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CotFotal Edge Domination Number of a Graph

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Abstract

A set F of a graph $G(V, E)$ is an edge dominating set if every edge in $E - F$ is adjacent to some edge in F. An edge domination number $\gamma^{'}(G)$ of G is the minimum cardinality of an edge dominating set. An edge dominating set F is called a cototal edge dominating set if the induced subgraph $\langle E - F \rangle$ doesnot contain isolated edge. The minimum cardinality of the cototal edge dominating set in G is its domination number and is denoted by $\gamma^{'}_{cot}(G)$. We investigate several properties of cototal edge dominating sets and give some bounds on the cototal edge domination number.

Keywords: Edge domination number, cototal edge domination number.

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1 Introduction

Let $G(V, E)$ be a graph with $p =| V |$ and $q =| E |$ denoting the number of vertices and edges respectively. All the graphs considered here are finite, non-trivial, undirected and connected without loops or multiple edges.

The degree of a vertex u is denoted by $d(u)$. The degree of an edge $e = uv$ of a graph G is the number defined by $deg(e) = degu + degv - 2$. The minimum(maximum) degree of an edge is denoted by $\delta^{'}(\Delta^{'})$. The induced subgraph of $X \subseteq E$ is denoted by $\langle X \rangle$. For a real number x, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. An edge independence number $\beta_1(G)$ is defined to be the number of edges in a maximum independent set of edges of G. A vertex of degree one is called a pendant vertex. An edge incident to pendant vertex is called the pendant edge. Let $\Omega_1(G)$ be the set of all pendant edges of G. As usual, P_p, C_p and K_p are respectively the path, cycle and complete graph of order p. $K_{m,n}$ is the complete bipartite graph with two partite sets containing m and n vertices. Let $t \geq 3, n \geq 1$ be two integers. We denote by W^n_t the graph $C_t + K_n$ as a generalized wheel. Note that for $n = 1, W^1_t = K_1 + C_{n-1}$ is a wheel. $B_{r,s}$ is a graph obtained by joining the centres of two stars $K_{1,r}$ and $K_{1,s}$ by an edge called as Bistar or double star. The subdivision graph of a graph G, denoted by $S(G)$, is a graph obtained from G by deleting every edge uv of G and replacing it by a vertex w of degree 2 that is joined to u and v.

Let $G(V, E)$ be a connected graph. A subset S of V is called a dominating set of G if every vertex in $V - S$ is adjacent to at least one vertex in S. The concept of edge domination was introduced by Mitchell and Hedetniemi [5, 7].

Definition 1.1. *A subset F of E is called an edge dominating set of G if every edge not in F is adjacent to some edge in F. The minimum cardinality of an edge dominating set of G is called an edge domination number and is denoted by*

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$\gamma^{'}(G)$.

Definition 1.2. *A dominating set D of G is a cototal dominating set if the induced subgraph $\langle V - D\rangle$ has no isolated vertices. The cototal domination number $\gamma_{cot}(G)$ of G is the minimum cardinality of a cototal dominating set.*

This concept was introduced by Kulli, Janakiram and Iyer in [6]. Any undefined term or notation in this paper can be found in Harary [4]. We need the following Theorems for our study on cototal edge domination number.

Theorem 1.1. *[3] For every n, $\gamma^{'}_{3,n} = n$.*

Theorem 1.2. *[2] For any connected graph G of even order $p, \gamma^{'}(G) = p/2$ if and only if G is isomorphic to K_p or $K_{p/2,p/2}$.*

In this paper, we determine the relation among the graph parameters, like edge independent number, maximum edge degree and cototal edge domination number of a graph. We have also derived some relations to determine cototal edge domination number of graph obtained by adding end edges to cycle, cartesian product, subdivision of graphs, corona and join of graphs.

Definition 1.3. *A set $F \subseteq E(G)$ is said to be cototal edge dominating set if F is an edge dominating set and induced subgraph $\langle E - F\rangle$ has no isolated edges. The minimum cardinality of cototal edge dominating set in G is the cototal edge domination number and is denoted by $\gamma^{'}_{cot}(G)$ of G.*

2 Main results

We list out cototal edge domination number of some standard graphs.

Theorem 2.1.

1. *For any spider G, the cototal edge domination number, $\gamma^{'}_{cot}(G) = |\Omega_1(G)|$.*
2. *For any octopus G, the cototal edge domination number, $\gamma^{'}_{cot}(G) = |\Omega_1(G)| + 2$.*
3. *For any generalized wheel W^n_t, with $t \geq 3, \gamma^{'}_{cot}(W^n_t) = \lceil t/2\rceil$.*
4. *For any wheel W_n, with $n \geq 3, \gamma^{'}_{cot}(W_n) = \lceil n/2\rceil$.*

Theorem 2.2. *For any path P_p with $p \geq 5$, the cototal edge domination number,*

$$\gamma^{'}_{cot}(P_p) = \begin{cases} \lfloor \frac{p}{3} \rfloor + 2 & \text{if } p \equiv 1 (mod 3), \\ \lfloor \frac{p}{3} \rfloor + 1 & \text{if } p \equiv 0 \text{ or } 2 (mod 3). \end{cases}$$

Proof. Let $P_p : v_1, v_2, ..., v_p$ be any path and let $e_i = v_i v_{i+1}$ be an edge. Let

$$S_1 = \begin{cases} S & \text{if } p \equiv 2(mod 3), \\ S \cup \{e_{p-1}\} & \text{if } p \equiv 0(mod 3), \\ S \cup \{e_{p-1}, e_{p-2}\} & \text{if } p \equiv 1(mod 3). \end{cases}$$

be an edge set where $S = \{e_j : j = 3k + 1 \text{ for } 0 \leq k \leq \lceil \frac{p}{3} \rceil - 1\}$. Clearly S_1 is an edge dominating set and the induced subgraph $\langle E - S_1\rangle$ has no isolated edges. Therefore $|S_1|$ will be the cototal edge dominating set with minimum cardinality. Hence the proof. □

Theorem 2.3. *For any cycle C_p with $p \geq 3$, the cototal edge domination number,*

$$\gamma^{'}_{cot}(C_p) = \begin{cases} \lfloor \frac{p}{3} \rfloor & \text{if } p \equiv 0(mod 3), \\ \lfloor \frac{p}{3} \rfloor + 1 & \text{if } p \equiv 1(mod 3), \\ \lfloor \frac{p}{3} \rfloor + 2 & \text{if } p \equiv 2(mod 3). \end{cases}$$

Proof. Let $C_p : v_1, v_2, ..., v_p$ be any cycle and let $e_i = v_i v_{i+1}$ be an edge. Let

$$S_1 = \begin{cases} S & \text{if } p \equiv 0 \text{ or } 1(mod 3), \\ S \cup \{e_p\} & \text{if } p \equiv 2(mod 3). \end{cases}$$

be an edge set where $S = \{e_j : j = 3k + 1 \text{ for } 0 \leq k \leq \lceil \frac{p}{3} \rceil - 1\}$. Clearly S_1 is an edge dominating set and the induced subgraph $\langle E - S_1 \rangle$ has no isolated edges. Therefore $|S_1|$ will be the cototal edge dominating set with minimum cardinality. Hence the proof. □

Theorem 2.4. *For any complete graph K_p with $p \geq 3$ vertices, the cototal edge domination number, $\gamma'_{cot}(K_p) = \lfloor \frac{p}{2} \rfloor$.*

Proof. Let S be a maximum matching of K_p. We know that the edge independence number $\beta_1(G)$ of a graph G is the maximum cardinality of an independent set of edges. Furthermore for an integer $p \geq 3$ and integers m and n with $1 \leq m \leq n$, we have $\beta_1(K_p) = \lfloor \frac{p}{2} \rfloor$. Clearly S is an edge dominating set. Also the induced subgraph $\langle E - S \rangle$ has no isolated edges. Therefore S is a cototal edge dominating set of K_p.

$$\begin{aligned} \gamma'_{cot}(K_p) &= |S|, \\ &= \left\lfloor \frac{p}{2} \right\rfloor. \end{aligned}$$

Hence the proof. □

Corollary 2.1. *Let G' be a graph obtained from K_p by adding the pendant edges to the vertices of K_p. Then the cototal edge domination number,*

$$\gamma'_{cot}(G') = \begin{cases} \gamma'_{cot}(K_p) & \text{if } p \text{ is even}, \\ \gamma'_{cot}(K_p) + 1 & \text{if } p \text{ is odd}. \end{cases}$$

Theorem 2.5. *For any complete bipartite graph $K_{m,n}$ with $2 \leq m \leq n$ the cototal edge domination number, $\gamma'_{cot}(K_{m,n})$ = m.*

Proof. Let v be any vertex on $K_{m,n}$such that $deg(v) = min(m, n)$. Let S be the set containing all the edges incident on v. It is clear that S is an edge dominating set. Also the induced subgraph $\langle E - S \rangle$ does not contain an isolated edge. Thus S is a cototal edge dominating set and the minimal cardinality $|S|$ of S will be the cototal edge domination number. Thus

$$\begin{aligned} \gamma'_{cot}(K_{m,n}) &\leq |S|, \\ &= deg(v). \end{aligned}$$

Therefore $\gamma'_{cot}(K_{m,n}) = m$. Hence the proof. □

Theorem 2.6. *Let F be a cototal edge dominating set of a connected graph G with $p \geq 5$ vertices. Then every pendant edge $e = uv$ with degree(support vertex) = 2 is in F.*

Proof. Let $e = uv$ be a pendant edge with degree of support vertex v is 2. Suppose $e \notin F$ and $e_1 \in N(e)$. Since F is a dominating set we have $e_1 \in F$. Also the induced subgraph $\langle E - F \rangle$ contains an isolated edge, a contradiction. Therefore every pendant edge with degree of support vertex equal to two, is in F. Hence the proof. □

The following Theorem gives a sharp upper bound for the cototal edge domination number of G.

Theorem 2.7. *For any connected graph G with $\delta(G) > 2$, the cototal edge domination number, $\gamma'_{cot}(G) \leq \beta_1(G)$.*

Proof. Let G be a graph with $\delta(G) > 2$. Let S be an edge independent set in G such that $|S| = \beta_1(G)$. Clearly every edge $e_1 \in E - S$ is adjacent to atleast one edge of S. Hence S is an edge dominating set. Since $\delta(G) > 2$, the induced subgraph $\langle E - S \rangle$ is connected. Therefore S is a cototal edge dominating set. Hence $\gamma'_{cot}(G) \leq \beta_1(G)$. Hence the proof. □

Let $S = \{e_1, e_2, \cdots e_s\}$ be a set of non-adjacent edges to e. The characterization of all graphs for which $\gamma'_{cot}(G) = q - \triangle'$ seems to be a difficult problem. In the next Theorem, we characterize few graphs for which $\gamma'_{cot}(G) = q - \triangle'$.

Theorem 2.8. *Let G be a graph with $\delta(G) \geq 2$ and $e = uv$ with $d(u), d(v) \geq 3$ be an edge with maximum degree. Then $\gamma'_{cot}(G) \leq q - \triangle'(G)$.*

Proof. Let $e = uv$ be an edge with maximum degree. Let $E_1 = \{e_1, e_2, \cdots e_r\}$ be the set of edges adjacent to e. Then $F = E - E_1$ is an edge dominating set and since $\delta(G) \geq 2$ the induced subgraph $\langle E - F \rangle$ has no isolated edges. Therefore F is a cototal edge dominating set of G. Hence $|F| \leq |E| - |E_1|$. Therefore $\gamma'_{cot}(G) \leq q - \triangle'(G)$. Equality holds in the following cases:

Case i) $S = 0$.
Let $e = uv$ be an edge with maximum degree. Since $d(u), d(v) > 2$, the induced subgraph $\langle E - \{e\} \rangle$ has no isolated edges. Because there are no non-adjacent edges to e and all the edges in G are dominated by $\{e\}$. Therefore $F = \{e\}$ is a cototal edge dominating set. Hence
$\gamma'_{cot}(G) = q - \triangle'(G)$.

Case ii) $|S| = 1$.
Let $e = uv$ be an edge with maximum degree. Since there is only one edge e_1 not adjacent to e, we get $F = \{e\} \cup \{e_1\}$ as an edge dominating set. Also $\langle E - F \rangle$ has no isolated edges. Therefore F is a cototal edge dominating set. Hence $\gamma'_{cot}(G) = q - \triangle'(G)$.

Hence the proof. □

Corollary 2.2. *If degree of each vertex in $\langle S \rangle$ is one and for all $w - \{u, v\}$ in $V(G)$, the other end vertices of the edges not in S are either u or v then*
$\gamma'_{cot}(G) = q - \triangle'(G) - 1$.

□

Let $e = uv$ be an edge satisfying $d(e) = q - 1$. Let $\mathcal{H}$ be a family of graphs isomorphic to $K_{1,r}$, $B_{r,s}$, where $r, s \geq 3$, the graphs formed by adding P_3 to $B_{r,s}$ such that end vertices of P_3 is u and v or the graph $\mathcal{G}''$ formed by adding P_3 to $B_{r,s}$ and removing pendant edges adjacent to u or adjacent to v or both of $B_{r,s}$. Few graphs which belong to $\mathcal{G}'$ or $\mathcal{G}''$ are shown in Figure 1. Now we have the following Theorem.

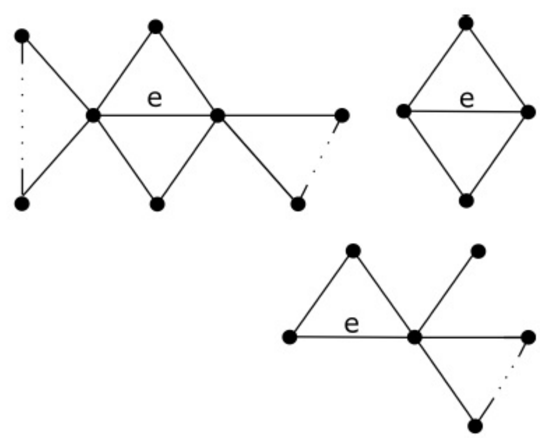

Figure 1: Illustration for the family $\mathcal{H}$

Theorem 2.9. *The cototal edge domination number of $\mathcal{H}$ is one if and only if the graph belongs to $\mathcal{H}$.*

Proof. The inequality $1 \leq \gamma'_{cot}(\mathcal{H})$ is obvious. If there exists an edge $e = uv \in \mathcal{H}$ where $d(e) = q - 1$ then $\gamma'_{cot}(\mathcal{H}) = 1$. Assume $\gamma'_{cot}(\mathcal{H}) = 1$. Then $F = \{e_1\}$ is a minimum cototal edge dominating set of G. Since F is a dominating set of G, e_1 must be an universal edge which belongs to the family $\mathcal{H}$. If $e \notin \mathcal{H}$ then there is a possibility where the induced subgraph $\langle E - F \rangle$ contains an isolated edge. It is a contradiction to the definition of cototal edge dominating set. Therefore $e \in \mathcal{H}$. Hence the proof. □

The following Theorem relates γ'_{cot} of few standard graphs G with their *subdivision* graph $S(G)$ in terms of vertices.

Theorem 2.10. *For any path P_p, $\gamma'_{cot}(P_p)$ + $\gamma'_{cot}(S(P_p)) \leq p+3$.*

Proof. Let S be the γ'_{cot} set of P_p. Therefore by Theorem 2.2,

$$\gamma'_{cot}(P_p) = |S| \tag{2.1}$$

Let S' be its γ'_{cot} set of *subdivision* of P_p. Then by Theorem 2.2 we get
$\gamma'_{cot}\Big(S(P_p)\Big) = |S'|$ that is

$$\gamma'_{cot}(S(P_p)) = \begin{cases} \frac{2p}{3} & \text{if } p \equiv 0(mod3), \\ \left\lfloor \frac{2p}{3} \right\rfloor + 2 & \text{if } p \equiv 1(mod3), \\ \left\lfloor \frac{2p}{3} \right\rfloor + 1 & \text{if } p \equiv 2(mod3). \end{cases} \tag{2.2}$$

Consider a particular case where $p \equiv 1(mod3)$. Adding equations (2.1) and (2.2), we get

$$\begin{aligned} \gamma'_{cot}(P_p) + \gamma'_{cot}(S(P_p)) &= |S| + |S'|, \\ &= \left\lfloor \frac{p}{3} \right\rfloor + \left\lfloor \frac{2p}{3} \right\rfloor + 4, \\ &\leq p+3. \end{aligned}$$

The other two cases are obvious. Hence the proof. □

Theorem 2.11. *For any Cycle C_p , $\gamma'_{cot}(C_p) + \gamma'_{cot}S((C_p)) \leq p+2$.*

Proof of this Theorem is similar to the Theorem 2.10.

□

The *corona* $G = H \circ K_1$ is a graph constructed from a copy of H, where for each vertex $v \in V(H)$, a new vertex v' and a pendant edge vv' are added. The following Theorem gives a sharp bound for the cototal edge domination number of $(G_{p_1} \circ G_{p_2})$.

Theorem 2.12. *Let G be a connected graph with p_1 vertices and K_{1,p_2} be any star. The cototal edge domination number of the corona of G and K_{1,p_2}, is given by $\gamma'_{cot}(G \circ K_{1,p_2}) = p_1$.*

Proof. Let $u_i \in V_1$ for $1 \leq i \leq p_1$ be a vertex set of G and v be a support vertex of K_{1,p_2}. Let $S = \{u_i v : \forall u_i \in G\}$ be an edge set in the *corona* $(G \circ K_{1,p_2})$. Then S is a minimum edge dominating set of $(G \circ K_{1,p_2})$. Clearly the induced subgraph $\langle E - S \rangle$ does not contain any isolated edge. Hence $S = \{e_1, e_2 \cdots e_{p_1}\}$ is a cototal edge dominating set with minimum cardinality. Therefore
$\gamma'_{cot}(G \circ K_{1,p_2})$ = p_1. Hence the proof. □

Theorem 2.13. *Let $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ be two connected graphs. The cototal edge domination number of the corona of G_1 and G_2, is given by $\gamma'_{cot}(G_1 \circ G_2) \leq p_1 + p_1 \lfloor \frac{p_2}{2} \rfloor$. Equality holds if G_2 is either a cycle or a complete graph.*

Proof. Let $u_i \in V_1$ for $1 \leq i \leq p_1$ and let $v_j \in V_2$ for $1 \leq j \leq p_2$ be the vertex set of G_1 and G_2 respectively. Let $S = \{u_i v_k : \forall u_i \in G_1$ and any one vertex $v_k \in V_2\}$ be an edge set in the *corona* $(G_1 \circ G_2)$. Let E_1 be an edge dominating set for the graph G_2. Then set S along with p_1 copies of E_1 in the *corona* will form an edge dominating set F of $(G_1 \circ G_2)$. Clearly $\langle E - F \rangle$ doesnot contain isolated edges. Therefore

$$\begin{aligned} \gamma'_{cot}(G_1 \circ G_2) &\leq |S| + p_1(|E_1|), \\ &\leq p_1 + p_1 \left\lfloor \frac{p_2}{2} \right\rfloor. \end{aligned}$$

Hence the proof. □

For disjoint graphs G_1 and G_2, the *join* $G = G_1 + G_2$ is the graph with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1) \wedge v \in V(G_2)\}$. In the following Theorem we give the result on the *join* of two paths.
Let $\varepsilon = 2p_1 - p_2$ be a constant used in the Theorem below.

Theorem 2.14. *Let P_{p_1} and P_{p_2} be any two paths of order p_1, p_2 respectively. Then the cototal edge domination number,*

$$\gamma'_{cot}(P_{p_1} + P_{p_2}) \leq \begin{cases} \lceil \frac{p_2}{2} \rceil + \left\lceil \frac{p_1 - \lfloor \frac{p_2}{2} \rfloor}{2} \right\rceil & \text{for } \varepsilon > 0, \\ p_1 + \frac{p_2 - 2p_1}{2} & \text{for } \varepsilon \leq 0. \end{cases}$$

Proof. Let P_{p_1} and P_{p_2} be the two paths labelled in order as $u_1e_1u_2e_2 \cdots u_{p_1-1}$
$e_{p_1-1}u_{p_1}$ and $v_1e'_1v_2e'_2 \cdots v_{p_2-1}e'_{p_2-1}v_{p_2}$ respectively. Let us consider the following cases:
Case i) Suppose $\varepsilon > 0$.
Choose an edge set $E_1 = \{u_iv_{2i} : 1 \leq i \leq \lfloor \frac{p_2}{2} \rfloor\}$. Then $|E_1| = \lfloor \frac{p_2}{2} \rfloor$. Let E_2 be the set of alternate edges in the path $u_{p_1}, u_{p_1-1}, \cdots, u_{\lfloor \frac{p_2}{2} \rfloor}$ in order such that $|E_2| = \left\lceil \frac{p_1 - \lfloor \frac{p_2}{2} \rfloor}{2} \right\rceil$. Then $E_1 \cup E_2$ forms a cototal edge dominating set in the *join* of the two paths P_{p_1} and P_{p_2}. Thus

$$\gamma'_{cot}(P_{p_1} + P_{p_2}) \leq |E_1| + |E_2|,$$
$$\leq \left\lfloor \frac{p_2}{2} \right\rfloor + \left\lceil \frac{p_1 - \lfloor \frac{p_2}{2} \rfloor}{2} \right\rceil.$$

Case ii) Suppose $|\varepsilon| \leq 0$.
Choose an edge set $E'_1 = \{u_iv_{2i} : 1 \leq i \leq p_1\}$ then $|E'_1| = p_1$. Let E'_2 be the edge dominating set of the path $v_{2p_1+1}, \cdots v_{p_2}$ in order such that $|E'_2| = \lceil \frac{p_2 - 2p_1}{2} \rceil$. Thus $E'_1 \cup E'_2$ forms a cototal edge dominating set in the *join* of the two paths P_{p_1} and P_{p_2} . Thus

$$\gamma'_{cot}(P_{p_1} + P_{p_2}) \leq |E'_1| + |E'_2|,$$
$$\leq p_1 + \left\lceil \frac{p_2 - 2p_1}{2} \right\rceil.$$

Hence the proof. □

In the following Theorem we give the result on the *join* of two cycles.

Theorem 2.15. *Let C_{p_1} and C_{p_2} be any two cycles of order p_1, p_2 respectively. Then the cototal edge domination number,*

$$\gamma'_{cot}(C_{p_1} + C_{p_2}) \leq \begin{cases} \lceil \frac{p_2}{2} \rceil + \left\lfloor \frac{p_1 - \lceil \frac{p_2}{2} \rceil + 1}{2} \right\rfloor & \text{for } \varepsilon \geq 0, \\ p_1 + \lceil \frac{\varepsilon}{3} \rceil & \text{for } \varepsilon < 0. \end{cases}$$

Proof. Let C_{p_1} and C_{p_2} be the two cycles labelled in order as, $u_1e_1u_2e_2 \cdots u_{p_1}$
$e_{p_1}u_1$ and $v_1e'_1v_2e'_2 \cdots v_{p_2}e'_{p_2}v_1$ respectively. Let us consider the following cases
Case i) Suppose $\varepsilon \geq 0$.
Consider the edge set $E_1 = \{u_iv_{2i-1} : 1 \leq i \leq \lceil \frac{p_2}{2} \rceil\}$. Let E_2 be the set of alternate edges in the path $u_{\lceil \frac{p_2}{2} \rceil + 1}, u_{\lceil \frac{p_2}{2} \rceil + 2}, \cdots u_{p_1}, u$ in order such that $|E_2| = \left\lceil \frac{p_1 - \lceil \frac{p_2}{2} \rceil + 1}{2} \right\rceil$. Then $E_1 \cup E_2$ forms a cototal edge dominating set in the *join* of two cycles C_{p_1} and C_{p_2}. Thus

$$\gamma'_{cot}(C_{p_1} + C_{p_2}) \leq |E_1| + |E_2|,$$
$$\leq \left\lceil \frac{p_2}{2} \right\rceil + \left\lfloor \frac{p_1 - \lceil \frac{p_2}{2} \rceil + 1}{2} \right\rfloor.$$

Case ii) Suppose $|\varepsilon| < 0$.
Choose an edge set $E'_1 = \{u_iv_{2i-1} : 1 \leq i \leq p_1\}$ then $|E'_1| = p_1$. Let E'_2 be the edge dominating set in the path $v_{2p_1}, v_{2p_1+1}, \cdots v_{p_2}$ in order such that $|E'_2| = \lceil \frac{p_2 - 2p_1}{3} \rceil$. Thus $E'_1 \cup E'_2$ forms a cototal edge dominating set in the *join* of the two cycles C_{p_1} and C_{p_2}. Thus

$$\gamma'_{cot}(C_{p_1} + C_{p_2}) \leq |E'_1| + |E'_2|,$$
$$\leq p_1 + \left\lceil \frac{p_2 - 2p_1}{3} \right\rceil.$$

Hence the proof. □

In the following two Theorems we are adding K_2 to a cycle C_p. For an edge $e = uv$ of a graph G with $deg(u) = 1$ and $deg(v) > 1$, we call e an end edge and u an end vertex.

Theorem 2.16. *Let G' be the graph obtained by adding k end edges u_1v_j for $j = 1, 2, \cdots k$ to a cycle C_p where $u_1 \in C_p$ and $\{v_1, v_2, \cdots v_k\} \notin C_p$. Then the cototal edge domination number, $\gamma'_{cot}(G') = \lceil \frac{p}{3} \rceil$.*

Proof. Let $C_p : u_1, u_2, \cdots, u_p$ be a cycle with p vertices and G' be the graph obtained by adding k end edges $\{u_1v_1, u_1v_2, \cdots u_1v_k\}$ such that $u_1 \in C_p$ and $\{v_1, v_2, \cdots v_k\} \notin C_p$. Let $e_i = u_iu_{i+1}$ be an edge on cycle.
Let $S = \{e_j : j = 3l + 1 \text{ for } 0 \leq l \leq \lfloor \frac{p}{3} \rfloor - 1\}$ and

$$S_1 = \begin{cases} S & \text{if } p \equiv 0 \text{ or } 2(mod3), \\ S \cup \{e_{p-2}\} & \text{if } p \equiv 1(mod3). \end{cases}$$

be an edge set on S. Then S_1 is an edge dominating set and the induced subgraph $\langle E - S_1 \rangle$ doesnot contain isolated edges. Therefore S_1 is a cototal edge dominating set and $|S_1|$ will be the cototal edge domination number for the graph G'. Hence the proof. □

Theorem 2.17. *Let G' be the graph obtained by adding k end edges u_iv_j to a cycle C_p with $d(u_i) \geq 3$ where $u_i \in C_p$, for $i = 1, 2, \cdots p, v_j \notin C_p$ for $j = 1, 2, \cdots k$. Then the cototal edge domination number, $\gamma'_{cot}(G') = \lceil \frac{p}{2} \rceil$.*

Proof. Let $C_p : u_1, u_2, \cdots, u_p$ be a cycle and G' be the graph obtained by adding k end edges u_iv_j where $u_i \in C_p$ for $i = 1, 2, \cdots p$ and $v_j \notin C_p$ for $j = 1, 2, \cdots k$. Let $e_i = u_iu_{i+1}$ be an edge of G'.
Let $S = \{e_j : j = 2l + 1 \text{ for } 0 \leq l \leq \lfloor \frac{p}{2} \rfloor - 1\}$ and

$$S_1 = \begin{cases} S \cup \{e_{p-1}\} & \text{if } p \equiv 0(mod2), \\ S \cup \{u_pv_1\} & \text{if } p \equiv 1(mod2). \end{cases}$$

be an edge set of G'. Then S_1 is an edge dominating set and the induced subgraph $\langle E - S_1 \rangle$ doesnot contain isolated edges. Therefore S_1 is a cototal edge dominating set and $|S_1|$ will be the cototal edge domination number for the graph G'. Hence the proof. □

The following corollary is the immediate consequence of the above Theorems.

Corollary 2.3. *Let G' be the graph obtained by adding k end edges u_iv_j to a cycle C_p of order $p \geq 3$ in any manner then from the above Theorems, we get $\lceil \frac{p}{3} \rceil \leq \gamma'_{cot}(G') \leq \lceil \frac{p}{2} \rceil$.*

3 Cartesian product of independent cototal edge domination number

In this section we define a new parameter "Independent cototal edge domination number" of a graph. An edge dominating set F is called an independent edge dominating set if no two edges of F are adjacent [1]. An independent edge domination number $\gamma'_i(G)$ of G is the minimum cardinality taken over all independent edge dominating sets of G. The cototal edge dominating set is said to be an independent cototal edge dominating set if the induced subgraph $\langle F \rangle$ is an independent edge set.

The cartesian product of G and H, denoted $G \times H$, has vertex set $V(G) \times V(H)$. Two vertices (u, v), (u', v'), in $V(G) \times V(H)$ are adjacent if either $u = u'$ and $vv' \in E(H)$, or $v = v'$ and $uu' \in E(G)$. The graph $P_n \times P_m$ has m copy of the graph P_n in m columns. Let $\gamma'_{icot}(P_n \times P_m)$ denotes the size of minimum independent cototal edge dominating set of two paths ($P_n \times P_m$ where $n \leq m$. In the next Theorem, we calculate γ'_{cot} for the product of two paths P_3 and P_m.

Theorem 3.18. *Let P_3 be a path of length 3 and P_m be any path with $m \geq 3$. Then independent cototal edge dominating number of the product of these two paths, $\gamma'_{icot}[P_3 \times P_m] = m$.*

Proof. Consider the independent edge set
$S = \Big\{\{(1, 2 + 3k), (1, 3 + 3k)\}, \{(2, 1 + 3k), (2, 2 + 3k)\}, \{(3, 2 + 3k), (3, 3 + 3k)\} / k = 0, 1, 2 \cdots \lfloor \frac{m}{3} \rfloor - 1\Big\}$, as shown in Figure 2.

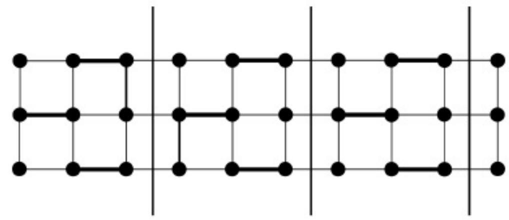

Figure 2: Cartesian product of $P_3 \times P_m$

Case i) Let $m \equiv 0(mod3)$.
Then S is an independent edge dominating set and the induced subgraph $\langle E - S \rangle$ of $P_3 \times P_m$ has no isolated edges. By Theorem 1.1, S is an independent cototal edge dominating set of $P_3 \times P_m$ with minimum cardinality. Therefore $|S| = m$.

Case ii) Let $m \equiv 1(mod3)$.
Then $S_1 = S \cup \{(1,n),(2,n)\}$ is an independent edge dominating set and the induced subgraph $\langle E - S_1 \rangle$ of $P_3 \times P_m$ has no isolated edges. By Theorem 1.1, S_1 is an independent cototal edge dominating set of $P_3 \times P_m$ with minimum cardinality. Therefore $|S_1| = |S| + 1 = m$.

Case iii) Let $m \equiv 2(mod3)$.
Then $S_2 = S \cup \Big\{\{(2,n-1),(3,n-1)\},\{(1,n),(2,n)\}\Big\}$ is an independent edge dominating set and the induced subgraph $\langle E - S_2 \rangle$ of $P_3 \times P_m$ has no isolated edges. By Theorem 1.1, S_2 is an independent cototal edge dominating set of $P_3 \times P_m$ with minimum cardinality. Therefore $|S_2| = |S| + 2 = m$.

Thus $\gamma'_{icot}[P_3 \times P_m] = m$. Hence the proof. □

In the next Theorem, we generalize γ'_{cot} for the *product* of two paths P_n and P_m.

Theorem 3.19. *Let P_n be a path with n vertices and P_m be any path with m vertices where $n \leq m$. Then independent cototal edge domination number of the cartesian product of two paths,*

$$\gamma'_{icot}[P_n \times P_m] = \begin{cases} \frac{nm}{3} & \text{if } m \equiv 0(mod3), \\ \left\lceil \frac{nm}{3} \right\rceil + 1 & \text{if } m \equiv 1 \text{ or } 2(mod3). \end{cases}$$

Proof. Consider an independent edge set as shown in Figure 3.
If i is odd, then

$$S = \begin{cases} \bigcup_{i=1,3,\cdots n} & \{(i,3k+2),(i,3k+3)\} : k = 0,1,\cdots \lfloor \frac{m}{3} \rfloor - 1, \\ \bigcup_{i=2,4,\cdots n-1} & \{(i,3k+1),(i,3k+2)\} : k = 0,1,\cdots \lfloor \frac{m}{3} \rfloor - 1. \end{cases}$$

If i is even, then

$$S = \begin{cases} \bigcup_{i=1,3,\cdots n-1} & \{(i,3k+2),(i,3k+3)\} : k = 0,1,\cdots \lfloor \frac{m}{3} \rfloor - 1, \\ \bigcup_{i=2,4,\cdots n} & \{(i,3k+1),(i,3k+2)\} : k = 0,1,\cdots \lfloor \frac{m}{3} \rfloor - 1. \end{cases}$$

Case i) Let $m \equiv 0(mod3)$.
Then S is an independent edge dominating set and the induced subgraph $\langle E - S \rangle$ of $P_n \times P_m$ has no isolated edges. Thus S is an independent cototal edge dominating set of $P_n \times P_m$. Therefore

$$\gamma'_{icot}[P_n \times P_m] \leq |S|, \\ \leq \frac{nm}{3}.$$

Case ii) Let $m \equiv 1(mod3)$.
Let us discuss the following subcases.

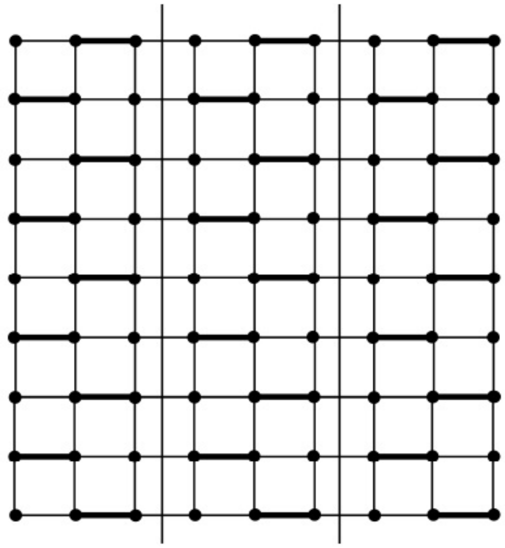

Figure 3: Cartesian product of $P_n \times P_m$

1. n is an odd number.
   Let $S_1 = \Big\{\{(1,m),(2,m)\},\{(3,m),(4,m)\},\cdots\{(n-2,m),$
   $(n-1,m)\}\Big\}$. Then $F_1 = S \cup S_1$ is an independent edge dominating set and the induced subgraph $\langle E - F_1\rangle$ of $P_n \times P_m$. Thus $|F_1| \leq |S| + |S_1|$.

2. n is an even number.
   Let $S_2 = \Big\{\{(1,m),(2,m)\},\{(3,m),(4,m)\},\cdots\{(n-2,m),$
   $(n-1,m)\}\Big\}$. Then $F_2 = S \cup S_2$ is an independent edge dominating set and the induced subgraph $\langle E - F_2\rangle$ of $P_n \times P_m$ has no isolated edges. Thus $|F_2| \leq |S| + |S_2|$.

Combining the above two subcases, we conclude that F_1 and F_2 form an independent edge dominating set of $P_n \times P_m$ in both the cases respectively. Therefore

$$\begin{aligned}\gamma'_{icot}[P_n \times P_m] &\leq |S_1 \cup S_2| + \left\lfloor \frac{n}{2} \right\rfloor,\\ &\leq \left\lceil \frac{nm}{3} \right\rceil + 1.\end{aligned}$$

Case iii) Let $m \equiv 2(mod 3)$.
For $n \geq 4$, we can partition the set of m columns of $P_n \times P_m$ into $B_i, i = 1,2,\cdots \lfloor \frac{m}{3} \rfloor$ blocks at the beginning and two columns at the end. The set S will dominate B_i blocks. In addition we dominate m and $m-1$ columns by a set isomorphic to S_R as shown in Figure 4. Let $n = 4q + l : 1 \leq q \leq \lfloor \frac{n}{4} \rfloor, 0 \leq l \leq 3$. Consider the following two cases to find S_R

i) If $q = 1$ then $S_R = \{R_l : 0 \leq l \leq 3\}$

ii) If $q > 1$ then $S_R = \Big\{(\lfloor \frac{n}{4} \rfloor - 1)R_0 + R_l : 0 \leq l \leq 3\Big\}$

Therefore $S_3 = S \cup S_R$ is an independent edge dominating set and the induced subgraph $\langle E - S_3\rangle$ of $P_n \times P_m$ has no isolated edges. Thus S_3 is a independent cototal edge dominating set of $P_n \times P_m$. Therefore,

$$\begin{aligned}\gamma'_{icot}[P_n \times P_m] &\leq |S_3|,\\ &\leq |S| + |S_R|,\\ &\leq \left\lceil \frac{nm}{3} \right\rceil + 1.\end{aligned}$$

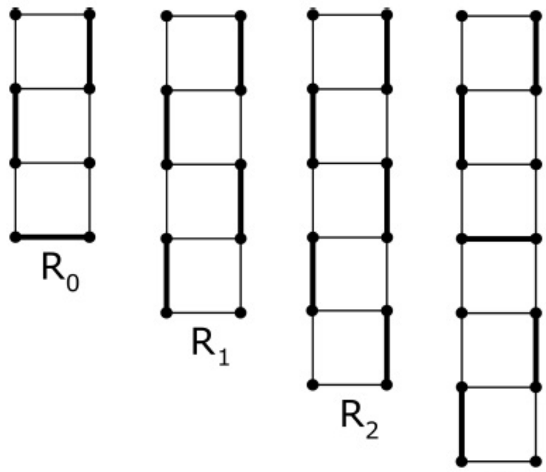

Figure 4: S_R

Hence the proof. □

Now let us study few more bounds on *cartesian product* of two cycles.

Theorem 3.20. *Let C_3 be a cycle of order 3 and C_m be any cycle of order m. Then independent cototal edge dominating number of the cartesian product of two cycles,*

$$\gamma'_{icot}[C_3 \times C_m] = \begin{cases} m & \text{if } m \text{ is even,} \\ m+1 & \text{if } m \text{ is odd.} \end{cases}$$

Proof. Consider an independent edge set.
$S = \Big\{\{(1,2k-1),(2,2k-1)\},\{(2,2k),(3,2k)\} : k = 1,2,\cdots \lceil \frac{m}{2} \rceil\Big\}$.

Case i) Suppose m is an even number.
Then S is an independent edge dominating set and the induced subgraph $\langle E - S\rangle$ of $C_3 \times C_m$ has no isolated edges. Thus S is an independent cototal edge dominating set of $C_3 \times C_m$. Therefore $|S| = m$.

Case ii) Suppose m is an odd number.
Then $S_1 = S \cup \Big\{(3,1),(3,m)\Big\}$ is an independent edge dominating set and the induced subgraph $\langle E - S_1\rangle$ of $C_3 \times C_m$ has no isolated edges. Thus S_1 is an independent cototal edge dominating set of $C_3 \times C_m$. Therefore $|S| = m+1$.

Hence the proof. □

Theorem 3.21. *Let C_4 be a cycle of order 4 and C_m be any cycle of order m where $m \geq 4$. Then independent cototal edge dominating number of the cartesian product of two cycles, $\gamma'_{icot}[C_4 \times C_m] \leq m + \lceil \frac{m}{2} \rceil$.*

Proof. Consider an independent edge set
$S = \Big\{\{(1,2k-1),(2,2k-1)\},\{(2,2k),(3,2k)\} : k = 1,2,\cdots \lceil \frac{m}{2} \rceil\Big\}$
and the set,

$$S_1 = \begin{cases} \Big\{(4,2i-1),(4,2i)\Big\} & \text{if } m \equiv 0(mod2), \\ \Big\{(4,2i-1),(4,2i)\Big\} \cup \Big\{(3,m),(4,m)\Big\} & \text{if } m \equiv 1(mod2), \end{cases}$$

for $i = 1,3,\cdots \lfloor \frac{m}{2} \rfloor$. Then $S \cup S_1$ is an independent edge dominating set and the induced subgraph $\langle E - (S \cup S_1)\rangle$ of $C_4 \times C_m$ has no isolated edges. Thus $S \cup S_1$ is an independent cototal edge dominating set of $C_4 \times C_m$.

$$\gamma'_{icot}[C_4 \times C_m] \leq |S \cup S_1|, \\ \leq m + \Big\lceil \frac{m}{2} \Big\rceil.$$

Hence the proof. □

Now let us study few more bounds for independent cototal edge dominating number on *cartesian product* of a path, cycle and complete graph.

Theorem 3.22. *Let P_3 be a path with 3 vertices and C_m be any cycle with m vertices where $m \leq 3$. Then independent cototal edge dominating number of the cartesian product,*

$$\gamma'_{icot}[P_3 \times C_m] \leq \begin{cases} m & \text{if } m \equiv 0(mod3), \\ m+1 & \text{if } m \equiv 1 \text{ or } 2(mod3). \end{cases}$$

Proof. Partition the set of m columns of $P_3 \times C_m$ into B_i blocks for $m \geq 6$. For $j = 1, 2, \cdots \lfloor \frac{m-3}{3} \rfloor$,

$$S = \begin{cases} \bigcup_{i=1,3} & \{(i, 3j-2), (i, 3j-1)\}, \\ \bigcup_{i=2} & \{(i, 3j-1), (i, 3j)\}. \end{cases}$$

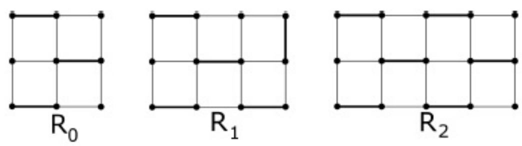

Figure 5: R_i

Case i) $m \equiv 0(mod3)$.
Then $S \cup R_0$, (as shown in R_0 of Figure 5) is an independent edge dominating set and the induced subgraph $\langle E - (S \cup R_0) \rangle$ of $P_3 \times C_m$ has no isolated edges. Thus $S \cup R_0$ is an independent cototal edge dominating set of $P_3 \times C_m$. Therefore

$$\gamma'_{icot}(P_3 \times C_m) \leq |S \cup R_0|,$$
$$\leq \frac{nm}{3} = m.$$

Case ii) $m \equiv 1(mod3)$,
Then $S \cup R_1$ (as shown in R_1 of Figure 5) is an independent edge dominating set and the induced subgraph $\langle E - (S \cup R_1) \rangle$ of $P_3 \times C_m$ has no isolated edges. Thus $S \cup R_1$ is an independent cototal edge dominating set of $P_3 \times C_m$. Therefore

$$\gamma'_{icot}[P_3 \times C_m] \leq |S| \cup |R_1|,$$
$$\leq \left\lceil \frac{nm}{3} \right\rceil,$$
$$\leq m+1.$$

Case iii) $m \equiv 2(mod3)$,
Then $S \cup R_2$ (as shown in R_2 of Figure 5) is an independent edge dominating set and the induced subgraph $\langle E - (S \cup R_2) \rangle$ of $P_3 \times C_m$ has no isolated edges. Thus $S \cup R_2$ is an independent cototal edge dominating set of $P_3 \times C_m$. Therefore

$$\gamma'_{icot}[P_3 \times C_m] \leq |S| + |R_2|,$$
$$\leq \lceil \frac{nm}{3} \rceil,$$
$$\leq m+1.$$

Hence the proof. □

Theorem 3.23. *Let P_n be a path with n vertices and K_m be any complete graph with m vertices where $n \leq m$. Then independent cototal edge domination number of the cartesian product,*

$$\gamma'_{icot}[P_n \times K_m] \leq n\left\lfloor \frac{m}{2} \right\rfloor.$$

Proof. Consider the following two cases,

Case i) m is an even number.
Consider the set

$$S = \bigcup_{i=1,2,\cdots n} \Big\{(i,1),(i,2)\Big\}, \Big\{(i,3),(i,4)\Big\}, \cdots \Big\{(i,m-1),(i,m)\Big\}.$$

If m is even then by Theorem 1.2, edge domination number of the K_m is $\frac{m}{2}$. Here we have n copies of K_m. Also S is an independent edge dominating set and the induced subgraph $\langle E - S \rangle$ of $P_n \times K_m$ has no isolated edges. Thus S is an independent cototal edge dominating set of $P_n \times K_m$. Therefore $|S| = n(\frac{m}{2})$.

Case ii) m is an odd number.
Let us discuss the following subcases:

1. If n is an odd number, then

$$S = \begin{cases} \bigcup_{i=1,3,\cdots n} & \Big\{(i,1),(i,2)\Big\}, \Big\{(i,3),(i,4)\Big\}, \cdots, \Big\{(i,m-2),(i,m-1)\Big\} \\ \bigcup_{i=2,4,\cdots n-1} & \Big\{(i,2),(i,3)\Big\}, \Big\{(i,4),(i,5)\Big\}, \cdots, \Big\{(i,m-1),(i,m)\Big\}. \end{cases}$$

2. If n is an even number, then

$$S = \begin{cases} \bigcup_{i=1,3,\cdots n-1} & \Big\{(i,1),(i,2)\Big\}, \Big\{(i,3),(i,4)\Big\}, \cdots, \Big\{(i,m-2),(i,m-1)\Big\} \\ \bigcup_{i=2,4,\cdots n} & \Big\{(i,2),(i,3)\Big\}, \Big\{(i,4),(i,5)\Big\}, \cdots, \Big\{(i,m-1),(i,m)\Big\}. \end{cases}$$

If m is odd then by Theorem 1.2, edge domination number of the K_m is lesser than $\frac{m}{2}$. Here we have n copies of K_m. Also S is an independent edge dominating set and the induced subgraph $\langle E - S \rangle$ of $P_n \times K_m$ has no isolated edges. Thus S is an independent cototal edge dominating set of $P_n \times K_m$. Therefore $|S| < n\lfloor \frac{m}{2} \rfloor$.

Combining the above two cases, we obtain the following result.

$$\gamma'_{icot}[P_n \times K_m] \leq n\left\lfloor \frac{m}{2} \right\rfloor$$

Hence the proof. □

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