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Caratheodory's Theorem for $\mathbb{B}^{-1}$-convex Sets

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Abstract

In this article, our main concept is $\mathbb{B}^{-1}-$convexity that is a new abstract convexity type. For the $\mathbb{B}^{-1}-$convex sets, Caratheodory's Theorem which is one of the most important results in convexity theory is proved and its corollary is given.

Keywords: Caratheodory's Theorem, $\mathbb{B}^{-1}-$convexity, $\mathbb{B}^{-1}-$convex sets, abstract convexity.

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1 Introduction

Caratheodory's Theorem is the fundamental dimensionality result in convexity theory, and it is the source of many other results in which dimensionality is prominent. It is used to prove Helly's Theorem, concerning intersections of convex sets, as well as various results about infinite systems of linear inequalities.

If S is a subset of $\mathbb{R}^n$, the convex hull of S can be obtained by forming all convex combinations of elements of S. According to the classical theorem of Caratheodory, it is not really necessary to form combinations involving more than $n+1$ elements at a time. One can limit attention to convex combinations $\lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_m x_m$ such that $m \leq n+1$ (or even to combinations such that $m = n+1$, if one does not insist on the vectors x_i being distinct).

$\mathbb{B}^{-1}$-convexity is an abstract convexity type ([5–7]). In 2012, $\mathbb{B}^{-1}$-convexity is introduced in [1]. Then, $\mathbb{B}^{-1}-$convex sets and their properties examined in [2, 4]. The applications of $\mathbb{B}^{-1}$-convexity to Mathematical Economy is investigated in [3]. Separation of $\mathbb{B}^{-1}-$convex sets by $\mathbb{B}^{-1}-$measurable maps is studied in [8].

In this paper, we examine Caratheodory's Theorem for $\mathbb{B}^{-1}-$convex sets. As being in classic convexity, this theorem is significant in $\mathbb{B}^{-1}-$convexity and it has applications to the Optimization Theory and Mathematical Economy. Since it is used for proving Helly's and Radon Theorems which are thought to be examined for $\mathbb{B}^{-1}-$convexity in next studies, we need to express Caratheodory's Theorem for $\mathbb{B}^{-1}-$convex sets.

The outline of this article is as follows: In Section 2, we recall some definitions and theorems about $\mathbb{B}^{-1}-$convexity. Then, we prove the Caratheodory's Theorem for $\mathbb{B}^{-1}-$convex sets and its corollary in last section.

2 $\mathbb{B}^{-1}-$convexity

For $r \in \mathbb{Z}^-$, the map $x \to \varphi_r(x) = x^{2r+1}$ is a homeomorphism from $K = \mathbb{R} \setminus \{0\}$ to itself; $\boldsymbol{x} = (x_1, x_2, ..., x_n) \to \Phi_r(\boldsymbol{x}) = (\varphi_r(x_1), \varphi_r(x_2), ..., \varphi_r(x_n))$ is homeomorphism from K^n to itself.

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For a finite nonempty set $A = \left\{x^{(1)}, x^{(2)}, ..., x^{(m)}\right\} \subset K^n$ the Φ_r-convex hull (shortly r-convex hull) of A, which we denote $Co^r(A)$ is given by

$$Co^r(A) = \left\{\Phi_r^{-1}\left(\sum_{i=1}^{m} t_i \Phi_r(x^{(i)})\right) : t_i \geq 0, \sum_{i=1}^{m} t_i = 1\right\} .$$

We denote by $\overset{m}{\underset{i=1}{\wedge}} x^{(i)}$ the greatest lower bound with respect to the coordinate-wise order relation of $x^{(1)}, x^{(2)}, ..., x^{(m)} \in \mathbb{R}^n$, that is:

$$\overset{m}{\underset{i=1}{\wedge}} x^{(i)} = \left(\min\left\{x_1^{(1)}, x_1^{(2)}, ..., x_1^{(m)}\right\}, ..., \min\left\{x_n^{(1)}, x_n^{(2)}, ..., x_n^{(m)}\right\}\right)$$

where, $x_j^{(i)}$ denotes jth coordinate of the point $x^{(i)}$.

Thus, we can define $\mathbb{B}^{-1}$-polytopes as follows:

Definition 2.1. *[1] The Kuratowski-Painleve upper limit of the sequence of sets $\{Co^r(A)\}_{r\in\mathbb{Z}^-}$, denoted by $Co^{-\infty}(A)$ where A is a finite subset of K^n, is called $\mathbb{B}^{-1}$-polytope of A.*

The definition of $\mathbb{B}^{-1}$-polytope can be expressed in the following form in $\mathbb{R}^n_{++} = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_i > 0,\ i = 1, 2, ..., n\}$.

Theorem 2.1. *[1] For all nonempty finite subsets $A = \left\{x^{(1)}, x^{(2)}, ..., x^{(m)}\right\} \subset \mathbb{R}^n_{++}$ we have*

$$Co^{-\infty}(A) = \lim_{r\to-\infty} Co^r(A) = \left\{\overset{m}{\underset{i=1}{\wedge}} t_i x^{(i)} : t_i \geq 1, \min_{1\leq i\leq m} t_i = 1\right\} .$$

Next, we give the definition of $\mathbb{B}^{-1}$-convex sets.

Definition 2.2. *[1] A subset U of K^n is called a $\mathbb{B}^{-1}$-convex if for all finite subsets $A \subset U$ the $\mathbb{B}^{-1}$-polytope $Co^{-\infty}(A)$ is contained in U.*

By Theorem 2.1, we can reformulate the above definition for subsets of $\mathbb{R}^n_{++}$:

Theorem 2.2. *[1] A subset U of $\mathbb{R}^n_{++}$ is $\mathbb{B}^{-1}$-convex if and only if for all $x^{(1)}, x^{(2)} \in U$ and all $\lambda \in [1, \infty)$ one has $\lambda x^{(1)} \wedge x^{(2)} \in U$.*

Definition 2.3. *Given a set $S \subset K^n$, the intersection of all the B^{-1}-convex subsets of K^n containing S is called the B^{-1}-convex hull of S and is denoted by $\mathbb{B}^{-1}[S]$.*

3 Caratheodory's Theorem for $\mathbb{B}^{-1}$-convex Sets

Lemma 3.1. *In $\mathbb{R}^n_{++}$, a set of the form $\prod_{i=1}^{n}[x_i, y_i]$ is a $\mathbb{B}^{-1}$-convex set.*

Proof. If $A \subset \prod_{i=1}^{n}[x_i, y_i]$ then $\Phi_r(A) \subset \prod_{i=1}^{n}\left[x_i^{2r+1}, y_i^{2r+1}\right]$, from the convexity of a product of intervals we obtain, after taking the inverse image by Φ_r, $Co^r(A) \subset \prod_{i=1}^{n}[x_i, y_i]$ and therefore $Co^{-\infty}(A) \subset \prod_{i=1}^{n}[x_i, y_i]$. □

We denote by $\langle L\rangle_m$, the family of nonempty subsets of L of cardinality at most m.

Theorem 3.3. *(Carathedory's Theorem) If L is a compact subset of $\mathbb{R}^n_{++}$ then*

$$Co^{-\infty}(L) = \bigcup_{A\in\langle L\rangle_{n+1}} Co^{-\infty}(A)$$

Consequently, for all subsets L of $\mathbb{R}^n_{++}$,

$$\mathbb{B}^{-1}[L] = \bigcup_{A\in\langle L\rangle_{n+1}} \mathbb{B}^{-1}[A] = \bigcup_{A\in\langle L\rangle_{n+1}} Co^{-\infty}(A) ;$$

and, if L is compact, $\mathbb{B}^{-1}[L] = Co^{-\infty}(L)$.

Proof. If $x \in Co^{-\infty}(L)$ then there is a sequence $\left(x_{r_k}\right)_{r_k \in \mathbb{N}}$ with $x_{r_k} \in Co^{-r_k}(L)$, $\forall k \in \mathbb{N}$ which converges to x. But from Caratheodorys theorem, there is, for each k, a set of points $x_k^1, x_k^2, ..., x_k^{n+1}$ in L and a set of numbers $\rho_k^1, \rho_k^2, ..., \rho_k^{n+1}$ in $[1, +\infty)$ such that

$$\sum_{j=1}^{n+1} \left(\rho_k^j\right)^{-2r_k+1} = 1$$

and

$$\Phi_{-r_k}\left(x_{r_k}\right) = \sum_{j=1}^{n+1} \left(\rho_k^j\right)^{-2r_k+1} \Phi_{-r_k}\left(x_k^j\right)$$

or, for $i = 1, 2, ..., n$,

$$x_{r_k,i} = \left(\sum_{j=1}^{n+1} \left(\rho_k^j x_{k,i}^j\right)^{-2r_k+1}\right)^{\frac{1}{-2r_k+1}}$$

Since L is compact we can without loss of generality assume that each of the sequences $\left(x_k^j\right)_{k \in \mathbb{N}}$, $j = 1, 2, ..., n+1$ converges in L to a point x^j, and also that each of the sequences ρ_k^j, $j = 1, 2, ..., n+1$ converges in L to a point ρ^j in $[1, +\infty)$. Taking into account that all the numbers involved are positive we have

$$\lim_{k \to \infty} \left(\sum_{j=1}^{n+1} \left(\rho_k^j x_{k,i}^j\right)^{-2r_k+1}\right)^{\frac{1}{-2r_k+1}} = \min_{1 \le j \le n+1} \left\{\rho^j x_i^j\right\}$$

moreover

$$\min_{1 \le j \le n+1} \left\{\rho^j\right\} = 1 \,.$$

Taking the limit componentwise we obtain $x = \wedge_{j=1}^{n+1} \rho^j x^j$, with $\rho^j \ge 1$ for all j and $\min_{1 \le j \le n+1} \left\{\rho^j\right\} = 1$. We have shown that $x \in Co^{-\infty}(A)$ with $A = \left\{x^1, x^2, ..., x^{n+1}\right\} \subset L$. The last formula follows from $\mathbb{B}^{-1}[A] = Co^{-\infty}(A)$ for all finite sets A, $\mathbb{B}^{-1}[L] = \bigcup_{A \in \langle L \rangle} Co^{-\infty}(A)$ and the first part applied to the finite sets $A \in \langle L \rangle$. ☐

Corollary 3.1. *If L is a compact subset of $\mathbb{R}_{++}^n$ then $\mathbb{B}^{-1}[L]$ is compact.*

Proof. If $L \subset \prod_{i=1}^n [a_i, b_i]$ then $Co^{-\infty}(L) \subset \prod_{i=1}^n [x_i, y_i]$; $Co^{-\infty}(L)$ is therefore compact. The equality $\mathbb{B}^{-1}[L] = Co^{-\infty}(L)$ concludes the proof. ☐

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