

Fractional integral Chebyshev inequality without synchronous functions condition

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Abstract

In this paper, we use the Riemann-Liouville fractional integrals to establish some new integral inequalities related to the Chebyshev inequality in the case where the synchronicity of the given functions is replaced by another condition. This paper generalises some recent results in the paper of [C.P. Niculescu and I. Roventa: An extension of Chebyshev's algebraic inequality, Math. Reports, 2013].

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1 Introduction

Let us consider the Chebyshev inequality [10]

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx \geq \left(\frac{1}{b-a} \int_a^b f(x)dx \right) \left(\frac{1}{b-a} \int_a^b g(x)dx \right), \quad (1.1)$$

where f and g are two integrable and synchronous functions on $[a, b]$ i.e. $(f(x) - f(y))(g(x) - g(y)) \geq 0, x, y \in [a, b]$.

Many researchers have given considerable attention to (1.1), see [2, 4, 7, 11–13, 15] and the references therein. For the fractional integration case, it has been proved in [1] that for any synchronous functions f and g on $[a, b]$, the fractional inequality

$$J^\alpha(1)J^\alpha fg(x) \geq J^\alpha f(x)J^\alpha g(x), x \in [a, b] \quad (1.2)$$

is valid.

For more information and applications on Chebyshev inequality, we refer the reader to [3, 5, 6, 9, 14, 16].

On the other hand, recently in [11], C.P. Niculescu and L. Roventa have proved that for two functions f and g of the space $L^\infty([a, b])$, the Chebyshev's inequality still works by assuming the condition:

$$\left(f(x) - \frac{1}{x-a} \int_a^b f(x)dx \right) \left(g(x) - \frac{1}{x-a} \int_a^b g(x)dx \right) \geq 0. \quad (1.3)$$

The main purpose of this paper is to establish some new results for (1.1) by using the Riemann-Liouville fractional integrals. We present our results in the case where the synchronicity of the given functions is replaced by another condition that is more general than that presented in [11]. For our results, Theorem 1 of [11] can be deduced as a special case.

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2 Preliminaries

In this section, we present some preliminaries on Riemann-Liouville fractional integration.

Definition 2.1. *The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a continuous function f on $[a, b]$ is defined as*

$$\begin{aligned} J_a^\alpha f(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad \alpha > 0, \quad a < t \leq b, \\ J_a^0 f(t) &= f(t), \end{aligned} \tag{2.4}$$

where $\Gamma(\alpha) := \int_0^\infty e^{-u} u^{\alpha-1} du$.

For $\alpha > 0, \beta > 0$, we have the following two properties:

$$J_a^\alpha J_a^\beta f(t) = J_a^{\alpha+\beta} f(t) \tag{2.5}$$

and

$$J_a^\alpha J_a^\beta f(t) = J_a^\beta J_a^\alpha f(t). \tag{2.6}$$

For more details, one can consult [8].

3 Main Results

Lemma 3.1. *Let f and g be two functions belonging to $L^\infty([a, b])$, then for all $x \in]a, b], \alpha \geq 1$, we have*

$$\begin{aligned} &\frac{1}{x-a} J_a^\alpha f(x)g(x) \\ &= \left(\frac{1}{x-a} \int_a^x f(s)ds \right) \left(\frac{1}{x-a} J_a^\alpha g(x) \right) \\ &\quad + \frac{1}{(x-a)\Gamma(\alpha)} \int_a^x \left[\left(f(t) - \frac{1}{t-a} \int_a^t f(s)ds \right) \left((x-t)^{\alpha-1}g(t) - \frac{1}{t-a} \int_a^t (x-s)^{\alpha-1}g(s)ds \right) \right] dt. \end{aligned} \tag{3.7}$$

Proof. We have:

$$\begin{aligned} &\int_a^x f(t)(x-t)^{\alpha-1}g(t)dt \\ &= \left(f(t) \int_a^t (x-s)^{\alpha-1}g(s)ds \right)_{t=a}^{t=x} - \int_a^x \left(f'(t) \int_a^t (x-s)^{\alpha-1}g(s)ds \right) dt \\ &= f(x) \int_a^x \left((x-s)^{\alpha-1}g(s)ds - \int_a^x ((t-a)f'(t)) \left(\frac{1}{t-a} \int_a^t (x-s)^{\alpha-1}g(s)ds \right) \right) dt. \end{aligned} \tag{3.8}$$

To integrate by part, let us take the quantities

$$u(t) = \frac{1}{t-a} \int_a^t (x-s)^{\alpha-1}g(s)ds, u(t)' = \frac{-1}{(t-a)^2} \int_a^t (x-s)^{\alpha-1}g(s)ds + \frac{(x-t)^{\alpha-1}}{(t-a)}g(t)$$

and

$$v'(t) = (t-a)f'(t), v(t) = \int_a^t (s-a)f'(s)ds = (t-a)f(t) - \int_a^t f(s)ds.$$

So, it yields that

$$\begin{aligned} &\int_a^x f(t)(x-t)^{\alpha-1}g(t)dt \\ &= f(x) \int_a^x (x-s)^{\alpha-1}g(s)ds \\ &\quad - \left[\left(\frac{1}{t-a} \int_a^t (x-s)^{\alpha-1}g(s)ds \right) \left((t-a)f(t) - \int_a^t f(s)ds \right) \right]_{t=a}^{t=x} \\ &\quad - \int_a^x \left[\left(\frac{1}{(t-a)^2} \int_a^t (x-s)^{\alpha-1}g(s)ds \right) \left((t-a)f(t) - \int_a^t f(s)ds \right) \right] dt \\ &\quad + \int_a^x \left[\left(\frac{(x-t)^{\alpha-1}}{(t-a)}g(t) \right) \left((t-a)f(t) - \int_a^t f(s)ds \right) \right] dt. \end{aligned} \tag{3.9}$$

Consequently,

$$\begin{aligned}
& \int_a^x f(t)(x-t)^{\alpha-1}g(t)dt \\
&= \frac{1}{x-a} \left(\int_a^x f(s)ds \right) \left(\int_a^x (x-s)^{\alpha-1}g(s)ds \right) \\
&\quad - \int_a^x \frac{1}{(t-a)^2} \left(\int_a^t (x-s)^{\alpha-1}g(s)ds \right) \left((t-a)f(t) - \int_a^t f(s)ds \right) dt \\
&\quad + \int_a^x \left(\frac{1}{t-a} (x-t)^{\alpha-1}g(t) \right) \left((t-a)f(t) - \int_a^t f(s)ds \right) dt.
\end{aligned} \tag{3.10}$$

Therefore,

$$\begin{aligned}
& \int_a^x f(t)(x-t)^{\alpha-1}g(t)dt \\
&= \frac{1}{x-a} \left(\int_a^x f(s)ds \right) \left(\int_a^x (x-s)^{\alpha-1}g(s)ds \right) \\
&\quad - \int_a^x \left[\frac{1}{(t-a)} \left(\int_a^t (x-s)^{\alpha-1}g(s)ds \right) \frac{1}{(t-a)} \left((t-a)f(t) - \int_a^t f(s)ds \right) \right] dt \\
&\quad + \int_a^x \left[\left(\frac{1}{t-a} (x-t)^{\alpha-1}g(t) \right) \left((t-a)f(t) - \int_a^t f(s)ds \right) \right] dt.
\end{aligned} \tag{3.11}$$

Hence,

$$\begin{aligned}
& \int_a^x f(t)(x-t)^{\alpha-1}g(t)dt \\
&= \frac{1}{x-a} \left(\int_a^x f(s)ds \right) \left(\int_a^x (x-s)^{\alpha-1}g(s)ds \right) \\
&\quad - \int_a^x \left[\frac{1}{(t-a)} \left(\int_a^t (x-s)^{\alpha-1}g(s)ds \right) \left(f(t) - \frac{1}{(t-a)} \int_a^t f(s)ds \right) \right] dt \\
&\quad + \int_a^x \left[(x-t)^{\alpha-1}g(t) \left(f(t) - \frac{1}{t-a} \int_a^t f(s)ds \right) \right] dt. \\
&= \frac{1}{x-a} \left(\int_a^x f(s)ds \right) \left(\int_a^x (x-s)^{\alpha-1}g(s)ds \right) \\
&\quad + \int_a^x \left[\left(f(t) - \frac{1}{t-a} \int_a^t f(s)ds \right) \left((x-t)^{\alpha-1}g(t) - \frac{1}{t-a} \int_a^t (x-s)^{\alpha-1}g(s)ds \right) \right] dt,
\end{aligned} \tag{3.12}$$

and then,

$$\begin{aligned}
& \frac{1}{\Gamma(\alpha)} \int_a^x f(t)(x-t)^{\alpha-1}g(t)dt \\
&= \frac{1}{x-a} \left(\int_a^x f(s)ds \right) \left(\frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1}g(s)ds \right) \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_a^x \left[\left(f(t) - \frac{1}{t-a} \int_a^t f(s)ds \right) \left((x-t)^{\alpha-1}g(t) - \frac{1}{t-a} \int_a^t (x-s)^{\alpha-1}g(s)ds \right) \right] dt.
\end{aligned} \tag{3.13}$$

So,

$$\begin{aligned}
& J_a^\alpha f(x)g(x) \\
&= \frac{1}{x-a} \left(\int_a^x f(s)ds \right) J_a^\alpha g(x) \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_a^x \left[\left(f(t) - \frac{1}{t-a} \int_a^t f(s)ds \right) \left((x-t)^{\alpha-1}g(t) - \frac{1}{t-a} \int_a^t (x-s)^{\alpha-1}g(s)ds \right) \right] dt.
\end{aligned} \tag{3.14}$$

Consequently, we obtain (3.7). \square

An immediate consequence of the previous Lemma is the following result:

Theorem 3.1. Let f and g be two functions of the space $L^\infty([a, b])$ and suppose that for any $\alpha \geq 1$ and for any $t, x \in]a, b]; t \leq x \leq b$, the inequality

$$\left(f(t) - \frac{1}{t-a} \int_a^t f(s) ds \right) \left((x-t)^{\alpha-1} g(t) - \frac{1}{t-a} \int_a^t (x-s)^{\alpha-1} g(s) ds \right) \geq 0$$

is satisfied.

Then, we have:

$$\begin{aligned} & \frac{1}{x-a} J_a^\alpha f(x) g(x) \\ & \geq \left(\frac{1}{x-a} \int_a^x f(s) ds \right) \left(\frac{1}{x-a} J_a^\alpha g(x) \right). \end{aligned} \quad (3.15)$$

Remark 3.1. Taking $\alpha = 1, x = b$ in Theorem 3.1, we obtain Theorem 1 of [11].

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