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The outer-independent edge-vertex domination in trees

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Abstract. Let $G = (V, E)$ be a finite simple graph. A vertex $v \in V$ is edge-vertex dominated by an edge $e \in E$ if e is incident with v or e is incident with a vertex adjacent to v. An edge-vertex dominating set of G is a subset $D \subseteq E$ such that every vertex of G is edge-vertex dominated by an edge of D. A subset $D \subseteq E$ is called an *outer-independent edge-vertex dominating set* of G if D is an edge-vertex dominating set of G and the set $V(G) \setminus I(D)$ is independent, where $I(D)$ is the set of vertices incident to an edge of D. The *outer-independent edge-vertex domination number* of G, denoted by $\gamma_{ev}^{oi}(G)$, is the smallest cardinality of an outer-connected edge-vertex dominating set of G. In this paper, we study outer-independent edge-vertex domination numbers. In particular, we prove that $\frac{n-l+1}{3} \leq \gamma_{ev}^{oi}(T) \leq \frac{2n-s-2}{3}$ for every tree T of order $n \geq 3$ with l leaves and s support vertices. We also characterize the trees attaining the bounds.

AMS Subject Classifications: 05C69.

Keywords: edge-vertex dominating set, outer independent edge-vertex dominating set.

Contents

1. Introduction and Terminology

Let $G = (V, E)$ be a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. The cardinality of V is called the *order* of G. The set $N(v) = \{u \in V(G) \mid uv \in E(G)\}$ is called the *open neighborhood* of $v \in V(G)$. The *degree* of $v \in V(G)$ is the cardinality of $N(v)$. We denote it by $deg_G(v)$. The *distance* between two distinct vertices in G is the length of a shortest path between them. The *diameter* of G is denoted by $diam(G)$. A *diametral path* of G is a path with the length which equals $diam(G)$.

Let T be a tree. A vertex v of T is called *leaf* if $deg_T(v) = 1$. A *support vertex* is a vertex adjacent to a leaf. A *weak support vertex* is a support vertex that is adjacent to exactly one leaf. A *rooted tree* T differentiates one vertex r called the root. For a vertex $v(\neq r) \in V(T)$, the parent of v is the neighbor of v placed on the unique (r, v)-path, while a child of v is any other neighbor of v. We denote the set of children of v by $C(v)$. A descendant of v is a vertex $w \neq v$ such that v is contained in the unique (r, w)-path. In particular, every child of v is also a descendant of v. We denote the set of descendants of v by $D(v)$. The subtree induced by $D(v) \cup \{v\}$ is denoted by T_v. The *star* is a complete bipartite graph $K_{1,t}$. The *double star* is the graph obtained by joining the centers of two stars $K_{1,p}$ and $K_{1,q}$. Subdividing an edge e is to delete e, add a new vertex x, and join x to the ends of e. A *healthy spider* $S_{t,t}$ is the graph obtained from a star $K_{1,t}$ by subdividing each edges of $K_{1,t}$. For a subset $S \subseteq V(G)$, $G - S$ denotes the subgraph of G induced by $V(G) \setminus S$.

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A vertex $v \in V(G)$ is edge-vertex dominated by an edge $e \in E(G)$ if e is incident with v or e is incident with a vertex adjacent to v (See [2]). An edge-vertex dominating set of G is a subset $D \subseteq E(G)$ such that every vertex of G is edge-vertex dominated by an edge of D (See [2]). A subset $D \subseteq E(G)$ is called an *outer-independent edge-vertex dominating set* (OIEVDS) of G if D is an edge-vertex dominating set of G and the set $V(G) \setminus I(D)$ is independent, where $I(D)$ is the set of vertices incident to an edge of D. The *outer-independent edge-vertex domination number* of G, denoted by $\gamma_{ev}^{oi}(G)$, is the smallest cardinality of an outer-connected edge-vertex dominating set of G. A $\gamma_{ev}^{oi}(G)$-set is an OIEVDS of G with the cardinality $\gamma_{ev}^{oi}(G)$.

Edge-vertex domination in graphs was introduced and studied in [2, 4]. Recently, variations of outer-independent and edge-vertex domination were given in [1, 5, 6]. In this paper, we study outer-independent edge-vertex domination numbers. We prove that $\frac{n-l+1}{3} \leq \gamma_{ev}^{oi}(T) \leq \frac{2n-s-2}{3}$ for every tree T of order $n \geq 3$ with l leaves and s support vertices. We also characterize the trees attaining the bounds.

Finally, we give a lemma whose proof follows from straightforward observation.

Lemma 1.1. *The following holds.*

1. *Every support vertex of T is incident to an edge of every $\gamma_{ev}^{oi}(T)$-set.*
2. *For every tree T with diameter at least three, there exists a $\gamma_{ev}^{oi}(T)$-set whose elements are not incident to any leaf.*

2. Main Result 1

In this section, we prove that if T is a tree of order $n \geq 3$ with l leaves, then $\frac{n-l+1}{3} \leq \gamma_{ev}^{oi}(T)$. We also give a characterization of all trees with $\frac{n-l+1}{3} = \gamma_{ev}^{oi}(T)$.

First of all, we introduce a family $\mathcal{T}$ of trees that be obtained from $T_1, \ldots, T_m$ $(m \geq 1)$ of trees such that T_1 is a path P_4 with two support vertices u, v, and let $S(T_1) = \{uv\}$. If $m \geq 2$, then T_{i+1} be obtained recursively from T_i by one of the following two operations for $1 \leq i \leq m-1$.

Operation $\mathcal{O}_1$:
(i) Attach a vertex by joining it to a vertex incident to edges of $S(T_i)$.
(ii) Set $S(T_{i+1}) = S(T_i)$.

Operation $\mathcal{O}_2$:
(i) Attach a path $P_3 := uvw$ by joining u to a leaf of T_i.
(ii) Set $S(T_{i+1}) = S(T_i) \cup \{uv\}$.

Proposition 2.1. *If a tree T belongs to $\mathcal{T}$, then $\gamma_{ev}^{oi}(T) = \frac{n-l+1}{3}$.*

Proof. We use the induction on the number of operations performed to construct the tree T. If $T = T_1 \cong P_4$, then $\gamma_{ev}^{oi}(T) = 1$. Let m be a positive integer. Suppose that every tree T' constructed by $m-1$ operations satisfies $\gamma_{ev}^{oi}(T') = \frac{n'-l'+1}{3}$. Let $T = T_{m+1}$ be a tree constructed by m operations.

First, we assume that T is obtained from T' by Operation $\mathcal{O}_1$. Then $n = n' + 1$ and $l = l' + 1$. It is easy to see that $S(T') = S(T)$ is an OIEVDS of T. Thus, $\gamma_{ev}^{oi}(T) = \gamma_{ev}^{oi}(T') = \frac{n'-l'+1}{3} = \frac{n-1-(l-1)+1}{3} = \frac{n-l+1}{3}$.

Second, we assume that T is obtained from T' by Operation $\mathcal{O}_2$. Then $n = n' + 3$ and $l = l'$. It is easy to see that $S(T) = S(T') \cup \{uv\}$ is an OIEVDS of T and $\gamma_{ev}^{oi}(T) = \gamma_{ev}^{oi}(T') + 1$ Thus, $\gamma_{ev}^{oi}(T) = \gamma_{ev}^{oi}(T') + 1 = \frac{n'-l'+1}{3} + 1 = \frac{n-3-l+1}{3} + 1 = \frac{n-l+1}{3}$. ■

Theorem 2.2. *Let T be a tree of order $n \geq 3$ with l leaves. Then $\frac{n-l+1}{3} \leq \gamma_{ev}^{oi}(T)$ with equality if and only if $T \in \mathcal{T}$.*

Proof. If $T = P_3$, then clearly $\frac{n-l+1}{3} = \frac{2}{3} < \gamma_{ev}^{oi}(T) = 1$. Assume that the order of T is at least 4. If T is a star, then $\frac{n-l+1}{3} = \frac{2}{3} < \gamma_{ev}^{oi}(T) = 1$. If T is a double star, then $\frac{n-l+1}{3} = 1 = \gamma_{ev}^{oi}(T)$. By using Operation $\mathcal{O}_1$ repeatably, we have $T \in \mathcal{T}$.

Now assume that $diam(T) \geq 4$. We use the induction on the order of T. Suppose that every tree T' of order $n'(< n)$ satisfies $\frac{n'-l'+1}{3} \leq \gamma_{ev}^{oi}(T')$ with equality only if $T' \in \mathcal{T}$.

Among all of diametrical paths in T, we choose $x_0x_1 \dots x_d$ so that it maximizes $deg_T(x_{d-1})$. Root T at x_0. We divide our consideration into four cases.

Case 1. $deg_T(x_{d-1}) \geq 3$.

Let $u\ (\neq x_d)$ be a leaf adjacent to x_{d-1}. Let $T' = T - \{u\}$. Then $n = n' + 1$ and $l = l' + 1$. It is easy to see that any $\gamma_{ev}^{oi}(T')$-set D is an OIEVDS of T. Applying the induction hypothesis to T', we have $\frac{n'-l'+1}{3} \leq \gamma_{ev}^{oi}(T')$. Thus, $\frac{n-l+1}{3} = \frac{n'-l'+1}{3} \leq \gamma_{ev}^{oi}(T') \leq \gamma_{ev}^{oi}(T)$. If $\frac{n-l+1}{3} = \gamma_{ev}^{oi}(T)$, then $\frac{n'-l'+1}{3} \leq \gamma_{ev}^{oi}(T')$ and $T' \in \mathcal{T}$. By Operation $\mathcal{O}_1$, we have $T \in \mathcal{T}$.

Case 2. $deg_T(x_{d-1}) = 2$ and $deg_T(x_{d-2}) \geq 3$.

Assume that there exists a support vertex $v \in C(x_{d-2}) \setminus \{x_{d-1}\}$. Let $T' = T - T_v$. Then $n = n' + 2$ and $l = l' + 1$. It is easy to see that $\gamma_{ev}^{oi}(T') \leq \gamma_{ev}^{oi}(T) - 1$. Applying the induction hypothesis to T', we have $\frac{n'-l'+1}{3} \leq \gamma_{ev}^{oi}(T')$. Thus, $\frac{n-2-(l-1)+1}{3} = \frac{n'-l'+1}{3} \leq \gamma_{ev}^{oi}(T') \leq \gamma_{ev}^{oi}(T) - 1$ and so $\frac{n-l+1}{3} < \gamma_{ev}^{oi}(T)$.

Assume that there exists a leaf $u \in C(x_{d-2})$. Let $T' = T - \{u\}$. Then $n = n' + 1$ and $l = l' + 1$. It is easy to see that $\gamma_{ev}^{oi}(T') = \gamma_{ev}^{oi}(T)$. Applying the induction hypothesis to T', we have $\frac{n'-l'+1}{3} \leq \gamma_{ev}^{oi}(T')$. Thus, $\frac{n-1-(l-1)+1}{3} = \frac{n'-l'+1}{3} \leq \gamma_{ev}^{oi}(T') \leq \gamma_{ev}^{oi}(T)$ and so $\frac{n-l+1}{3} \leq \gamma_{ev}^{oi}(T)$. If $\frac{n-l+1}{3} = \gamma_{ev}^{oi}(T)$, then $\frac{n'-l'+1}{3} \leq \gamma_{ev}^{oi}(T')$ and $T' \in \mathcal{T}$. By Operation $\mathcal{O}_1$, we have $T \in \mathcal{T}$.

Case 3. $deg_T(x_{d-1}) = 2$, $deg_T(x_{d-2}) = 2$ and $deg_T(x_{d-3}) \geq 3$.

Let $T' = T - T_{x_{d-2}}$. Then $n = n' + 3$ and $l = l' + 1$. It is easy to see that $\gamma_{ev}^{oi}(T') \leq \gamma_{ev}^{oi}(T) - 1$. Applying the induction hypothesis to T', we have $\frac{n'-l'+1}{3} \leq \gamma_{ev}^{oi}(T')$. Thus, $\frac{n-3-(l-1)+1}{3} = \frac{n'-l'+1}{3} \leq \gamma_{ev}^{oi}(T') \leq \gamma_{ev}^{oi}(T) - 1$ and so $\frac{n-l+1}{3} < \gamma_{ev}^{oi}(T)$.

Case 4. $deg_T(x_{d-1}) = 2$, $deg_T(x_{d-2}) = 2$ and $deg_T(x_{d-3}) = 2$.

Let $T' = T - T_{x_{d-2}}$. Then $n = n' + 3$ and $l = l'$. It is easy to see that $\gamma_{ev}^{oi}(T') \leq \gamma_{ev}^{oi}(T) - 1$. Applying the induction hypothesis to T', we have $\frac{n'-l'+1}{3} \leq \gamma_{ev}^{oi}(T')$. Thus, $\frac{n-3-l+1}{3} = \frac{n'-l'+1}{3} \leq \gamma_{ev}^{oi}(T') \leq \gamma_{ev}^{oi}(T) - 1$ and so $\frac{n-l+1}{3} \leq \gamma_{ev}^{oi}(T)$. If $\frac{n-l+1}{3} = \gamma_{ev}^{oi}(T)$, then $\frac{n'-l'+1}{3} \leq \gamma_{ev}^{oi}(T')$ and $T' \in \mathcal{T}$. By Operation $\mathcal{O}_2$, we have $T \in \mathcal{T}$. ■

3. Main Result 2

In this section, we prove that if T is a tree of order $n \geq 3$ with s support vertices, then $\gamma_{ev}^{oi}(T) \leq \frac{2n-s-2}{3}$. We also give a characterization of all trees with $\gamma_{ev}^{oi}(T) = \frac{2n-s-2}{3}$.

Theorem 3.1. *Let T be a tree of order $n \geq 3$ with s support vertices. Then $\gamma_{ev}^{oi}(T) \leq \frac{2n-s-2}{3}$ with equality if and only if T is a healthy spider.*

Proof. If $T = P_3$, then clearly $\gamma_{ev}^{oi}(T) = 1$ and T is a healthy spider. Assume that the order of T is at least 4. If T is a star, then $\gamma_{ev}^{oi}(T) = 1 < \frac{2n-s-2}{3}$. If T is a double star, then $\gamma_{ev}^{oi}(T) = 1 < \frac{2n-s-2}{3} = \frac{2n-4}{3}$.

Now assume that $diam(T) \geq 4$. We use the induction on the order of T. Suppose that every tree T' of order $n'(< n)$ satisfies $\gamma_{ev}^{oi}(T') \leq \frac{2n'-s'-2}{3}$ with equality only if T' is a healthy spider. Among all of diametrical paths in T, we choose $x_0x_1 \dots x_d$ so that it maximizes $deg_T(x_{d-1})$. Root T at x_0. We divide our consideration into three cases.

Case 1. $deg_T(x_{d-1}) \geq 3$.

Let $u\ (\neq x_d)$ be a leaf adjacent to x_{d-1}. Let $T' = T - \{u\}$. Then $n = n' + 1$ and $s = s'$. It is easy to see that any $\gamma_{ev}^{oi}(T')$-set is an OIEVDS of T. So, $\gamma_{ev}^{oi}(T) \leq \gamma_{ev}^{oi}(T')$. Applying the induction hypothesis to T', we have $\gamma_{ev}^{oi}(T') \leq \frac{2n'-s'-2}{3}$. Thus, $\gamma_{ev}^{oi}(T) \leq \gamma_{ev}^{oi}(T') \leq \frac{2n'-s'-2}{3} < \frac{2n-s-2}{3}$.

Case 2. $deg_T(x_{d-1}) = 2$ and $deg_T(x_{d-2}) = 2$.

Let $T' = T - \{x_{d-2}, x_{d-1}, x_d\}$. It is easy to see that $\gamma_{ev}^{oi}(T) = \gamma_{ev}^{oi}(T') + 1$, $s - 1 \leq s' \leq s$ and $n = n' + 3$. Applying the induction hypothesis to T', we have $\gamma_{ev}^{oi}(T') \leq \frac{2n'-s'-2}{3}$. Thus, $\gamma_{ev}^{oi}(T) - 1 = \gamma_{ev}^{oi}(T') \leq \frac{2n'-s'-2}{3} \leq \frac{2n-6-s+1-2}{3}$ and so $\gamma_{ev}^{oi}(T) < \frac{2n-s-2}{3}$.

Case 3. $deg_T(x_{d-1}) = 2$ and $deg_T(x_{d-2}) \geq 3$.

Assume that there exists a leaf $c \in C(x_{d-2})$. Let $T' = T - \{v\}$. By the argument as in Case 1, we have $\gamma_{ev}^{oi}(T) < \frac{2n-s-2}{3}$.

Assume that there exists a support vertex $c \in C(x_{d-2}) \setminus \{x_{d-1}\}$. By the assumption, c is weak and has a child w. Let $T' = T - T_c$. It is easy to see that $\gamma_{ev}^{oi}(T) = \gamma_{ev}^{oi}(T') + 1$, $s = s' + 1$ and $n = n' + 2$. Applying the induction hypothesis to T', we have $\gamma_{ev}^{oi}(T') \leq \frac{2n'-s'-2}{3}$. Thus, $\gamma_{ev}^{oi}(T) - 1 = \gamma_{ev}^{oi}(T') \leq \frac{2n'-s'-2}{3} = \frac{2n-4-s+1-2}{3}$ and so $\gamma_{ev}^{oi}(T) \leq \frac{2n-s-2}{3}$.

Now we assume that $T_{x_{d-2}}$ is a healthy spider $S_{t,t}$. Let $T' = T - V(T_{x_{d-2}})$. It is easy to see that $\gamma_{ev}^{oi}(T) \leq \gamma_{ev}^{oi}(T') + t$, $s - t \leq s'$ and $n = n' + 2t + 1$. If $|V(T')| \geq 3$, then by the induction hypothesis on T', we have $\gamma_{ev}^{oi}(T') \leq \frac{2n'-s'-2}{3}$. Thus, $\gamma_{ev}^{oi}(T) - t \leq \gamma_{ev}^{oi}(T') \leq \frac{2n'-s'-2}{3} = \frac{2n-4t-2-s+t-2}{3}$ and so $\gamma_{ev}^{oi}(T) < \frac{2n-s-2}{3}$. If $|V(T')| = 2$, then clearly $\gamma_{ev}^{oi}(T) = \frac{2n-s-2}{3}$ and T is a healthy spider. ■

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