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A Note on Global Bipartite Domination in Graphs

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Abstract

In this paper we introduce the concept of the *global bipartite domination number* $\gamma_{gb}(G)$ of a connected bipartite graph G and study some of its general properties. Moreover we determine the global bipartite domination number of certain classes of graphs.

Keywords: Domination, global bipartite domination, global bipartite domination number.

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1 Introduction

In this paper we consider simple, connected and bipartite graphs. All notations and definitions not given here can be found in [1, 3]. A *graph* is an ordered pair $G = (V(G), E(G))$, where $V(G)$ is a finite nonempty set and $E(G)$ is a collection of 2- point subsets of V. The sets $V(G)$ and $E(G)$ are the vertex set and edge set of G respectively. The *degree* of a vertex v in G is the number of edges incident at v. The set of all neighbors of v is the *open neighborhood* of v, denoted by $N(v)$. Let P_n, C_n, K_n and $K_{m,n}$denote path, cycle, complete graph and complete bipartite graph respectively. The sudivision of the graph G is the graph $S(G)$ obtained from G by subdividing each edge of G. The corona $G \circ K_1$ of G is the graph obtained from G by adding a pendant edge to each vertex of G. A set $A \subseteq V(G)$ of vertices in a graph $G = (V, E)$ is called a *dominating set*, if every vertex $v \in V$ is either an element of A or adjacent to an element of A. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set in G.

2 Main results

We introduce a new concept, namely, **Global Bipartite Dominating Set** of a simple bipartite graph. Then we define the global bipartite domination number of G.

Definition 2.1. *Let G be a connected bipartite graph with bipartition (X, Y), with $|X| = m$ and $|Y| = n$. The relative complement of G in $K_{m,n}$ denoted by $\widehat{G}$ is the graph obtained by deleting all edges of G from $K_{m,n}$ (i.e., $K_{m,n} \setminus G$). A global bipartite dominating set (GBDS) of G is a set S of vertices of G such that it dominates G and its relative complement $\widehat{G}$. The global bipartite domination number, $\gamma_{gb}(G)$ is the minimum cardinality of a global bipartite dominating set of G.*

Theorem 2.1. *For any connected spanning subgraph G of $K_{m,n}$, $\gamma(G) \leq \gamma_{gb}(G) \leq m + n$.*

Proof. A global bipartite dominating set of G is a dominating set of G and so $\gamma(G) \leq \gamma_{gb}(G)$. The set of all vertices of G is clearly a GBDS of G so, $\gamma_{gb}(G) \leq m + n$. Therefore $\gamma(G) \leq \gamma_{gb}(G) \leq m + n$. □

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Remark 2.1. *The bounds in Theorem 2.1 are sharp. For the complete bipartite graph $G = K_{m,n}$, $\gamma_{gb}(K_{m,n}) = m + n$. For P_4, $\gamma(P_4) = \gamma_{gb}(P_4) = 2$. So $K_{m,n}$ has the largest possible GBD number. Also the bounds in Theorem 2.1 are strict. For the graph $K_{2,3} - e$, $\gamma(K_{2,3} - e) = 2$ and $\gamma_{gb}(K_{2,3} - e) = 4$.*

Theorem 2.2. *If G and $\widehat{G}$ does not contain isolated vertices, then $\gamma_{gb}(G) \leq min\{m, n\}$, where G is a spanning subgraph of $K_{m,n}$.*

Proof. Let (X, Y) be the bipartition of G with $|X| = m \leq |Y| = n$. Since G and $\widehat{G}$ does not contain isolated vertices, X is a G.B.D.S. of G. Therefore $\gamma_{gb}(G) \leq m$. □

Theorem 2.3. *For any positive integers m and n, $\gamma_{gb}(K_{m,n}) = m + n$.*

Proof. Let G be a complete bipartite graph with partitions X and Y. Then $uv \in E(G)$ for every $u \in X$ and $v \in Y$. Let $\widehat{G}$ denotes the relative complement of G in $K_{m,n}$. Then $\widehat{G}$ contains $m + n$ isolated vertices. Hence every global bipartite dominating set of G must contain all vertices of $\widehat{G}$ and so $\gamma_{gb}(G) \geq slant m + n$. Now $V(G)$ is a global bipartite dominating set of G. Hence $\gamma_{gb}(G) = m + n$. □

Theorem 2.4. *For a spanning subgraph G of $K_{m,n}$, a vertex v is in every global bipartite dominating set of G if and only if v is an isolated vertex in $\widehat{G}$.*

Proof. If $|V(G)| \leq 3$, the proof is trivial. So let $|V(G)| > 3$. If v is an isolated vertex in $\widehat{G}$, then v is in every global bipartite dominating set of G. Conversely if v is not an isolated vertex in $\widehat{G}$, then there exist atleast two vertices u and w such that u is adjacent to v in G and w is adjacent to v in $\widehat{G}$. So $V(G) \setminus \{v\}$ is a global bipartite dominating set of G. □

Theorem 2.5. *Let G be a connected bipartite graph with partite sets X and Y. Let $S = V_1 \cup V_2$ be a GBDS of G, where $V_1 \subseteq X$ and $V_2 \subseteq Y$. Then if $V_1 = \phi$, then $V_2 = Y$ and if $V_2 = \phi$, then $V_1 = X$.*

Proof. Let $S = V_1 \cup V_2$, where $V_1 \subseteq X$ and $V_2 \subseteq Y$. If $V_1 = \phi$, then $S \subseteq Y$. Since G is bipartite, the vertices in Y are not adjacent and so $S \supseteq Y$. Therefore $S = V_2 = Y$. Similarly, we can prove that if $V_2 = \phi$ then $V_1 = X$. □

Theorem 2.6. *Let (X, Y) be the bipartition of a connected graph G. Then X is a GBDS of G if and only if $|N(y)| < |X|$, $\forall y \in Y$.*

Proof. Let X be a GBDS of G. If possible assume that there exists a vertex $y \in Y$ such that $|N(y)| = |X|$. Then y is an isolated vertex in $\widehat{G}$, contradiction to the fact that X is a GBDS of G. Conversely, since G is connected, X is dominating set of G. So it is sufficient to show that X dominates $\widehat{G}$ also. Let $y \in Y$, then $N(y)$ is a proper subset of X. So y is adjacent to at least one vertex of X in $\widehat{G}$. This completes the proof. □

Theorem 2.7. *Let G be a connected sub graph of $K_{m,n}$. Then $\gamma_{gb}(G) = m + n - 1$ if and only if $G \cong K_{m,n} - e$.*

Proof. Let $G \cong K_{m,n} - e$. where $e = uv \in E(K_{m,n})$. So $uv \notin E(G)$ and hence $uv \in E(\widehat{G})$. Since $\widehat{G}$ contains $m + n - 2$ isolated vertices, every global bipartite dominating set of G contains all vertices of $V(G) - \{u, v\}$ and at least one of u and v. Thus

$$\gamma_{gb}(G) \geq m + n - 1 \tag{2.1}$$

Since $V(G) - \{u\}$ is a GBDS of G, it follows that

$$\gamma_{gb}(G) \leq m + n - 1 \tag{2.2}$$

Thus by (1) and (2)we obtain $\gamma_{gb}(G) = m + n - 1$.
Conversely assume that $\gamma_{gb}(G) = m + n - 1$. To prove $G \cong K_{m,n} - e$. We observe that $\gamma_{gb}(K_{m,n}) = m + n$ and $\gamma_{gb}(K_{m,n} - e) = m + n - 1$. Let G be a proper subgraph of $K_{m,n} - e$ containing $m + n$ vertices. Then $\widehat{G}$ contains atmost $m + n - 3$ isolated vertices. In that case $\widehat{G}$ contains a path uvw. Then $V(G) - \{u, w\}$ is a GBDS of G. So $\gamma_{gb}(G) \leq m + n - 2$. This completes the proof. □

Theorem 2.8. *Let G be a graph with bipartition (X, Y). If G has a γ-set $S = V_1 \cup V_2$, where $V_1 \subseteq X$ and $V_2 \subseteq Y$ then S is a γ_{gb}-set of G if and only if $\bigcap_{x \in V_1} N(x) \subseteq V_2$ and $\bigcap_{y \in V_2} N(y) \subseteq V_1$.*

Proof. Let $\bigcap_{x\in V_1} N(x) \subseteq V_2$ and $\bigcap_{y\in V_2} N(y) \subseteq V_1$. Since S is a γ- set of G, it suffices to show that S dominates the relative compliment of G. Let $u \in X$. If $u \in \bigcap_{y\in V_2} N(y)$, then $u \in V_1$. If $u \notin \bigcap_{y\in V_2} N(y)$ then u is adjacent to atleast one vertex of V_2 in $\widehat{G}$. Similarly, we can prove that if $v \in Y$ then $v \in V_2$ or v is adjacent to atleast one vertex of V_1 in $\widehat{G}$. Conversely, let S dominates $\widehat{G}$. Let x be an arbitrary vertex in X. If $x \in \bigcap_{y\in V_2} N(y)$, then in $\widehat{G}$, x is not adjacent to any vertex of V_2. Since S dominates $\widehat{G}$, we can deduce that $x \in V_1$. If $x \notin \bigcap_{y\in V_2} N(y)$, then x is adjacent to atleast one element of V_2 in $\widehat{G}$. Hence the proof. □

Corollary 1. *Let G be a connected bipartite graph with n vertices, $n \geq 4$. Then $\gamma_{gb}(G \circ K_1) = n$, where $G \circ K_1$ denotes the corona of the graphs G and K_1.*

Proof. If $G \cong K_{1,n}$, the proof is trivial. Otherwise, let (X, Y) be the bipartition of $G \circ K_1$. Let $S = V_1 \cup V_2$, where $V_1 \subseteq X$ and $V_2 \subseteq Y$, be the set of all pendant vertices of $G \circ K_1$. Clearly S is γ-set of $G \circ K_1$. Also $\bigcap_{x\in V_1} N(x) = \phi$ and $\bigcap_{y\in V_2} N(y) = \phi$. Therefore the proof follows immediately from theorem 2.8. □

Corollary 2. *For $n \geq 10$, $\gamma_{gb}(P_n) = \gamma(P_n) = \lceil \frac{n}{3} \rceil$.*

Proof. Let $V(P_n) = \{1, 2, 3, \ldots, n\}$. Then $X = \{x : x \text{ is even}, x \leq n\}$, $Y = \{y : y \text{ is odd}, y \leq n\}$ is the bipartition of P_n. Let $S_1 = \{i : i \equiv 1(mod\, 3), i \leq n\}$ and $S_2 = \{i : i + 1 \equiv 0(mod\, 3), i \leq n\}$. Then either S_1 or S_2 is a γ-set of P_n. Also for $i = 1, 2$, $\bigcap_{x\in S_i\cap X} N(x) = \phi$ and $\bigcap_{y\in S_i\cap Y} N(y) = \phi$. Thus the proof follows from theorem 2.8. □

Corollary 3. *For an even integer $n \geq 10$, $\gamma_{gb}(C_n) = \gamma(C_n) = \lceil \frac{n}{3} \rceil$.*

Proof. The proof is exactly similar to corollary 2. □

Theorem 2.9. *For any two positive integers a and b with $a < b$, there exists a graph G such that $\gamma(G) = a$ and $\gamma_{gb}(G) = b$.*

Proof. Consider the graph $K_{b-a,a}$, with partite sets $W = \{w_1, w_2, \ldots, w_{b-a}\}$ and $U = \{u_1, u_2, \ldots, u_a\}$. Let G be the graph obtained from $K_{b-a,a}$ by adding new vertices $v_1, v_2, \ldots, v_a$and join v_i with u_i for $i = 1, 2, \ldots, a$. Let S be a dominating set of G. Since for each i, v_i is adjacent to u_i only, $|S| \geq a$. Now U is a dominating set of G. So $|S| \leq a$. Hence $\gamma(G) = a$. In $\widehat{G}$, the vertices $w_1, w_2, \ldots, w_{b-a}$ are isolated. So W is a subset of every γ_{gb}-set of G. Therefore the set $\{u_1, u_2, \ldots, u_a, w_1, w_2, \ldots, w_{b-a}\}$ is a γ_{gb}-set of G. Hence $\gamma_{gb}(G) = b$.

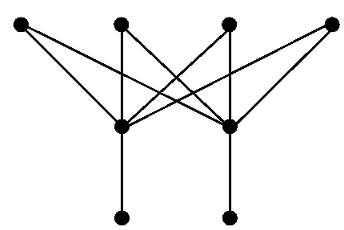

Figure 1: Graph G with $\gamma = 2$ and $\gamma_{gb} = 6$

□

Lemma 2.1. *If G is an r-regular connected bipartite graph with bipartition (X, Y) then $|X| = |Y|$.*

Proof. Each edge in G contributes exactly one to the degree sums $r|X|$ and $r|Y|$. Therefore $r|X| = r|Y| = |E| \Rightarrow |X| = |Y|$. □

Theorem 2.10. *If G is an $n - 1$-regular bipartite graph, then $\gamma_{gb}(G) = n$.*

Proof. Since G is $n-1$ regular, $\widehat{G}$ has n components and all of them are P_2. So $\gamma(\widehat{G}) = n$. Then by theorem 2.8, we can find a γ-set of $\widehat{G}$ such that it dominates G also. Therefore $\gamma_{gb}(G) = n$. □

Theorem 2.11. *Let G be a healthy spider with $2n+1$ vertices, then $\gamma_{gb}(G) = n+1$.*

Proof. Let S be a γ-set of G, then $|S| = n$ and $u \notin S$ (see Figure 2). So S dominates all vertices except u in $\widehat{G}$. So $S \cup \{u\}$ is a γ_{gb}-set of G. This completes the proof.

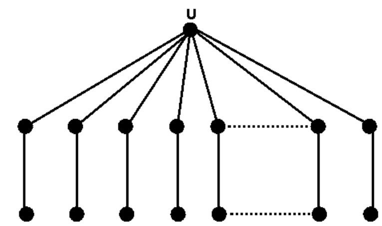

Figure 2: Healthy Spider

□

Theorem 2.12. *If G is a wounded spider with $n+k+1$ vertices, then $\gamma_{gb}(G) = k+1$.*

Proof. Observe that $\gamma(G) = k+1$. Also the set $S = \{1, 2, 3, \ldots, k, u\}$ is a γ_{gb}-set of G (see Figure 3).

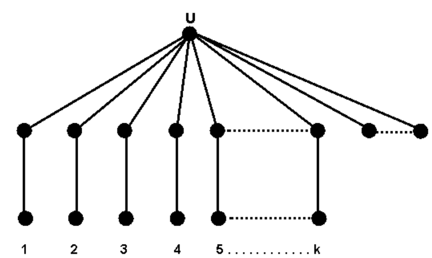

Figure 3: Wounded Spider

□

Theorem 2.13. *$\gamma_{gb}(B_n) = 4$, where B_n is the book graph on $2n+1$ vertices.*

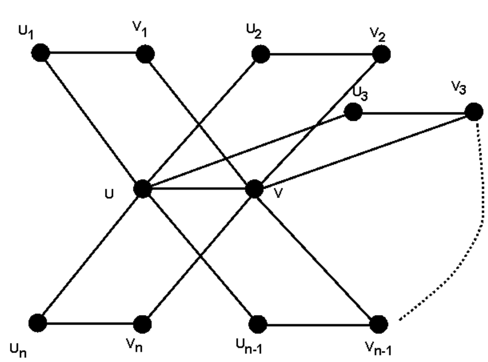

Figure 4: Book Graph

Proof. Let the vertices of B_n be labelled as shown in figure 4. Then $X = \{v, u_1, u_2, \ldots, u_n\}$, $Y = \{u, v_1, v_2, \ldots, v_n\}$ is the bipartition of B_n. Clearly the set $\{u, v\}$ is the γ-set of B_n. Also $\{u, v, u_1, v_1\}$ is a γ-set of $\widehat{B_n}$. Therefore $\gamma_{gb}(B_n) = 4$. □

Theorem 2.14. *$\gamma_{gb}(S(K_n)) = n$, where $S(K_n)$ is the subdivision of the complete graph K_n.*

Proof. Let X be the set of all old vertices and Y be the set of all new vertices of $S(K_n)$. Then (X, Y) is a bipartition of $S(K_n)$. In $S(K_n)$, the degree of each vertex in X is $n-1$ and the degree of each vertex in Y is 2. We construct a γ-set of $S(K_n)$ as follows: Let $S \subseteq X$ such that $|S| = n-2$. Then S dominates all but one vertex u in Y. Also $N(u) = \{x, y\}$ and $X - S = \{x, y\}$. So $S \cup \{u\}$ is a γ-set of $S(K_n)$. Since $S \cup \{u\}$ does not dominate x and y in $\widehat{G}$, this set is not a γ_{gb}-set. So $S \cup \{u, v\}$, where $v \notin N(x) \cup N(y)$, is a γ_{gb}-set of $S(K_n)$. Therefore $\gamma_{gb}(S(K_n)) = n$. □

References

[1] R. Balakrishnan and K. Ranganathan, *A Textbook of Graph theory*, Springer, 2012.

[2] E.J. Cockayne, R.M. Dawes and S.T. Hedetniemi, Total Domination in Graphs, *Networks* 10 :211-219, 1980.

[3] F. Harary, *Graph theory*, Addison-Wesley, Reading, MA,1972.

[4] T.W. Haynes, S.T. Hedetniemi and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[5] T.W. Haynes, S.T. Hedetniemi and P. J. Slater, *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York, 1998.

[6] Michael A. Henning and Anders Yeo, *Total Domination in Graphs*, Springer, New York, 2013.

[7] V.R. Kuli and B.Janakiram, The total global domination number of a graph. *Indian Journal of Pure and Applied Mathematics* 27(1996):537-542.

[8] E. Sampathkumar, The global domination number of a graph, *J. Math. Phys.Sci*, 23:377-385, (1989).

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