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Certain properties of a subclass of harmonic convex functions of complex order defined by Multiplier transformations

K. Thilagavathi[a] and K. Vijaya[a,*] and N. Magesh [b]

[a]Department of Mathematics,School of advanced Sciences, VIT University, Vellore-632014, Tamil Nadu, India.

[b]P.G. and Research Department of Mathematics, Govt Arts College(Men), Krishnagiri - 635 001, Tamil Nadu, India.

Abstract

In this paper, we investigate some properties of harmonic univalent functions of complex order using multiplier transformation.Such as Coefficient bounds, extreme points, distortion bounds, convolution conditions and convex combination are determined for functions in this family. Further, we obtain the closure property of this class under integral operator. Consequently, many of our results are either extensions or new approaches to those corresponding to previously known results.

Keywords: Harmonic functions, analytic functions, univalent functions, starlike functions of complex order,Multiplier transformation..

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1 Introduction

A continuous function $f = u + iv$ is a complex- valued harmonic function in a complex domain Ω if both u and v are real and harmonic in Ω. In any simply-connected domain $D \subset \Omega$, we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and orientation preserving in D is that $|h'(z)| > |g'(z)|$ in D. See Clunie and Sheil-Small [3].

Denote by $\mathcal{S_H}$ the family of functions $f = h + \overline{g}$ which are harmonic, univalent and orientation preserving in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$ so that f is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$. Thus, for $f = h + \overline{g} \in \mathcal{S_H}$, the functions h and g analytic $\mathcal{U}$ can be expressed in the following forms:

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \;\; g(z) = \sum_{k=1}^{\infty} b_k z^k \;\; (|b_1| < 1),$$

and $f(z)$ is then given by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k} \quad (|b_1| < 1). \tag{1.1}$$

We note that the family $\mathcal{S_H}$ of orientation preserving, normalized harmonic univalent functions reduces to the well known class $\mathcal{S}$ of normalized univalent functions if the co-analytic part of f is identically zero, i.e. $g \equiv 0$.

Also, we denote by $T\mathcal{S_H}$ the subfamily of $\mathcal{S_H}$ consisting of harmonic functions of the form $f = h + \overline{g}$ such that h and g are of the form

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k, \quad g(z) = \sum_{k=1}^{\infty} |b_k| z^k. \tag{1.2}$$

*Corresponding author.
E-mail address: kvijaya@vit.ac.in (K. Vijaya), kthilagavathi@vit.ac.in(K. Thilagavathi) and nmagi2000@yahoo.co.in (N. Magesh)

In [3] Clunie and Sheil-Small, investigated the class $\mathcal{S_H}$ as well as its geometric subclasses and its properties. Since then, there have been several studies related to the class $\mathcal{S_H}$ and its subclasses. In particular, Avci and Zlotkiewicz [2], Silverman [9], Jahangiri [5, 6] and others have investigated various subclasses of $\mathcal{S_H}$ and its properties. Furthermore, Yalçin and Öztürk [11] and Murugusundaramoorthy[7] have considered a class $T\mathcal{S}^*_{\mathcal{H}}(\gamma)$ of harmonic starlike functions of complex order based on a corresponding study of Nasr and Aouf [8] for analytic case. (see [4, 13]).

For $f \in S$ the differential operator $D^n(n \in N_0)$ of f was introduced by salagean for $f = h + \overline{g}$ Jagangiri et al[] defined the modified salagean operator of f as

$$D^n f(z) = D^n h(z) + (-1)^n \overline{D^n g(z)} \tag{1.3}$$

$$D^n h(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \quad D^n g(z) = \sum_{k=1}^{\infty} k^n b_k z^k. \tag{1.4}$$

Next,for functions $f \in A$ Cho and Srivastava defined Multiplier transformation.For $f = h + \overline{g}$ given by (1) we define the modified Multiplier transformation of f.

$$I^0_\gamma f(z) = D^0 f(z) = h(z) + \overline{g(z)} \tag{1.5}$$

$$I^1_\gamma f(z) = \frac{\gamma D^0 f(z) + D^1 f(z)}{\gamma + 1} \tag{1.6}$$

$$I^n_\gamma f(z) = I^1_\gamma (I^{n-1}_\gamma f(z)), \ (n \in N_0) \tag{1.7}$$

$$I^n_\gamma f(z) = z + \sum_{k=2}^{\infty} (\frac{k+\gamma}{1+\gamma})^n a_k z^k + (-1)^n \sum_{k=1}^{\infty} (\frac{k-\gamma}{1+\gamma})^n \overline{b_k z^k} \tag{1.8}$$

Also if f is given by (1) then we have

$$I^n_\gamma f(z) = f \tilde{*} \underbrace{(\phi_1(z) + \overline{\phi_2(z)}) \tilde{*} ...... \tilde{*} (\phi_1(z) + \overline{\phi_2(z)})}_{n-times} = h * \underbrace{(\phi_1(z) * ...(\phi_1(z)}_{n-times} + \overline{g + \underbrace{(\phi_2(z) * ...(\phi_2(z)))}_{n-times}} \tag{1.9}$$

Where $*$ denotes the usual Hadamard product or convolution of power series and

$$\phi_1(z) = \frac{(1+\gamma)z - \gamma z^2}{(1+\gamma)(1-z)^2}, \ \phi_2(z) = \frac{(\gamma-1)z - \gamma z^2}{(1+\gamma)(1-z)^2} \tag{1.10}$$

By specializing the parameters γ and n we obtain the following operators studied by various authors for $f \in A$

$$(i) I^n_0 f(z) = D^n f(z) \ (ii) I^n_\lambda f(z) \ (iii) I^n_1 = I^n f(z) \tag{1.11}$$

Motivated by the earlier works of [4, 7, 11–13] now we define the class of harmonic convex functions of complex order in the following definition.

Definition 1.1. *For* $0 \leq \gamma < 1, 0 \leq \lambda \leq \frac{\gamma}{(1+\gamma)}$ *or* $\lambda \geq \frac{1}{1+\gamma}$ *and* $b \in \mathbb{C} \setminus \{0\}$, *let* $\mathcal{SC_H}(b, \gamma, \lambda, n)$ *denote the family of harmonic functions* $f \in \mathcal{S_H}$ *of the form (1.1) which satisfy the condition*

$$\Re \left(1 + \frac{1}{b} \left(\frac{\mathcal{F}(z)}{\mathcal{G}(z)} - 1 \right) \right) \geq \gamma, \tag{1.12}$$

where

$$\mathcal{F}(z) = \lambda(z^3(I^n_\gamma h(z))''' - \overline{z^3(I^n_\gamma g(z))'''}) + (2\lambda+1)z^2(I^n_\gamma(h(z))'' + (1-4\lambda)\overline{z^2(I^n_\gamma g(z))''} + z(I^n_\gamma h(z))' + (1-2\lambda)\overline{z(I^n_\gamma g(z))'}$$

and

$$\mathcal{G}(z) = \lambda(z^2(I^n_\gamma(h(z))'' + \overline{z^2(I^n_\gamma g(z))''}) + z(I^n_\gamma h(z))' + (2\lambda-1)\overline{z(I^n_\gamma g(z))'}$$

for $z \in \mathcal{U}$. *Further, we define the subclass* $T\mathcal{SC_H}(b, \gamma, \lambda, n)$ *of* $\mathcal{SC_H}(b, \gamma, \lambda, n)$ *consisting of functions* $f = h + \bar{g}$ *of the form (1.2).*

We observe that for $b = 1$ the class was introduced and studied by first author with Öztürk [12], the class $\mathcal{SC}_{\mathcal{H}}(1, \gamma, 0, 0) = \mathcal{SC}_{\mathcal{H}}(\gamma)$ is given in [5, 6] and $\mathcal{SC}_{\mathcal{H}}(1, 0, 0, 0) = \mathcal{SC}_{\mathcal{H}}$ see [2].

In this paper, we investigate coefficient conditions, extreme points and distortion bounds for functions in the families $T\mathcal{SC}_{\mathcal{H}}(b, \gamma, \lambda, n)$. We also examine their convolution and convex combination properties and neighborhood result. Further, we obtain the closure property of this class under integral operator. We remark that the results so obtained for these general families can be viewed as extensions and generalizations for various subclasses of $\mathcal{S}_{\mathcal{H}}$ as listed previously in this section.

2 Main results

3 Coefficient inequalities

Our first theorem gives a sufficient condition for functions in $\mathcal{SC}_{\mathcal{H}}(b, \gamma, \lambda, n)$.

Theorem 3.1. *Let $f = h + \bar{g}$ be so that h and g are given by (1.1). If*

$$\sum_{k=1}^{\infty} \frac{(\frac{k+\gamma}{1+\gamma})^n k(k\lambda - \lambda + 1)[(k-1) + |b|(1-\gamma)]}{(1-\gamma)|b|} |a_k| + \sum_{k=1}^{\infty} \frac{(\frac{k-\gamma}{1+\gamma})^n k(k\lambda + \lambda - 1)[(k+1) - |b|(1-\gamma)]}{(1-\gamma)|b|} |b_k| \leq 2, \tag{3.13}$$

where $a_1 = 1,\ 0 \leq \gamma < 1,\ 0 \leq \lambda \leq \frac{\gamma}{1+\gamma}$ or $\lambda \geq \frac{1}{1+\gamma}$. Then $f \in \mathcal{SC}_{\mathcal{H}}(b, \gamma, \lambda, n)$ and f is sense preserving, univalent harmonic in $\mathcal{U}$.

Proof. We show that $f \in \mathcal{SC}_{\mathcal{H}}(b, \gamma, \lambda, n)$. We only need to show that if (3.13) holds then the condition (1.12) is satisfied. In view of (1.1) the condition (1.12) takes the form

$$\Re\left(\frac{(1-\gamma) + \sum_{k=2}^{\infty} \frac{(\frac{k+\gamma}{1+\gamma})^n k(k\lambda-\lambda+1)[(k-1)+b(1-\gamma)]}{b} |a_k| \frac{z^k}{z} - \sum_{k=1}^{\infty} \frac{(\frac{k-\gamma}{1+\gamma})^n k(k\lambda+\lambda-1)[(k+1)-b(1-\gamma)]}{b} |b_k| \frac{\bar{z}^k}{z}}{1 + \sum_{k=2}^{\infty} (\frac{k+\gamma}{1+\gamma})^n k(k\lambda - \lambda + 1)|a_k| \frac{z^k}{z} + \sum_{k=1}^{\infty} (\frac{k-\gamma}{1+\gamma})^n k(k\lambda + \lambda - 1)|b_k| \frac{\bar{z}^k}{z}}\right) = \Re\frac{1 + A(z)}{1 + B(z)}.$$

Setting

$$\frac{1 + A(z)}{1 + B(z)} = \frac{1 + \omega(z)}{1 - \omega(z)},$$

we will have $\Re\frac{1+A(z)}{1+B(z)} > 0$ if $|\omega(z)| < 1$,

$$\begin{aligned}
\omega(z) &= \frac{A(z) - B(z)}{2 + A(z) + B(z)} \\
&= \frac{-\gamma + \sum_{k=2}^{\infty} (\frac{k+\gamma}{1+\gamma})^n k(k\lambda - \lambda + 1) \left[\frac{[(k-1)+b(1-\gamma)]}{b} - 1\right] |a_k| z^{k-1} - \sum_{k=1}^{\infty} (\frac{k-\gamma}{1+\gamma})^n k(k\lambda + \lambda - 1) \left[\frac{[(k+1)-b(1-\gamma)}{b} + 1\right] |b_k| \frac{\bar{z}^k}{z}}{2 - \gamma + \sum_{k=2}^{\infty} (\frac{k+\gamma}{1+\gamma})^n k(k\lambda - \lambda + 1) \left[\frac{[(k-1)+b(1-\gamma)]}{b} + 1\right] |a_k| z^{k-1} - \sum_{k=1}^{\infty} (\frac{k-\gamma}{1+\gamma})^n k(k\lambda + \lambda - 1) \left[\frac{[(k+1)-b(1-\gamma)}{b} - 1\right] |b_k| \frac{\bar{z}^k}{z}}
\end{aligned}$$

This last expression is bounded above by 1 if and only if

$$\sum_{k=2}^{\infty} \frac{k(\frac{k+\gamma}{1+\gamma})^n (k\lambda - \lambda + 1)[(k-1) + |b|(1-\gamma)]}{|b|} |a_k| + \sum_{k=1}^{\infty} \frac{k(\frac{k+\gamma}{1+\gamma})^n (k\lambda + \lambda - 1)[(k+1) - |b|(1-\gamma)]}{|b|} |b_k| \leq (1-\gamma).$$

Or, equivalently

$$\sum_{k=1}^{\infty}\frac{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)[(k-1)+|b|(1-\gamma)]}{(1-\gamma)|b|}|a_k| + \sum_{k=1}^{\infty}\frac{k(\frac{k-\gamma}{1+\gamma})^n(k\lambda+\lambda-1)[(k+1)-|b|(1-\gamma)]}{(1-\gamma)|b|}|b_k| \leq 2$$

If $z_1 \neq z_2$, then for $\lambda \geq \frac{1}{1+\gamma}$ or $0 \leq \lambda \leq \frac{\gamma}{1+\gamma}$

$$\begin{aligned}\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| &\geq 1-\left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right| = 1-\left|\frac{\sum_{k=1}^{\infty} b_k(z_1^k-z_2^k)}{(z_1-z_2)+\sum_{k=2}^{\infty} a_k(z_1^k-z_2^k)}\right| \\ &\geq 1-\frac{\sum_{k=1}^{\infty} k|b_k|}{1-\sum_{k=2}^{\infty} k|a_k|} \geq 1-\frac{\sum_{k=1}^{\infty}\frac{k(\frac{k-\gamma}{1+\gamma})^n(k\lambda+\lambda-1)[(k+1)-|b|(1-\gamma)]}{(1-\gamma)|b|}|b_k|}{1-\sum_{k=2}^{\infty}\frac{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)[(k-1)+|b|(1-\gamma)]}{(1-\gamma)|b|}|a_k|} \\ &\geq 0,\end{aligned}$$

which proves univalence. Note that f is sense preserving in $\mathcal{U}$, for $0 \leq \lambda \leq \frac{\gamma}{1+\gamma}$ or $\lambda \geq \frac{1}{1+\gamma}$. This is because

$$\begin{aligned}|h'(z)| &\geq 1-\sum_{k=2}^{\infty} k|a_k||z|^{k-1} > 1-\sum_{k=2}^{\infty} k|a_k| \\ &\geq 1-\sum_{k=2}^{\infty}\frac{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)[(k-1)+|b|(1-\gamma)]}{(1-\gamma)|b|}|a_k| \\ &\geq \sum_{k=1}^{\infty}\frac{k(\frac{k-\gamma}{1+\gamma})^n(k\lambda+\lambda-1)[(k+1)-|b|(1-\gamma)]}{(1-\gamma)|b|}|b_k| \\ &> \sum_{k=1}^{\infty}\frac{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda+\lambda-1)[(k+1)-|b|(1-\gamma)]}{(1-\gamma)|b|}|b_k||z|^{k-1} \\ &\geq \sum_{k=1}^{\infty} k(\frac{k+\gamma}{1+\gamma})^n|b_k||z|^{k-1} \geq |g'(z)|.\end{aligned}$$

The function

$$\begin{aligned}f(z) = z &+ \sum_{k=2}^{\infty}\frac{(1-\gamma)|b|}{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)[(k-1)+|b|(1-\gamma)]}x_kz^k \\ &+ \sum_{k=1}^{\infty}\frac{(1-\gamma)|b|}{k(\frac{k-\gamma}{1+\gamma})^n(k\lambda+\lambda-1)[(k+1)-|b|(1-\gamma)]}\overline{y_kz^k},\end{aligned} \tag{3.14}$$

where $\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k| = 1$, show that the coefficient bound given by (3.13) is sharp. The functions of the form (3.14) are in $\mathcal{SC}_{\mathcal{H}}(b,\gamma,\lambda,n)$ because

$$\begin{aligned}&\sum_{k=1}^{\infty}\left(\frac{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)[(k-1)+|b|(1-\gamma)]}{(1-\gamma)|b|}|a_k| + \frac{k(\frac{k-\gamma}{1+\gamma})^n(k\lambda+\lambda-1)[(k+1)-|b|(1-\gamma)]}{(1-\gamma)|b|}|b_k|\right) \\ &= 1+\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k| = 2.\end{aligned}$$

□

Theorem 3.2. *Let $f = h+\bar{g}$ be so that h and g are given by (1.2). Then $f \in \mathcal{TSC}_{\mathcal{H}}(b,\gamma,\lambda,n)$ if and only if*

$$\sum_{k=1}^{\infty}\frac{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)[(k-1)+|b|(1-\gamma)]}{(1-\gamma)|b|}|a_k| + \sum_{k=1}^{\infty}\frac{k(\frac{k-\gamma}{1+\gamma})^n(k\lambda+\lambda-1)[(k+1)-|b|(1-\gamma)]}{(1-\gamma)|b|}|b_k| \leq 2, \tag{3.15}$$

where $a_1 = 1,\ 0 \leq \gamma < 1,\ 0 \leq \lambda \leq \frac{\gamma}{1+\gamma}$ or $\lambda \geq \frac{1}{1+\gamma}$ and $b \in \mathbb{C}\setminus\{0\}$.

Proof. The 'if part' follows from Theorem 3.1 upon noting that the functions $\mathcal{TSC}_{\mathcal{H}}(b,\gamma,\lambda,n) \subset \mathcal{SC}_{\mathcal{H}}(b,\gamma,\lambda,n)$. For the 'only if' part, we show that $f \in \mathcal{TSC}_{\mathcal{H}}(b,\gamma,\lambda,n)$. Then for $z = re^{i\theta}$ in $\mathcal{U}$ we obtain

$$\begin{aligned}
&\Re\left(1+\frac{1}{b}\left(\frac{\mathcal{F}(z)}{\mathcal{G}(z)}-1\right)-\gamma\right)\\
=\ &\Re\left(\frac{(1-\gamma)z-\sum\limits_{k=2}^{\infty}\frac{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)[(k-1)+b(1-\gamma)]}{b}|a_k|z^k-\sum\limits_{k=1}^{\infty}\frac{k(\frac{k-\gamma}{1+\gamma})^n(k\lambda+\lambda-1)[(k+1)-b(1-\gamma)]}{b}|b_k|\overline{z}^k}{z-\sum\limits_{k=2}^{\infty}k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)|a_k|z^k+\sum\limits_{k=1}^{\infty}k(\frac{k-\gamma}{1+\gamma})^n(k\lambda+\lambda-1)|b_k|\overline{z}^k}\right)\\
\geq\ &\frac{(1-\gamma)-\sum\limits_{k=2}^{\infty}\frac{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)[(k-1)+b(1-\gamma)]}{b}|a_k|r^{k-1}-\sum\limits_{k=1}^{\infty}\frac{k(\frac{k-\gamma}{1+\gamma})^n(k\lambda+\lambda-1)[(k+1)-b(1-\gamma)]}{b}|b_k|r^{k-1}}{z-\sum\limits_{k=2}^{\infty}k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)|a_k|r^{k-1}+\sum\limits_{k=1}^{\infty}k(\frac{k-\gamma}{1+\gamma})^n(k\lambda+\lambda-1)|b_k|r^{k-1}}>0,
\end{aligned}$$

The above inequality must hold for all $z \in \mathcal{U}$. In particular, letting $z = r \to 1^-$ yields the required condition. □

As special cases of Theorem 3.2, we obtain the following two corollaries.

Corollary 3.1. *Let $f = h + \bar{g} \in \mathcal{TSC}_{\mathcal{H}}(b,\gamma,0,n)$ if and only if*

$$\sum_{k=1}^{\infty}\frac{n(\frac{k+\gamma}{1+\gamma})^n[(k-1)+|b|(1-\gamma)]}{(1-\gamma)|b|}|a_k|+\sum_{k=1}^{\infty}\frac{n(\frac{k-\gamma}{1+\gamma})^n[(k+1)-|b|(1-\gamma)]}{(1-\gamma)|b|}|b_k|\leq 2.$$

Corollary 3.2. *Let $f = h + \bar{g} \in \mathcal{TSC}_{\mathcal{H}}(b,\gamma,1,n)$ if and only if*

$$\sum_{k=1}^{\infty}\frac{n^2(\frac{k+\gamma}{1+\gamma})^n[(k-1)+|b|(1-\gamma)]}{(1-\gamma)|b|}|a_k|+\sum_{k=1}^{\infty}\frac{n^2(\frac{k-\gamma}{1+\gamma})^n[(k+1)-|b|(1-\gamma)]}{(1-\gamma)|b|}|b_k|\leq 2.$$

4 Extreme points and Distortion bounds

In this section, our first theorem gives the extreme points of the closed convex hulls of $\mathcal{TSC}_{\mathcal{H}}(b,\gamma,\lambda,n)$.

Theorem 4.3. *Let $f = h + \bar{g} \in \mathcal{TSC}_{\mathcal{H}}(b,\gamma,\lambda,n)$ if and only if f can be expressed as*

$$f(z)=\sum_{k=1}^{\infty}(X_k h_k(z)+Y_k g_k(z)),\ \ z\in\mathcal{U}, \tag{4.16}$$

where $h_1(z) = z$,

$$h_k(z)=z-\frac{(1-\gamma)|b|}{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)[(k-1)+|b|(1-\gamma)]}z^k\quad(k=2,3,...)$$

and

$$g_k(z)=z+\frac{(1-\gamma)|b|}{k(\frac{k-\gamma}{1+\gamma})^n(k\lambda+\lambda-1)[(k+1)-|b|(1-\gamma)]}\overline{z}^k\quad(k=1,2,3,...),$$

$$\sum_{k=1}^{\infty}(X_k+Y_k)=1,\ X_k\geq 0,\ Y_k\geq 0.$$

In particular, the extreme points of $\mathcal{TSC}_{\mathcal{H}}(b,\gamma,\lambda,n)$ are $\{h_k\}$ and $\{g_k\}$.

Proof. For functions f of the form (4.16), we have

$$\begin{aligned}
f(z) &= \sum_{k=1}^{\infty}(X_k h_k(z)+Y_k g_k(z))\\
&= \sum_{k=1}^{\infty}(X_k+Y_k)\,z-\sum_{k=2}^{\infty}\frac{(1-\gamma)|b|}{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)[(k-1)+|b|(1-\gamma)]}X_k z^k\\
&\quad+\sum_{k=1}^{\infty}\frac{(1-\gamma)|b|}{k(\frac{k-\gamma}{1+\gamma})^n(k\lambda+\lambda-1)[(k+1)-|b|(1-\gamma)]}Y_k\overline{z}^k.
\end{aligned}$$

Then

$$\sum_{k=2}^{\infty} \frac{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)[(k-1)+|b|(1-\gamma)]}{(1-\gamma)|b|} \left(\frac{(1-\gamma)|b|}{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)[(k-1)+|b|(1-\gamma)]} \right) X_k$$
$$+ \sum_{k=1}^{\infty} \frac{k(\frac{k-\gamma}{1+\gamma})^n(k\lambda+\lambda-1)[(k+1)-|b|(1-\gamma)]}{(1-\gamma)|b|} \left(\frac{(1-\gamma)|b|}{k(\frac{k-\gamma}{1+\gamma})^n(k\lambda+\lambda-1)[(k+1)-|b|(1-\gamma)]} \right) Y_k$$
$$= \sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k = 1 - X_1 \le 1$$

and so $f \in clcoT\mathcal{SC}_{\mathcal{H}}(b,\gamma,\lambda)$.

Conversely, suppose that $f \in clcoT\mathcal{SC}_{\mathcal{H}}(b,\gamma,\lambda,n)$. Letting

$$X_1 = 1 - \sum_{k=2}^{\infty} X_k - \sum_{k=1}^{\infty} Y_k$$

where

$$X_k = \frac{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)[(k-1)+|b|(1-\gamma)]}{(1-\gamma)|b|}|a_k|, \ k = 2,3,\ldots,$$

and

$$Y_k = \frac{k(\frac{k-\gamma}{1+\gamma})^n(k\lambda+\lambda-1)[(k+1)-|b|(1-\gamma)]}{(1-\gamma)|b|}|b_k|, \ k = 1,2,\ldots,$$

we obtain the require representation, since

$$\begin{aligned}
f(z) &= z - \sum_{k=2}^{\infty} |a_k| z^k + \sum_{k=1}^{\infty} |b_k| \bar{z}^k \\
&= z - \sum_{k=2}^{\infty} \frac{(1-\gamma)|b|X_k}{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)[(k-1)+|b|(1-\gamma)]} z^k \\
&+ \sum_{k=1}^{\infty} \frac{(1-\gamma)|b|Y_k}{k(\frac{k-\gamma}{1+\gamma})^n(k\lambda+\lambda-1)[(k+1)-|b|(1-\gamma)]} \bar{z}^k \\
&= z - \sum_{k=2}^{\infty} (z-h_k(z))X_k - \sum_{k=1}^{\infty} (z-g_k(z))Y_k \\
&= \left(1 - \sum_{k=2}^{\infty} X_k - \sum_{k=1}^{\infty} Y_k\right) z + \sum_{k=2}^{\infty} h_k(z)X_k + \sum_{k=1}^{\infty} g_k(z)Y_k \\
&= \sum_{k=1}^{\infty} (X_k h_k(z) + Y_k g_k(z)).
\end{aligned}$$

□

The following theorem gives the distortion bounds for functions in $T\mathcal{SC}_{\mathcal{H}}(b,\gamma,\lambda,n)$ which yields a covering result for this family.

Theorem 4.4. *Let $f \in T\mathcal{SC}_{\mathcal{H}}(b,\gamma,\lambda,n)$ then*

$$|f(z)| \le (1+|b_1|)r + r^2 \left(\frac{(1-\gamma)|b|}{2(\lambda+1)[1+b(1-\gamma)]} - \frac{(2\lambda-1)(\frac{2+\gamma}{1+\gamma})^n[2-b(1-\gamma)]}{2(\lambda+1)[1+b(1-\gamma)]}|b_1| \right)$$

and

$$|f(z)| \ge (1-|b_1|)r - r^2 \left(\frac{(1-\gamma)|b|}{2(\lambda+1)[1+b(1-\gamma)]} - \frac{(2\lambda-1)(\frac{2+\gamma}{1+\gamma})^n[2-b(1-\gamma)]}{2(\lambda+1)[1+b(1-\gamma)]}|b_1| \right).$$

Proof. Let $f \in \mathcal{TSC_H}(b,\gamma,\lambda,n)$, Taking the absolute value of f and then by Theorem3.14, we obtain

$$\begin{aligned}
|f(z)| &\leq (1+|b_1|)r + \sum_{k=2}^{\infty}(|a_k|+|b_k|)r^k \\
&\leq (1+|b_1|)r + r^2\sum_{k=2}^{\infty}(|a_k|+|b_k|) \\
&\leq (1+|b_1|)r + \frac{(1-\gamma)|b|}{2(\lambda+1)[1+b(1-\gamma)]}\left(\sum_{k=2}^{\infty}\frac{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)[(k-1)+|b|(1-\gamma)]}{(1-\gamma)|b|}|a_k|\right. \\
&\qquad \left.+\frac{k(\frac{k-\gamma}{1+\gamma})^n(k\lambda+\lambda-1)[(k+1)-|b|(1-\gamma)]}{(1-\gamma)|b|}|b_k|\right)r^2 \\
&\leq (1+|b_1|)r + \frac{(1-\gamma)|b|}{2(\lambda+1)[1+b(1-\gamma)]}\left(1-\frac{(2\lambda-1)(\frac{2+\gamma}{1+\gamma})^n[2-b(1-\gamma)]}{(1-\gamma)|b|}|b_1|\right)r^2 \\
&= (1+|b_1|)r + \left(\frac{(1-\gamma)|b|}{2(\lambda+1)[1+b(1-\gamma)]}-\frac{(2\lambda-1)(\frac{2+\gamma}{1+\gamma})^n[2-b(1-\gamma)]}{2(\lambda+1)[1+b(1-\gamma)]}|b_1|\right)r^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
|f(z)| &\geq (1-|b_1|)r - \sum_{k=2}^{\infty}(|a_k|+|b_k|)r^k \\
&\geq (1-|b_1|)r - r^2\sum_{k=2}^{\infty}(|a_k|+|b_k|) \\
&\leq (1-|b_1|)r - \frac{(1-\gamma)|b|}{2(\lambda+1)[1+b(1-\gamma)]}\left(\sum_{k=2}^{\infty}\frac{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)[(k-1)+|b|(1-\gamma)]}{(1-\gamma)|b|}|a_k|\right. \\
&\qquad \left.+\frac{k(\frac{k-\gamma}{1+\gamma})^n(k\lambda+\lambda-1)[(k+1)-|b|(1-\gamma)]}{(1-\gamma)|b|}|b_k|\right)r^2 \\
&\geq (1-|b_1|)r - \frac{(1-\gamma)|b|}{2(\lambda+1)[1+b(1-\gamma)]}\left(1-\frac{(2\lambda-1)(\frac{2+\gamma}{1+\gamma})^n[2-b(1-\gamma)]}{(1-\gamma)|b|}|b_1|\right)r^2 \\
&= (1-|b_1|)r - \left(\frac{(1-\gamma)|b|}{2(\lambda+1)[1+b(1-\gamma)]}-\frac{(2\lambda-1)(\frac{2+\gamma}{1+\gamma})^n[2-b(1-\gamma)]}{2(\lambda+1)[1+b(1-\gamma)]}|b_1|\right)r^2.
\end{aligned}$$

The upper and lower bounds given in Theorem 4.4 are respectively attained for the following functions.

$$f(z) = z + |b_1|\bar{z} + \frac{1}{\Gamma(2)}\left(\frac{(1-\gamma)|b|}{2(\lambda+1)[1+b(1-\gamma)]}-\frac{(2\lambda-1)(\frac{2+\gamma}{1+\gamma})^n[2-b(1-\gamma)]}{2(\lambda+1)[1+b(1-\gamma)]}|b_1|\right)\bar{z}^2$$

and

$$f(z) = (1-|b_1|)z - \frac{1}{\Gamma(2)}\left(\frac{(1-\gamma)|b|}{2(\lambda+1)[1+b(1-\gamma)]}-\frac{(2\lambda-1)(\frac{2+\gamma}{1+\gamma})^n[2-b(1-\gamma)]}{2(\lambda+1)[1+b(1-\gamma)]}|b_1|\right)z^2,$$

□

The following covering result follows from the left hand inequality in Theorem 4.4.

Corollary 4.3. *If $f \in \mathcal{TSC_H}(b,\gamma,\lambda,n)$, then*

$$\left\{\omega : |\omega| < 1 - \frac{(1-\gamma)|b|}{2(\lambda+1)[1+b(1-\gamma)]} - \left[1-\frac{(2\lambda-1)[2-b(1-\gamma)]}{2(\lambda+1)[1+b(1-\gamma)]}\right]|b_1|\right\}.$$

5 Convolution and Convex Combinations

In this section we show that the class $TSC_{\mathcal{H}}(b,\gamma,\lambda,n)$ is closed under convolution and convex combinations. Now we need the following definition of convolution of two harmonic functions. For $f(z) = z - \sum_{k=2}^{\infty}|a_k|z^k + \sum_{k=1}^{\infty}|b_k|\overline{z}^k$ and $F(z) = z - \sum_{k=2}^{\infty}|A_k|z^k + \sum_{k=1}^{\infty}|B_k|\overline{z}^k$, we define the convolution of two harmonic functions f and F as

$$(f*F)(z) = f(z)*F(z) = z - \sum_{k=2}^{\infty}|a_k||A_k|z^k + \sum_{k=1}^{\infty}|b_k||B_k|\overline{z}^k. \tag{5.17}$$

Using the definition, we show that the class $TSC_{\mathcal{H}}(b,\gamma,\lambda,n)$ is closed under convolution.

Theorem 5.5. *For $0 \le \delta < \gamma < 1$, let $f \in TSC_{\mathcal{H}}(b,\gamma,\lambda,n)$ and $F \in TSC_{\mathcal{H}}(b,\delta,\lambda,n)$. Then $f*F \in TSC_{\mathcal{H}}(b,\gamma,\lambda,n) \subset TSC_{\mathcal{H}}(b,\delta,\lambda,n)$.*

Proof. Let $f(z) = z - \sum_{k=2}^{\infty}|a_k|z^k + \sum_{k=1}^{\infty}|b_k|\overline{z}^k$ and $F(z) = z - \sum_{k=2}^{\infty}|A_k|z^k + \sum_{k=1}^{\infty}|B_k|\overline{z}^k$ be in $TSC_{\mathcal{H}}(b,\delta,\lambda)$. Then the convolution $f*F$ is given by (5.17). From the assertion that $f*F \in TSC_{\mathcal{H}}(b,\delta,\lambda)$, we note that $|A_k| \le 1$ and $|B_k| \le 1$. In view of Theorem 3.2 and the inequality $0 \le \delta \le \gamma < 1$, we have

$$\begin{aligned}
&\sum_{k=2}^{\infty}\frac{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)[(k-1)+|b|(1-\delta)]}{(1-\delta)|b|}|a_k||A_k| \\
&\qquad + \sum_{k=1}^{\infty}\frac{k(\frac{k-\gamma}{1+\gamma})^n(k\lambda+\lambda-1)[(k+1)-|b|(1-\delta)]}{(1-\delta)|b|}|b_k||B_k| \\
&\le \sum_{k=2}^{\infty}\frac{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)[(k-1)+|b|(1-\delta)]}{(1-\delta)|b|}|a_k| \\
&\qquad + \sum_{k=1}^{\infty}\frac{k(\frac{k-\gamma}{1+\gamma})^n(k\lambda+\lambda-1)[(k+1)-|b|(1-\delta)]}{(1-\delta)|b|}|b_k| \\
&\le \sum_{k=2}^{\infty}\frac{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)[(k-1)+|b|(1-\gamma)]}{(1-\gamma)|b|}|a_k| \\
&\qquad + \sum_{k=1}^{\infty}\frac{k(\frac{k-\gamma}{1+\gamma})^n(k\lambda+\lambda-1)[(k+1)-|b|(1-\gamma)]}{(1-\gamma)|b|}|b_k| \\
&\le 1.
\end{aligned}$$

by Theorem 3.2, $f \in TSC_{\mathcal{H}}(b,\gamma,\lambda)$. By the same token, we then conclude that $f*F \in TSC_{\mathcal{H}}(b,\gamma,\lambda,n) \subset TSC_{\mathcal{H}}(b,\delta,\lambda,n)$. □

Next, we show that the class $TSC_{\mathcal{H}}(b,\gamma,\lambda,n)$ is closed under convex combination of its members.

Theorem 5.6. *The class $TSC_{\mathcal{H}}(b,\gamma,\lambda,n)$ is closed under convex combinations.*

Proof. For i=1,2,3,.... Suppose that $f_i(z) \in TSC_{\mathcal{H}}(b,\gamma,\lambda,n)$ where f_i given by

$$f_i(z) = z - \sum_{k=2}^{\infty}|a_{i,k}|z^k + \sum_{k=1}^{\infty}|b_{i,k}|\overline{z}^k.$$

For $\sum_{i=1}^{\infty} t_i = 1, 0 \le t_i \le 1$, the convex combinations of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{k=2}^{\infty}\left(\sum_{i=1}^{\infty} t_i|a_{i,k}|\right)z^k + \sum_{k=1}^{\infty}\left(\sum_{i=1}^{\infty} t_i|b_{i,k}|\right)\overline{z}^k.$$

Since,

$$\sum_{k=2}^{\infty} \frac{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)[(k-1)+|b|(1-\gamma)]}{(1-\gamma)|b|}|a_{i,k}| + \sum_{k=1}^{\infty} \frac{k(\frac{k-\gamma}{1+\gamma})^n(k\lambda+\lambda-1)[(k+1)-|b|(1-\gamma)]}{(1-\gamma)|b|}|b_{i,k}| \leq 1.$$

from the above equation we obtain

$$\begin{aligned}
&\sum_{k=2}^{\infty} \frac{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)[(k-1)+|b|(1-\gamma)]}{(1-\gamma)|b|}\sum_{i=1}^{\infty} t_i|a_{i,k}| \\
&\qquad + \sum_{k=1}^{\infty} \frac{k(\frac{k-\gamma}{1+\gamma})^n(k\lambda+\lambda-1)[(k+1)-|b|(1-\gamma)]}{(1-\gamma)|b|}\sum_{i=1}^{\infty} t_i|b_{i,k}| \\
&= \sum_{i=1}^{\infty} t_i \left\{ \sum_{k=2}^{\infty} \frac{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)[(k-1)+|b|(1-\gamma)]}{(1-\gamma)|b|}|a_{i,k}| \right. \\
&\qquad \left. + \sum_{k=1}^{\infty} \frac{k(\frac{k-\gamma}{1+\gamma})^n(k\lambda+\lambda-1)[(k+1)-|b|(1-\gamma)]}{(1-\gamma)|b|}|b_{i,k}| \right\} \\
&\leq \sum_{i=1}^{\infty} t_i = 1
\end{aligned}$$

This is the condition required by (3.14) and so $\sum_{i=1}^{\infty} t_i f_i(z) \in T\mathcal{SC}_{\mathcal{H}}(b,\gamma,\lambda,n)$. □

6 Class Preserving Integral Operator

In this section, we consider the closure property of the class $T\mathcal{SC}_{\mathcal{H}}(b,\gamma,\lambda,n)$ under the Bernardi integral operator $\mathcal{L}_c[f(z)]$ which is defined by

$$\mathcal{L}_c[f(z)] = \frac{c+1}{z^c}\int_0^z \xi^{c-1} f(\xi)d\xi \ (c > -1).$$

Theorem 6.7. *Let $f(z) \in T\mathcal{SC}_{\mathcal{H}}(b,\gamma,\lambda,n)$, then $\mathcal{L}_c[f(z)] \in T\mathcal{SC}_{\mathcal{H}}(b,\gamma,\lambda,n)$.*

Proof. From the representation of $\mathcal{L}_c[f(z)]$, if follows that

$$\begin{aligned}
\mathcal{L}_c[f(z)] &= \frac{c+1}{z^c}\int_0^z \xi^{c-1}h(\xi)d\xi + \overline{\frac{c+1}{z^c}\int_0^z \xi^{c-1}g(\xi)d\xi} \\
&= \frac{c+1}{z^c}\int_0^z \xi^{c-1}\left(\xi - \sum_{k=2}^{\infty}|a_k|\xi^k\right)d\xi + \overline{\frac{c+1}{z^c}\int_0^z \xi^{c-1}\left(\sum_{k=1}^{\infty}|b_k|\xi^k\right)d\xi} \\
&= z - \sum_{k=2}^{\infty} A_k z^k + \sum_{k=1}^{\infty} B_k z^k,
\end{aligned}$$

where $A_k = \frac{c+1}{c+k}\,|a_k|$ and $B_k = \frac{c+1}{c+k}\,|b_k|$. Hence

$$\sum_{k=2}^{\infty} \frac{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)[(k-1)+|b|(1-\gamma)]}{(1-\gamma)|b|}\left(\frac{c+1}{c+k}|a_k|\right)$$
$$+\sum_{k=1}^{\infty} \frac{k(\frac{k-\gamma}{1+\gamma})^n|k\lambda+\lambda-1|[(k+1)-|b|(1-\gamma)]}{(1-\gamma)|b|}\left(\frac{c+1}{c+k}|b_k|\right)$$
$$\leq \sum_{k=2}^{\infty} \frac{k(\frac{k+\gamma}{1+\gamma})^n(k\lambda-\lambda+1)[(k-1)+|b|(1-\gamma)]}{(1-\gamma)|b|}|a_k|$$
$$+\sum_{k=1}^{\infty} \frac{k(\frac{k-\gamma}{1+\gamma})^n|k\lambda+\lambda-1|[(k+1)-|b|(1-\gamma)]}{(1-\gamma)|b|}|b_k| \leq 1,$$

since $f \in T\mathcal{SC}_{\mathcal{H}}(b,\gamma,\lambda)$, therefore by Theorem 3.2, $\mathcal{L}_c[f(z)] \in T\mathcal{SC}_{\mathcal{H}}(b,\gamma,\lambda)$. □

Remark 6.1. *Specializing the parameter, the result discussed in this paper leads many subclasses discussed in [4, 5, 7, 11–13].*

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