A_{(X, \mathcal{T})}^*$ and $x \notin A_{(\mathcal{I}_2, \mathcal{T})}^*$. Then there exist open sets U and V in \mathcal{T} containing x such that $U \cap A \in \mathcal{I}_1$ and $V \cap A \in \mathcal{I}_2$. Let us take $G = U \cap V$. Clearly $x \in G$ and $G \in \mathcal{T}$. Also

$$G \cap A = (U \cap V) \cap A = (U \cap A) \cap (V \cap A) \in \mathcal{I}_1 \cap \mathcal{I}_2.$$

Thus there exists an open set $G \in \mathcal{T}(x)$ such that $G \cap A \in \mathcal{I}_1 \cap \mathcal{I}_2$. Therefore $x \notin A_{(\mathcal{I}_1 \cap \mathcal{I}_2, \mathcal{T})}^*$ and hence $A_{(\mathcal{I}_1 \cap \mathcal{I}_2, \mathcal{T})}^* \subseteq A$. This implies that A is closed in $\mathcal{T}_{\mathcal{I}_1 \cap \mathcal{I}_2}$. Thus $\mathcal{T}_{\mathcal{I}_1} \cap \mathcal{T}_{\mathcal{I}_2} \subseteq \mathcal{T}_{\mathcal{I}_1 \cap \mathcal{I}_2}$. \square

We note that this result is not true in case of intersection of infinitely many ideals. If $\{\mathcal{I}_\alpha\}$ is a collection of ideals on (X, \mathcal{T}) , then we have only $\mathcal{T}_{\cap \mathcal{I}_\alpha} \subseteq \cap \mathcal{T}_{\mathcal{I}_\alpha}$. The equality holds if and only if (X, \mathcal{T}) is an Alexandroff space [11, Theorem 3.3].

3 Hausdorff Spaces in Ideal Topological Spaces

In this section we define Hausdorff space in the context of ideal topological spaces, compare it with the one available in the literature and prove some results. We start with the definition of \mathcal{I} -open sets and Hausdorff spaces in the context of ideal topological spaces given by Dontchev [2].

Definition 3.3. [3] Let $(X, \mathcal{T}, \mathcal{I})$ be an ideal topological space. A subset A of X is said to be \mathcal{I} -open if $A \subseteq \text{int}(A^*)$.

Definition 3.4. [2] An ideal topological space $(X, \mathcal{T}, \mathcal{I})$ is called \mathcal{I} -Hausdorff if for every two distinct points x, y of X , there exist disjoint \mathcal{I} -open sets U, V in (X, \mathcal{T}) such that $x \in U$ and $y \in V$.

According to this definition, if (X, \mathcal{T}) is a topological space and if $\mathcal{I} = \mathcal{P}(X)$, then (X, \mathcal{T}) is not \mathcal{I} -Hausdorff even if (X, \mathcal{T}) is Hausdorff. Indeed, if A is any subset of X , then $A \in \mathcal{I}$ and hence $A^* = \emptyset$ which implies that no set other than the empty set is \mathcal{I} -open; so one cannot find two disjoint \mathcal{I} -open sets containing two distinct points. Furthermore, the set $X = \{\frac{1}{n}\}_{n \in \mathbb{N}} \cup \{0\}$ under the usual metric of \mathbb{R} is a metric space and hence it is Hausdorff; but when $\mathcal{I} = \{\emptyset\}$, X is not \mathcal{I} -Hausdorff. Example 3.1 in [4] shows that an \mathcal{I} -Hausdorff space need not be Hausdorff. So an \mathcal{I} -Hausdorff space need not be a Hausdorff space and a Hausdorff space need not be an \mathcal{I} -Hausdorff space according to the definition available in the literature.

However, according to the theory of \mathcal{I} -Hausdorff space we are going to develop in this paper, every Hausdorff space is an \mathcal{I} -Hausdorff space and there are \mathcal{I} -Hausdorff spaces which are not Hausdorff space. To avoid confusions in the notations we write \mathcal{S} -Hausdorff instead of writing \mathcal{I} -Hausdorff in the new sense.

Definition 3.5. Let (X, \mathcal{T}) be a topological space and \mathcal{I} be an ideal on X . Then (X, \mathcal{T}) is said to be \mathcal{S} -Hausdorff with respect to the ideal \mathcal{I} if for every pair of distinct points x and y in X , there exist two open sets U_1 and U_2 in \mathcal{T} such that $x \in U_1, y \in U_2$ and $U_1 \cap U_2 \in \mathcal{I}$.

From the very definition itself, it follows that every Hausdorff space is \mathcal{S} -Hausdorff whatever be the ideal \mathcal{I} on it, as $\emptyset \in \mathcal{I}$; if $X = \{1, 2\}$, $\mathcal{T} = \{\emptyset, X, \{1\}\}$ and $\mathcal{I} = \{\emptyset, \{1\}\}$, then X is \mathcal{S} -Hausdorff with respect to \mathcal{I} whereas it is not Hausdorff in the classical sense. Thus the class of \mathcal{S} -Hausdorff spaces is strictly larger than the class of Hausdorff spaces. Whenever there is no ambiguity we just write \mathcal{S} -Hausdorff leaving the tail "with respect to the ideal \mathcal{I} ".

In view of the discussion below Definition 3.4, there are many \mathcal{S} -Hausdorff spaces in our context which are not \mathcal{I} -Hausdorff space according to Definition 3.4. Example 3.1 in [4] shows that an \mathcal{I} -Hausdorff space need not be an \mathcal{S} -Hausdorff space in our context. From this we conclude that our definition of Hausdorffness is different from, and more natural than, the one available in the literature.

Theorem 3.2. Let $(X, \mathcal{T}, \mathcal{I})$ be an ideal topological space. Then X is \mathcal{S} -Hausdorff with respect to \mathcal{I} if and only if the following holds:

If $x, y \in X$ with $x \neq y$, then there exist sets $V_1, V_2 \in \mathcal{T}$ and $I_1, I_2 \in \mathcal{I}$ such that $x \in V_1 - I_1, y \in V_2 - I_2$ and $(V_1 - I_1) \cap (V_2 - I_2) \in \mathcal{I}$.

Proof. Assume that X is \mathcal{S} -Hausdorff. Let $x, y \in X$ such that $x \neq y$. Since X is \mathcal{S} -Hausdorff, there exist open sets V_1 and V_2 in \mathcal{T} such that $x \in V_1, y \in V_2$ and $V_1 \cap V_2 \in \mathcal{I}$. By taking $I_1 = I_2 = \emptyset$ we see that the statement holds.

To prove the converse, let $x, y \in X$ such that $x \neq y$. Then there exist sets $V_1, V_2 \in \mathcal{T}$ and $I_1, I_2 \in \mathcal{I}$ such that $x \in V_1 - I_1, y \in V_2 - I_2$ and $(V_1 - I_1) \cap (V_2 - I_2) \in \mathcal{I}$. We claim that $V_1 \cap V_2 \in \mathcal{I}$.

As $(V_1 - I_1) \cap (V_2 - I_2) = (V_1 \cap V_2) - (I_1 \cup I_2)$, we have

$$V_1 \cap V_2 = [(V_1 - I_1) \cap (V_2 - I_2)] \cup [(V_1 \cap V_2) \cap (I_1 \cup I_2)].$$

Since $I_1, I_2 \in \mathcal{I}$, we have $I_1 \cup I_2 \in \mathcal{I}$. Since $(V_1 \cap V_2) \cap (I_1 \cup I_2) \subseteq (I_1 \cup I_2)$ and $(V_1 - I_1) \cap (V_2 - I_2) \in \mathcal{I}$, we have $V_1 \cap V_2 \in \mathcal{I}$. Thus X is \mathcal{S} -Hausdorff. \square

If \mathcal{I} is an ideal on a set X and if $Y \subseteq X$, then $\mathcal{I}_Y = \{A \cap Y / A \in \mathcal{I}\}$ is an ideal on Y .

Theorem 3.3. *If (X, \mathcal{T}) is an \mathcal{I} -Hausdorff space with respect to the ideal \mathcal{I} and if $Y \subseteq X$, then (Y, \mathcal{T}_Y) is \mathcal{I} -Hausdorff space with respect to the ideal \mathcal{I}_Y where \mathcal{T}_Y is the subspace topology on Y inherited from \mathcal{T} .*

Proof. Let $y_1, y_2 \in Y$ be such that $y_1 \neq y_2$. Since X is \mathcal{I} -Hausdorff, there exist open sets G_1, G_2 in \mathcal{T} such that $y_1 \in G_1, y_2 \in G_2$ and $G_1 \cap G_2 \in \mathcal{I}$. Let $H_1 = Y \cap G_1$ and $H_2 = Y \cap G_2$. Then H_1 and H_2 are open sets in \mathcal{T}_Y containing y_1 and y_2 respectively. As $H_1 \cap H_2 = (G_1 \cap G_2) \cap Y$, we have $H_1 \cap H_2 \in \mathcal{I}_Y$. Therefore Y is \mathcal{I} -Hausdorff with respect to the ideal \mathcal{I}_Y . \square

In the crisp topological theory, the intersection of two Hausdorff topologies on a set X need not be Hausdorff. That is, if \mathcal{T}_1 and \mathcal{T}_2 are two Hausdorff topologies on X , then $\mathcal{T}_1 \cap \mathcal{T}_2$ need not be Hausdorff. But if \mathcal{T}_1 and \mathcal{T}_2 are Hausdorff topologies on a set X and if there exists a topology \mathcal{T} and two ideals \mathcal{I}_1 and \mathcal{I}_2 on X such that $\mathcal{T}_1 = \mathcal{T}_{\mathcal{I}_1}$ and $\mathcal{T}_2 = \mathcal{T}_{\mathcal{I}_2}$, then $\mathcal{T}_1 \cap \mathcal{T}_2$ is a Hausdorff topology on X (See Theorem 3.6).

First we give a necessary and sufficient condition for a space (X, \mathcal{T}) to be \mathcal{I} -Hausdorff with respect to an ideal \mathcal{I} .

Theorem 3.4. *Let (X, \mathcal{T}) be a topological space and \mathcal{I} be an ideal on X . Then (X, \mathcal{T}) is \mathcal{I} -Hausdorff if and only if $(X, \mathcal{T}_{\mathcal{I}})$ is Hausdorff.*

Proof. Let x and y be two distinct points in an \mathcal{I} -Hausdorff space (X, \mathcal{T}) . Then there exist open sets U and V in \mathcal{T} such that $x \in U, y \in V$ and $U \cap V \in \mathcal{I}$. Let us take $U_1 = U - ((U \cap V) - \{x\})$ and $U_2 = V - ((U \cap V) - \{y\})$. Clearly $x \in U_1$ and $y \in U_2$. Since $U \cap V \in \mathcal{I}$, $(U \cap V) - \{x\} \in \mathcal{I}$; therefore it is closed in $\mathcal{T}_{\mathcal{I}}$ and hence U_1 is open in $\mathcal{T}_{\mathcal{I}}$. Similarly U_2 is open in $\mathcal{T}_{\mathcal{I}}$. Thus we get two open sets U_1 and U_2 in $\mathcal{T}_{\mathcal{I}}$ such that $x \in U_1, y \in U_2$ and $U_1 \cap U_2 = \emptyset$. Therefore, $(X, \mathcal{T}_{\mathcal{I}})$ is Hausdorff.

Conversely, let x and y be two distinct points in (X, \mathcal{T}) . Since $(X, \mathcal{T}_{\mathcal{I}})$ is Hausdorff, there exist open sets U and V in $\mathcal{T}_{\mathcal{I}}$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$. As $U \in \mathcal{T}_{\mathcal{I}}$, U^c is closed in $\mathcal{T}_{\mathcal{I}}$ and hence $(U^c)^* \subseteq U^c$. Since $x \notin U^c$, we have $x \notin (U^c)^*$. Thus there exists $U_1 \in \mathcal{T}$ containing x such that $U_1 \cap U^c \in \mathcal{I}$. Let $I_1 = U_1 \cap U^c$. Clearly $x \in U_1 - I_1 \subseteq U$. Similarly there exists $U_2 \in \mathcal{T}$ and $I_2 \in \mathcal{I}$ such that $y \in U_2 - I_2 \subseteq V$. Since $U \cap V = \emptyset$, we have $(U_1 - I_1) \cap (U_2 - I_2) = \emptyset$. It follows that $U_1 \cap U_2 \subseteq I_1 \cup I_2$ and hence $U_1 \cap U_2 \in \mathcal{I}$. Thus we get open sets U_1 and U_2 in \mathcal{T} such that $x \in U_1, y \in U_2$ and $U_1 \cap U_2 \in \mathcal{I}$. Therefore (X, \mathcal{T}) is \mathcal{I} -Hausdorff. \square

Theorem 3.5. *Let \mathcal{I}_1 and \mathcal{I}_2 be ideals on (X, \mathcal{T}) . If (X, \mathcal{T}) is \mathcal{I} -Hausdorff with respect to \mathcal{I}_1 and \mathcal{I}_2 , then (X, \mathcal{T}) is \mathcal{I} -Hausdorff with respect to the ideal $\mathcal{I}_1 \cap \mathcal{I}_2$.*

Proof. Let x and y be two distinct points in (X, \mathcal{T}) . By Theorem 3.4, $(X, \mathcal{T}_{\mathcal{I}_1})$ and $(X, \mathcal{T}_{\mathcal{I}_2})$ are Hausdorff. Since $(X, \mathcal{T}_{\mathcal{I}_1})$ is Hausdorff, as in the proof of Theorem 3.4, there exist $U_1, V_1 \in \mathcal{T}$ and $I_1, J_1 \in \mathcal{I}_1$ such that $x \in U_1 - I_1, y \in V_1 - J_1$ and $(U_1 - I_1) \cap (V_1 - J_1) = \emptyset$. Similarly there exist $U_2, V_2 \in \mathcal{T}$ and $I_2, J_2 \in \mathcal{I}_2$ such that $x \in U_2 - I_2, y \in V_2 - J_2$ and $(U_2 - I_2) \cap (V_2 - J_2) = \emptyset$.

Let $W_1 = U_1 \cap U_2$ and $W_2 = V_1 \cap V_2$. Clearly $x \in W_1, y \in W_2$ and $W_1, W_2 \in \mathcal{T}$. Let $I = (I_1 \cup J_1) \cap (I_2 \cup J_2)$. Since $I \subseteq I_1 \cup J_1$ and $I \subseteq I_2 \cup J_2$, we have $I \in \mathcal{I}_1 \cap \mathcal{I}_2$. We claim that $W_1 \cap W_2 \in \mathcal{I}_1 \cap \mathcal{I}_2$. As $(U_1 - I_1) \cap (V_1 - J_1) = \emptyset$ and $(U_2 - I_2) \cap (V_2 - J_2) = \emptyset$, we have $U_1 \cap V_1 \subseteq I_1 \cup J_1$ and $U_2 \cap V_2 \subseteq I_2 \cup J_2$. Since $W_1 \cap W_2 = (U_1 \cap V_1) \cap (U_2 \cap V_2)$, we have $W_1 \cap W_2 \subseteq I$ and hence $W_1 \cap W_2 \in \mathcal{I}_1 \cap \mathcal{I}_2$. Therefore (X, \mathcal{T}) is \mathcal{I} -Hausdorff with respect to the ideal $\mathcal{I}_1 \cap \mathcal{I}_2$. \square

Theorem 3.6. *Let \mathcal{T}_1 and \mathcal{T}_2 be Hausdorff topologies on a set X . Let there be a topology \mathcal{T} and two ideals \mathcal{I}_1 and \mathcal{I}_2 on X such that $\mathcal{T}_1 = \mathcal{T}_{\mathcal{I}_1}$ and $\mathcal{T}_2 = \mathcal{T}_{\mathcal{I}_2}$. Then $\mathcal{T}_1 \cap \mathcal{T}_2$ is a Hausdorff topology on X .*

Proof. Since $\mathcal{T}_1 = \mathcal{T}_{\mathcal{I}_1}, \mathcal{T}_2 = \mathcal{T}_{\mathcal{I}_2}$, by Theorem 3.4, (X, \mathcal{T}) is \mathcal{I} -Hausdorff with respect to the ideals \mathcal{I}_1 and \mathcal{I}_2 . Also by Theorem 3.5, (X, \mathcal{T}) is \mathcal{I} -Hausdorff with respect to the ideal $\mathcal{I}_1 \cap \mathcal{I}_2$; by Theorem 3.4, $(X, \mathcal{T}_{\mathcal{I}_1 \cap \mathcal{I}_2})$ is Hausdorff. By Theorem 2.1, $\mathcal{T}_{\mathcal{I}_1 \cap \mathcal{I}_2} = \mathcal{T}_{\mathcal{I}_1} \cap \mathcal{T}_{\mathcal{I}_2}$. Therefore $(X, \mathcal{T}_{\mathcal{I}_1} \cap \mathcal{T}_{\mathcal{I}_2})$ is Hausdorff. In other words, $\mathcal{T}_1 \cap \mathcal{T}_2$ is a Hausdorff topology on X . \square

If \mathcal{A} and \mathcal{B} are collections of subsets of X and Y , then the collection $\mathcal{A} \times \mathcal{B} = \{A \times B / A \in \mathcal{A}, B \in \mathcal{B}\}$ is called the product of \mathcal{A} and \mathcal{B} . If \mathcal{I}_1 and \mathcal{I}_2 are ideals on X_1 and X_2 , then $\mathcal{I}_1 \times \mathcal{I}_2$ need not be an ideal on $X_1 \times X_2$. Indeed, if $X = \{1, 2, 3\}, \mathcal{I} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$, then $\mathcal{I} \times \mathcal{I}$ is not an ideal on $X \times X$ as $\{(1, 1), (2, 2)\}$

is not in $\mathcal{I} \times \mathcal{I}$ whereas $\{(1,1)\}, \{(2,2)\}$ are in $\mathcal{I} \times \mathcal{I}$. However with ideals \mathcal{I}_1 on X and \mathcal{I}_2 on Y we can associate an ideal $\mathcal{I}_1 \otimes \mathcal{I}_2$ on $X \times Y$ in a natural way. The ideal $\mathcal{I}_1 \otimes \mathcal{I}_2$ is in fact the smallest ideal containing $\mathcal{I}_1 \times \mathcal{I}_2$ which can be obtained as the intersection of all ideals containing $\mathcal{I}_1 \times \mathcal{I}_2$.

It is well known that the product of two Hausdorff spaces is a Hausdorff space in crisp topological theory. But this is not true in ideal topological theory. That is, if (X_1, \mathcal{T}_1) is \mathcal{S} -Hausdorff with respect to the ideal \mathcal{I}_1 and (X_2, \mathcal{T}_2) is \mathcal{S} -Hausdorff with respect to the ideal \mathcal{I}_2 , then $(X_1 \times X_2, \mathcal{T}_1 \times \mathcal{T}_2)$ need not be \mathcal{S} -Hausdorff with respect to the ideal $\mathcal{I}_1 \otimes \mathcal{I}_2$ as seen in the following example.

Example 3.1. Let $X_1 = \{1, 3\}, X_2 = \{1, 4\}$; let $\mathcal{T}_1 = \{\emptyset, X_1, \{1\}\}, \mathcal{T}_2 = \{\emptyset, X_2, \{1\}\}$; then \mathcal{T}_1 and \mathcal{T}_2 are topologies on X_1 and X_2 ; let $\mathcal{I}_1 = \{\emptyset, \{1\}\}, \mathcal{I}_2 = \{\emptyset, \{1\}\}$; then \mathcal{I}_1 and \mathcal{I}_2 are ideals on X_1 and X_2 . Clearly the product topology $\mathcal{T}_1 \times \mathcal{T}_2$ is the collection

$$\{\emptyset, X_1 \times X_2, \{(1,1)\}, \{(1,1), (1,4)\}, \{(1,1), (3,1)\}, \{(1,1), (1,4), (3,1)\}\}$$

and the ideal $\mathcal{I}_1 \otimes \mathcal{I}_2$ is the collection $\{\emptyset, \{(1,1)\}\}$. As we cannot separate the points $(1,4)$ and $(3,1)$, $X_1 \times X_2$ is not \mathcal{S} -Hausdorff with respect to the ideal $\mathcal{I}_1 \otimes \mathcal{I}_2$.

We note an interesting observation in the comparison of Hausdorff theory between the crisp and ideal topological theory. One point sets in Hausdorff spaces are closed in crisp theory whereas it is not so in the theory of ideal topology. For example, if $X = \{1, 2, 3\}, \mathcal{T} = \{\emptyset, X, \{1\}, \{1, 2\}\}$ and $\mathcal{I} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$, then $\{1\}, \{2\}$ are not closed in (X, \mathcal{T}) ; but X is \mathcal{S} -Hausdorff with respect to \mathcal{I} .

4 Regular and Normal Spaces

Now we define regular and normal space in the context of ideal topological spaces and prove some results.

Definition 4.6. Let (X, \mathcal{T}) be a topological space and \mathcal{I} be an ideal on X . Let singleton sets be closed in X . Then (X, \mathcal{T}) is said to be \mathcal{S} -regular with respect to the ideal \mathcal{I} if given $x \in X$ and a closed set B not containing x , there exist two open sets U_1 and U_2 in \mathcal{T} such that $x \in U_1, B \subseteq U_2$ and $U_1 \cap U_2 \in \mathcal{I}$.

As $\emptyset \in \mathcal{I}$, every regular space is \mathcal{S} -regular with respect to the ideal \mathcal{I} whatever be the ideal \mathcal{I} . An \mathcal{S} -regular space with respect to an ideal \mathcal{I} need not be regular. For example, any uncountable set X with cocountable topology is not regular; but it is \mathcal{S} -regular space with respect to the ideal \mathcal{I} where $\mathcal{I} = \mathcal{P}(X)$.

In Theorem 3.4 we have proved that a space (X, \mathcal{T}) is \mathcal{S} -Hausdorff with respect to the ideal \mathcal{I} if and only if $(X, \mathcal{T}_{\mathcal{I}})$ is Hausdorff. But in the case of regular spaces it is not so. That is, if (X, \mathcal{T}) is \mathcal{S} -regular with respect to an ideal \mathcal{I} , then $(X, \mathcal{T}_{\mathcal{I}})$ need not be regular. For example, in \mathbb{R} with usual topology, let \mathcal{I} be the collection of all subsets of $\{1, \frac{1}{2}, \frac{1}{3}, \dots\}$. Since \mathbb{R} with usual topology is regular, it is \mathcal{S} -regular with respect to the ideal \mathcal{I} ; but it is not $\mathcal{T}_{\mathcal{I}}$ -regular because we cannot separate the point 0 and a closed set $\{1, \frac{1}{2}, \frac{1}{3}, \dots\}$. However, $(X, \mathcal{T}_{\mathcal{I}})$ is regular if the following additional condition is satisfied.

C1: For any $A \in \mathcal{I}$ and $x \notin A$, there exists U in \mathcal{T} such that $x \in U$ and $\bar{U} \cap A = \emptyset$.

Theorem 4.7. If (X, \mathcal{T}) is \mathcal{S} -regular with respect to the ideal \mathcal{I} and if **C1** holds, then $(X, \mathcal{T}_{\mathcal{I}})$ is regular.

Proof. Let F be closed in $\mathcal{T}_{\mathcal{I}}$ and $x \notin F$.

Suppose F is closed in \mathcal{T} , by \mathcal{S} -regularity, there exist $U_1, U_2 \in \mathcal{T}$ such that $x \in U_1, F \subseteq U_2$ and $U_1 \cap U_2 \in \mathcal{I}$. If needed replacing U_1 by $U_1 \cap F^c$, we can assume $U_1 \cap F = \emptyset$. Let $I = U_1 \cap U_2, V_1 = U_1$ and $V_2 = U_2 - I$. Since $I \cap F = \emptyset$ and $F \subseteq U_2, F \subseteq V_2$. Clearly $V_1 \cap V_2 = \emptyset$. Thus we obtained two open sets V_1, V_2 in $\mathcal{T}_{\mathcal{I}}$ such that $x \in V_1, F \subseteq V_2$ and $V_1 \cap V_2 = \emptyset$. Hence $(X, \mathcal{T}_{\mathcal{I}})$ is regular in this case.

Now we prove the general case. Since F is closed in $\mathcal{T}_{\mathcal{I}}$, we have $F_{(\mathcal{I}, \mathcal{T})}^* \subseteq F$ and hence $x \notin F_{(\mathcal{I}, \mathcal{T})}^*$. Then there exists $U \in \mathcal{T}(x)$ such that $U \cap F \in \mathcal{I}$. Let $I = U \cap F$ and $F_1 = F - I$. Clearly $U \cap F_1 = \emptyset$. Let F_2 be the closure of F_1 with respect to \mathcal{T} . Since $x \in U$ and $U \cap F_1 = \emptyset$, we have $x \notin F_2$. Therefore F_2 is closed set in \mathcal{T} such that $x \notin F_2$. By the particular case discussed above, there exist $V_1, V_2 \in \mathcal{T}_{\mathcal{I}}$ such that $x \in V_1, F_2 \subseteq V_2$ and $V_1 \cap V_2 = \emptyset$. Since $x \notin I \in \mathcal{I}$, by the condition **C1**, there exists an open set $U \in \mathcal{T}$ such that $x \in U$ and $\bar{U} \cap I = \emptyset$. Let us take $W_1 = U$ and $W_2 = \bar{U}^c$. Also $x \in W_1, I \subseteq W_2$ and $W_1 \cap W_2 = \emptyset$. Let $G_1 = V_1 \cap W_1$ and $G_2 = V_2 \cup W_2$. Clearly $x \in G_1$ and $G_1 \cap G_2 = \emptyset$. Since $F_2 \subseteq V_2$ and $I \subseteq W_2$, we have $F \subseteq G_2$. Thus we get open sets G_1, G_2 in $\mathcal{T}_{\mathcal{I}}$ such that $x \in G_1, F \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$ and hence $(X, \mathcal{T}_{\mathcal{I}})$ is regular. \square

We can weaken condition **C1** by the following condition:

C2: For any $A \in \mathcal{I}$ and $x \notin A$, there exist $U, V \in \mathcal{T}_{\mathcal{I}}$ such that $x \in U$, $A \subseteq V$ and $U \cap V = \emptyset$.

Theorem 4.8. *If (X, \mathcal{T}) is \mathcal{I} -regular with respect to the ideal \mathcal{I} and if **C2** holds, then $(X, \mathcal{T}_{\mathcal{I}})$ is regular.*

If $x \notin I \in \mathcal{I}$, by the condition **C2**, there exist W_1, W_2 in $\mathcal{T}_{\mathcal{I}}$ such that $x \in W_1$, $I \subseteq W_2$ and $W_1 \cap W_2 = \emptyset$. Replacing the sets W_1 and W_2 in the proof of Theorem 4.7 by these W_1 and W_2 , we get the proof.

If U is a set that exists in condition **C1**, then U and \overline{U}^c serve as the open sets in condition **C2**. Thus **C2** is weaker than **C1**. The following example shows that **C2** is strictly weaker than **C1**.

Example 4.2. *Let $X = \{1, 2, 3, 4\}$, $\mathcal{T} = \{\emptyset, X, \{1\}, \{2\}, \{1, 2\}\}$ and $\mathcal{I} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. Let $A = \{2\}$, $x = 3$. As the only open set containing x in \mathcal{T} is X , condition **C1** is not satisfied whereas it is easy to verify that condition **C2** is satisfied.*

The following theorem can be proved analogous to Theorem 3.2.

Theorem 4.9. *Let \mathcal{I} be an ideal on (X, \mathcal{T}) . Then X is \mathcal{I} -regular with respect to \mathcal{I} if and only if the following holds:*

If $x \in X$ and a closed set B not containing x , then there exist sets $V_1, V_2 \in \mathcal{T}$ and $I_1, I_2 \in \mathcal{I}$ such that $x \in V_1 - I_1$, $B \subseteq V_2 - I_2$ and $(V_1 - I_1) \cap (V_2 - I_2) \in \mathcal{I}$.

Theorem 4.10. *Let \mathcal{I}_1 and \mathcal{I}_2 be ideals on (X, \mathcal{T}) . If (X, \mathcal{T}) is \mathcal{I} -regular with respect to \mathcal{I}_1 and \mathcal{I}_2 , then (X, \mathcal{T}) is \mathcal{I} -regular with respect to the ideal $\mathcal{I}_1 \cap \mathcal{I}_2$.*

Proof. Let $x \in X$ and B a closed set in (X, \mathcal{T}) not containing x . Since X is \mathcal{I} -regular with respect to \mathcal{I}_1 , there exist two open sets U_1, V_1 in \mathcal{T} such that $x \in U_1$, $B \subseteq V_1$ and $U_1 \cap V_1 \in \mathcal{I}_1$. Similarly there exist two open sets U_2, V_2 in \mathcal{T} such that $x \in U_2$, $B \subseteq V_2$ and $U_2 \cap V_2 \in \mathcal{I}_2$. Let $U = U_1 \cap U_2$ and $V = V_1 \cap V_2$. Clearly $x \in U$ and $B \subseteq V$. Since $U \cap V = (U_1 \cap V_1) \cap (U_2 \cap V_2)$, we have $U \cap V \in \mathcal{I}_1 \cap \mathcal{I}_2$. Thus there exist two open sets U, V in \mathcal{T} such that $x \in U$, $B \subseteq V$ and $U \cap V \in \mathcal{I}_1 \cap \mathcal{I}_2$ and hence (X, \mathcal{T}) is \mathcal{I} -regular with respect to the ideal $\mathcal{I}_1 \cap \mathcal{I}_2$. □

Theorem 4.11. *Let $(X, \mathcal{T}, \mathcal{I})$ be an ideal topological space. If X is an \mathcal{I} -regular space with respect to \mathcal{I} and if $Y \subseteq X$, then (Y, \mathcal{T}_Y) is \mathcal{I} -regular space with respect to the ideal \mathcal{I}_Y where \mathcal{T}_Y is the subspace topology on Y inherited from \mathcal{T} .*

Proof. Let A be a closed set in (Y, \mathcal{T}_Y) and $x \notin A$. Since A is closed in Y , we have $A = Y \cap F$ where F is closed in X . As F is closed in X and $x \notin F$, there exist U_1, U_2 in \mathcal{T} such that $x \in U_1$, $F \subseteq U_2$ and $U_1 \cap U_2 \in \mathcal{I}$. Let $V_1 = Y \cap U_1$ and $V_2 = Y \cap U_2$. Clearly $x \in V_1$, $A \subseteq V_2$ and V_1, V_2 are open sets in \mathcal{T}_Y . As $V_1 \cap V_2 = (U_1 \cap U_2) \cap Y$, we have $V_1 \cap V_2 \in \mathcal{I}_Y$. Therefore Y is \mathcal{I} -regular with respect to the ideal \mathcal{I}_Y . □

Theorem 4.12. *Every \mathcal{I} -regular space is \mathcal{I} -Hausdorff space with respect to the same ideal.*

Now we define normal space in the context of ideal topological spaces and prove some results.

Definition 4.7. *Let (X, \mathcal{T}) be a topological space and \mathcal{I} be an ideal on X . Let singleton sets be closed in X . Then (X, \mathcal{T}) is said to be \mathcal{I} -normal with respect to the ideal \mathcal{I} if given two disjoint closed sets A and B , there exist two open sets U_1 and U_2 in \mathcal{T} such that $A \subseteq U_1$, $B \subseteq U_2$ and $U_1 \cap U_2 \in \mathcal{I}$.*

As $\emptyset \in \mathcal{I}$, every normal space is \mathcal{I} -normal space with respect to the ideal \mathcal{I} whatever be the ideal \mathcal{I} on X . The converse is not true. For example, any infinite set X with cofinite topology is not normal; but it is \mathcal{I} -normal space with respect to the ideal \mathcal{I} where $\mathcal{I} = \mathcal{P}(X)$.

The following theorem can be proved analogous to Theorem 3.2.

Theorem 4.13. *Let $(X, \mathcal{T}, \mathcal{I})$ be an ideal topological space. Then X is \mathcal{I} -normal with respect to \mathcal{I} if and only if the following holds:*

If A and B be two closed sets such that $A \cap B = \emptyset$, then there exist sets $V_1, V_2 \in \mathcal{T}$ and $I_1, I_2 \in \mathcal{I}$ such that $A \subseteq V_1 - I_1$, $B \subseteq V_2 - I_2$ and $(V_1 - I_1) \cap (V_2 - I_2) \in \mathcal{I}$.

The following theorem can be proved analogous to Theorem 4.10.

Theorem 4.14. Let \mathcal{I}_1 and \mathcal{I}_2 be ideals on (X, \mathcal{T}) . If (X, \mathcal{T}) is \mathcal{S} -normal with respect to \mathcal{I}_1 and \mathcal{I}_2 , then (X, \mathcal{T}) is \mathcal{S} -normal with respect to the ideal $\mathcal{I}_1 \cap \mathcal{I}_2$.

Theorem 4.15. Every \mathcal{S} -normal space is \mathcal{S} -regular with respect to the same ideal.

Theorem 4.16. Let $(X, \mathcal{T}, \mathcal{I})$ be an ideal topological space. If X is an \mathcal{S} -normal space with respect to \mathcal{I} and if Y is a closed subset of (X, \mathcal{T}) , then (Y, \mathcal{T}_Y) is \mathcal{S} -normal with respect to the ideal \mathcal{I}_Y where \mathcal{T}_Y is the subspace topology on Y inherited from \mathcal{T} .

Proof. Let Y be a closed subset of X and let A and B be disjoint closed sets in (Y, \mathcal{T}_Y) . Then A and B are disjoint closed sets in (X, \mathcal{T}) . By \mathcal{S} -normality, there exist U_1, U_2 in \mathcal{T} such that $A \subseteq U_1, B \subseteq U_2$ and $U_1 \cap U_2 \in \mathcal{I}$. Let $V_1 = Y \cap U_1$ and $V_2 = Y \cap U_2$. Clearly $A \subseteq V_1, B \subseteq V_2$ and V_1, V_2 are open sets in \mathcal{T}_Y . As $V_1 \cap V_2 = (U_1 \cap U_2) \cap Y$, we have $V_1 \cap V_2 \in \mathcal{I}_Y$. Therefore Y is \mathcal{S} -normal with respect to the ideal \mathcal{I}_Y . \square

Conclusion

We defined and discussed the separation axioms in ideal topological spaces in a new way which is more natural than the previous versions. We proved a property that holds in ideal topological theory which does not hold in the classical theory of topology and also established a property that holds in the classical theory which does not hold in the ideal topological theory. This makes the ideal topological theory interesting and independent. Many concepts available in the classical theory may be discussed using the theory developed in this paper.

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