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An efficient modification of PIM by using Chebyshev polynomials

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Abstract

In this article, an efficient modification of the Picard iteration method (PIM) is presented by using Chebyshev polynomials. Special attention is given to study the convergence of the proposed method. The proposed modification is tested for some examples to demonstrate reliability and efficiency of the introduced method. A comparison between our numerical results against the conventional numerical method, fourth-order Runge-Kutta method (RK4) is given. From the presented examples, we found that the proposed method can be applied to wide class of non-linear ordinary differential equations.

Keywords: Picard iteration method, Chebyshev polynomials, Runge-Kutta method, Convergence analysis.

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1 Introduction

Many different approximate methods have recently introduced to solve non-linear problems of differential equations, such as, variational iteration method ([3], [8], [18], [19], [22]), Adomian decomposition method ([1], [10], [23]), homotopy perturbation method ([6], [20]) and spectral collocation method ([6], [17]). The Adomian decomposition method provides solutions as a series by employing the so-called Adomian's polynomials which are related to the derivatives of the nonlinearities; therefore, these nonlinearities must be analytical functions of the dependent variables and this has often been ignored in the literature, for the existence and the uniqueness of solutions to, for example, initial-value problems in ODEs is ensured under much milder conditions ([4], [14]). However, the decomposition method may be formulated in a manner that does not require that the nonlinearities be differentiable with respect to the dependent variables and their derivatives [15]. Other techniques also require that the nonlinearities be analytical functions of the dependent variable and provide either convergent series or asymptotic expansions to the solution include perturbation methods [13], the homotopy perturbation technique and the homotopy analysis procedure [21].

By way of contrast, iterative techniques for solving a large class of linear or non-linear differential equations without the tangible restriction of sensitivity to the degree of the non-linear term and also it reduces the size of calculations besides, its interactions are direct and straightforward. These techniques include the well-known Picard fixed-point iterative procedure.

In this paper, we present a modification of PIM. This modification depends on the useful properties of the Chebyshev polynomials. Special attention is given to study the convergence analysis of the proposed method. Convergence analysis is reliable enough to estimate the maximum absolute error of the solution given by PIM. To guarantee this study, effectively employ this modification to a certain class of non-linear ODEs. Therefore, this modification of PIM has been widely used for solving non-linear problems to overcome the shortcoming of other methods.

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The rest of this paper is organized as follows: Section 2 is assigned to the analysis of the standard PIM. Section 3 is assigned to the convergence study of the proposed method. In section 4, some test problems have been solved by the modified PIM, to illustrate the efficiency of the proposed method. In finally, the paper ends with the conclusions in section 5.

2 Picard iteration method

To illustrate the analysis of PIM, we limit ourselves to consider the following non-linear first order ODE in the type ([5], [9], [16])

$$u'(x) = R\,u + N(u), \qquad u(0) = c, \quad 0 < x < a, \tag{2.1}$$

here R is a linear bounded operator i.e., it is possible to find a number $m_1 > 0$ such that $||R\,u|| \leq m_1||u||$. The non-linear term $N(u)$ is Lipschitz continuous with $|N(u) - N(v)| \leq m_2\,|u - v|, \forall\, x \in J = [0, a]$, for any constant $m_2 > 0$.
The PIM gives the possibility to write the solution of Eq.(2.1) in the following iteration formula

$$u_p(x) = u(0) + \int_0^x [R\,u_{p-1}(\tau) + N(u_{p-1}(\tau))\,]d\tau, \qquad p \geq 1. \tag{2.2}$$

The successive approximations u_p, $p \geq 0$, of the solution $u(x)$ will be readily obtained upon using any selective function u_0. The initial values of the solution are usually used for selecting the zeroth approximation u_0. In this technique we obtain a sequence of components of the solution $u(x)$. Consequently, the exact solution may be obtained by using

$$u(x) = \lim_{p\to\infty} u_p(x). \tag{2.3}$$

3 Convergence analysis

In this section, the sufficient conditions are presented to guarantee the convergence of PIM, when applied to solve non-linear ODEs, where the main point is that we prove the convergence of the recurrence sequence ([2], [12]), which is generated by using PIM.

Lemma 3.1. *Let $A : U \to V$ be a bounded linear operator and let $\{u_p\}$ be a convergent sequence in U with limit u, then $u_p \to u$ in U implies that $A(u_p) \to A(u)$ in V [12].*

Now, to prove the convergence of the sequence of solution using the Picard iteration method, we will rewrite Eq.(2.2) in an operator form as follows

$$u_p = A[u_{p-1}], \tag{3.4}$$

where the operator A takes the following form

$$A\,[u] = u(0) + \int_0^x [R\,u + N(u)\,]\,d\tau. \tag{3.5}$$

Theorem 3.1. *Assume that X be a Banach space and $A : X \to X$ is a nonlinear mapping, and suppose that*

$$||\,A[u] - A[v]\,|| \leq \alpha\,||u - v||, \quad \forall\ u, v \in X, \tag{3.6}$$

for any constant $\alpha = (m_1 + m_2)a$ $(0 < \alpha < 1)$ where m_1, m_2 and a are defined above. Then A has a unique fixed point. Furthermore, the sequence (2.2) using PIM with an arbitrary choice of $u(0) \in X$, converges to the fixed point of A and

$$||u_p - u_q|| \leq \frac{\alpha^q}{1-\alpha}\,||u_1 - u_0||. \tag{3.7}$$

Proof. Denoting $(C[J], ||.||)$ Banach space of all continuous functions on J with the norm defined by

$$||u(x)|| = \max_{x\in J} |u(x)|.$$

We are going to prove that the sequence $\{u_p\}$ is a Cauchy sequence in this Banach space

$$\begin{aligned}\|u_p - u_q\| &= \max_{x\in J} \mid u_p - u_q \mid \\ &= \max_{x\in J}\Big|\int_0^x [\,R\,(u_{p-1}-u_{q-1}) + N(u_{p-1}) - N(u_{q-1})\,]\,d\tau\Big| \\ &\le \max_{x\in J}\int_0^x [\,|R\,(u_{p-1}-u_{q-1})| + |N(u_{p-1}) - N(u_{q-1})|\,]\,d\tau \\ &\le \max_{x\in J}\int_0^x [\,(m_1+m_2)(u_{p-1}-u_{q-1})\,]\,d\tau \\ &\le \alpha||u_{p-1}-u_{q-1}||.\end{aligned}$$

Let, $p = q+1$ then

$$\|u_{q+1}-u_q\| \le \alpha\,\|u_q - u_{q-1}\| \le \alpha^2\,\|u_{q-1}-u_{q-2}\| \le ... \le \alpha^q\,\|u_1-u_0\|.$$

From the triangle inequality we have

$$\begin{aligned}||u_p - u_q|| &\le ||u_{q+1}-u_q|| + ||u_{q+2}-u_{q+1}|| + ... + ||u_p - u_{p-1}|| \\ &\le [\,\alpha^q + \alpha^{q+1} + ... + \alpha^{p-1}\,]\,||u_1-u_0|| \\ &\le \alpha^q[\,1+\alpha+\alpha^2+...+\alpha^{p-q-1}\,]\,||u_1-u_0|| \\ &\le \alpha^q[\,\frac{1-\alpha^{p-q-1}}{1-\alpha}\,]\,||u_1-u_0||.\end{aligned}$$

Since $0<\alpha<1$ so, $(1-\alpha^{p-q-1})<1$ then

$$||u_p-u_q|| \le \frac{\alpha^q}{1-\alpha}||u_1-u_0||.$$

But $||u_1-u_0||<\infty$ so, as $q\to\infty$ then $||u_p-u_q||\to 0$. We conclude that $\{u_p\}$ is a Cauchy sequence in $C[J]$ so, the sequence converges and the proof is complete. □

Theorem 3.2. *The maximum absolute error of the approximate solution u_p to problem (2.1) is estimated to be*

$$\max_{t\in J}|u_{exact}-u_p| \le \beta, \tag{3.8}$$

where $\beta = \dfrac{\alpha^q\,a\,[\,m_1\,||u_0||+k\,]}{1-\alpha}, \qquad k=\max_{x\in J}|N(u_0)|.$

Proof. From Theorem 1 and inequality (3.7) we have

$$||u_p-u_q|| \le \frac{\alpha^q}{1-\alpha}||u_1-u_0||,$$

as $p\to\infty$ then $u_p\to u_{\text{exact}}$ and

$$||u_1-u_0|| = \max_{x\in J}\Big|\int_0^x [\,R\,u_0 + N(u_0)\,]\,d\tau\Big| \le \max_{x\in J}\int_0^x [\,|R\,u_0| + |N(u_0)|\,]\,d\tau \le a\,[\,m_1\,||u_0||+k\,],$$

so, the maximum absolute error in the interval J is

$$||u_{\text{exact}}-u_p|| = \max_{x\in J}|u_{\text{exact}}-u_p| \le \beta.$$

This completes the proof. □

Our main goal in this paper is concerned with the implementation of PIM and its modification which have efficiently used to solve a certain class of ODEs. To achieve this goal, at the beginning of implementation of PIM, we use the orthogonal Chebyshev polynomials to expand the functions in the non-homogeneous term in the considered differential equation [17].

4 Solution procedure using the modified PIM

In this section, an efficient modification of PIM is presented by using Chebyshev polynomials. The well known Chebyshev polynomials [17] are defined on the interval $[-1,1]$ and can be determined with the aid of the following recurrence formula

$$T_{n+1}(z) = 2z\,T_n(z) - T_{n-1}(z), \qquad n = 1,2,\ldots.$$

The first three Chebyshev polynomials are $T_0(z) = 1,\ T_1(z) = z,\ T_2(z) = 2z^2 - 1$.

Theorem 4.3. *The error in approximating $f(x)$ by the sum of its first m terms is bounded by the sum of the absolute values of all the neglected coefficients. If*

$$f_m(x) = \sum_{k=0}^{m} c_k\,T_k(x), \tag{4.9}$$

then, for all $f(x)$, all m, and all $x \in [-1,1]$, we have

$$E_T(m) \equiv |f(x) - f_m(x)| \leq \sum_{k=m+1}^{\infty} |c_k|. \tag{4.10}$$

Proof. The Chebyshev polynomials are bounded by one, that is, $|T_k(x)| \leq 1$ for all $x \in [-1,1]$ and for all k. This implies that the k-th term is bounded by $|c_k|$. Subtracting the truncated series from the infinite series, bounding each term in the difference, and summing the bounds gives the theorem. □

For more details about the definition of the Chebyshev polynomials and its properties see ([7], [11], [17]). Now, in order to use these polynomials on the interval $[0,1]$ we define the so called shifted Chebyshev polynomials by introducing the change of variable $z = 2x - 1$. Let the shifted Chebyshev polynomials $T_n(2x-1)$ be denoted by $T^*_n(x)$. Then $T^*_n(x)$ can be obtained as follows

$$T^*_{n+1}(x) = 2(2x-1)\,T^*_n(x) - T^*_{n-1}(x), \qquad n = 1,2,\ldots. \tag{4.11}$$

Now, we use the shifted Chebyshev expansion to expand $f(x)$ in the following form

$$f(x) \approx f_m(x) = \sum_{k=0}^{m} c_k\,T^*_k(x), \tag{4.12}$$

where the constant coefficients c_k are defined by

$$c_k = \frac{2}{\pi\,h_k} \int_0^1 \frac{f(x)\,T^*_k(x)}{\sqrt{x - x^2}}\,dx, \qquad h_0 = 2, \quad h_k = 1, \quad k = 1,2,\ldots. \tag{4.13}$$

Now, the proposed modification will implement to solve the following two initial non-linear ordinary differential equations.

Model problem 1

Consider the following non-linear ordinary differential equation

$$u''(x) + x\,u'(x) + x^2\,u^3(x) = f(x), \quad x \in [0,1], \tag{4.14}$$

where $f(x) = (2 + 6x^2)e^{x^2} + x^2\,e^{3x^2}$ and subject to the following initial conditions

$$u(0) = 1, \qquad u'(0) = 0. \tag{4.15}$$

The exact solution of this problem is $u(x) = e^{x^2}$.
The procedure of the solution follows the following two steps:
Step 1. Expand the function $f(x)$ using shifted Chebyshev polynomials:

Using the above consideration, the function $f(x)$ can be approximated by eight terms ($m = 8$) of the shifted Chebyshev expansion (4.12) as follows

$$f_C(x) \approx 2.00232 - 0.358488\,x + 18.0328\,x^2 - 86.4534\,x^3 + 416.556\,x^4 - 1042.66\,x^5 + 1502.72x^6 - 1134.64x^7 + 366.624x^8.$$

Step 2. Implementation of PIM:

To solve Eq.(4.14) by the PIM we reduce this equation to the following system of first order ODEs

$$u'(x) = v(x), \tag{4.16}$$

$$v'(x) = -x\, v(x) - x^2\, u^3(x) + f(x), \tag{4.17}$$

with the following initial conditions $u(0) = 1,\ v(0) = 0$.
Now, the PIM gives the possibility to write the solution of the system (4.16)-(4.17) with the aid of the following iteration formula

$$u_{n+1}(x) = u_0 + \int_0^x v_n(\tau)d\tau, \qquad n \geq 0, \tag{4.18}$$

$$v_{n+1}(x) = v_0 - \int_0^x [\tau\, v_n(\tau) + \tau^2\, u_n^3(\tau) - f(\tau)\,]d\tau, \qquad n \geq 0. \tag{4.19}$$

We start with initial approximations $u_0 = 1$, $v_0 = 0$, and by using the above iteration formulae (4.18)-(4.19), we can directly obtain the components of the solution.
Now, the first three components of the solution $u(x)$ of Eq.(4.14) by using (4.18)-(4.19) are

$$u_0(x) = 1,$$

$$u_1(x) = 1,$$

$$u_2(x) = 1 + 1.00116x^2 - 0.059748x^3 + 1.4194x^4 - 4.32267x^5 + 13.8852x^6 - 24.8252x^7$$
$$+26.8343x^8 - 15.7589x^9 + 4.0736x^{10} + ...,$$

$$u_3(x) = 1 + 1.00116x^2 - 0.059748x^3 + 1.25254x^4 - 4.31371x^5 + 13.6959x^6 - 24.3106x^7 + 25.3466x^8$$
$$-13.3453x^9 + 1.68833x^{10} + 1.28936x^{11} - 0.308606x^{12} + ...$$

Now, also to perform PIM, we can expand the function $f(x)$ using Taylor series at the point $x = x_0$ as follows

$$f(x) \approx \sum_{k=0}^{m} \frac{f^{(k)}(x_0)}{k!}\,(x - x_0)^k, \tag{4.20}$$

for an arbitrary integer number m.
If we expand the function $f(x)$ by the Taylor series (4.20) about the point $x_0 = 0$ with eight terms, we have

$$f_T(x) \approx 2 + 9\,x^2 + 10\,x^4 + 7.83\,x^6 + 5.58333\,x^8 + O(x^9).$$

So, the first three components of the solution by using (4.18)-(4.19) are

$$u_0(x) = 1,$$
$$u_1(x) = 1,$$
$$u_2(x) = 1 + x^2 + 0.666667x^4 + 0.333333x^6 + 0.139881x^8 + 0.062037x^{10},$$
$$u_3(x) = 1 + x^2 + 0.5x^4 + 0.244444x^6 + 0.104167x^8 + 0.0496032x^{10} - 0.00469978x^{12}\,.$$

Also, to solve the same problem (4.14) using the fourth-order Runge-Kutta method, we used its corresponding system of ODEs (4.16)-(4.17).

The absolute errors between the function $f(x)$ and its approximation by using the Taylor expansion (Top) and the Chebyshev expansion (Bottom) are presented in figure 1.
The absolute error between the exact solution $u(x)$ and the approximate solution $u_C(x) = u_4(x)$ (after four iterations) and using the Chebyshev expansion for $f(x)$ with $m = 8$ is presented in figure 2(Right). Also, the absolute error between the exact solution $u(x)$ and the approximate solution $u_T(x) = u_4(x)$ (after four iterations) using the Taylor expansion for $f(x)$ with eight terms is presented in figure 2(Left).

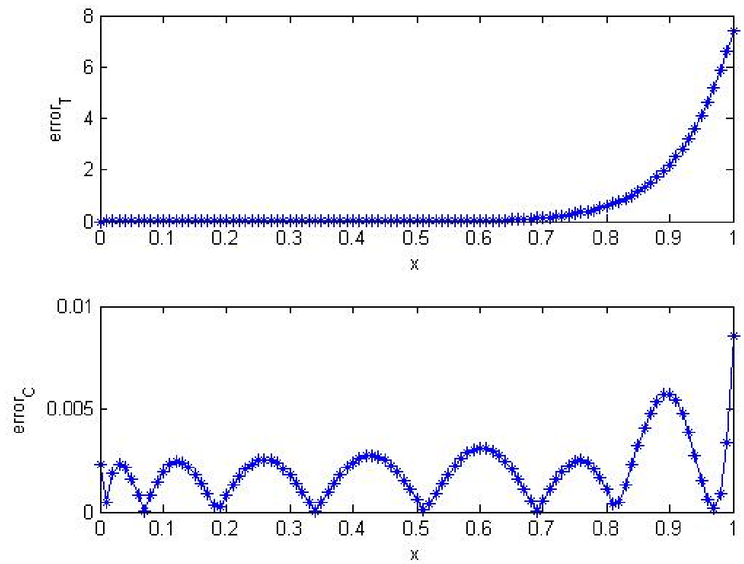

Figure 1: The absolute error: $|f(x) - f_T(x)|$ (Top) and $|f(x) - f_C(x)|$ (Bottom).

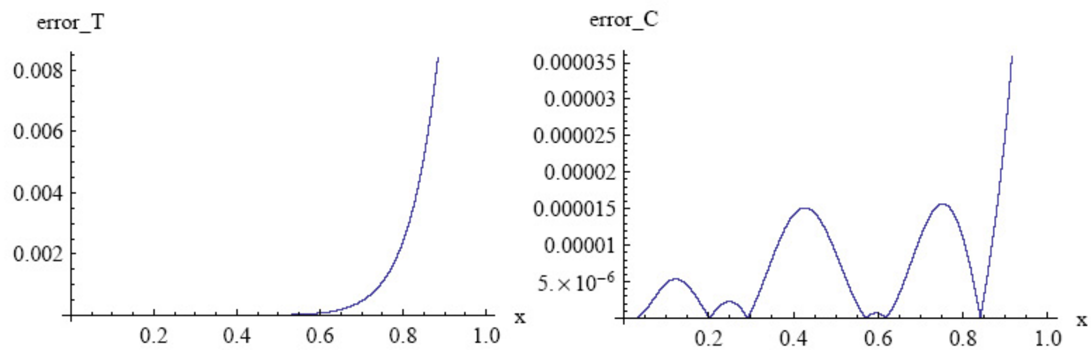

Figure 2: The absolute error $|u(x) - u_T(x)|$ (Left) and $|u(x) - u_C(x)|$ (Right).

Also, the figure 3 presents a comparison between the exact solution $u(x)$, with the numerical solution u_{RK4} using fourth-order Runge-Kutta and the approximate solution of our proposed method $u_C(x)$. From this figure, we can see that the two methods are in excellent agreement with the exact solution.

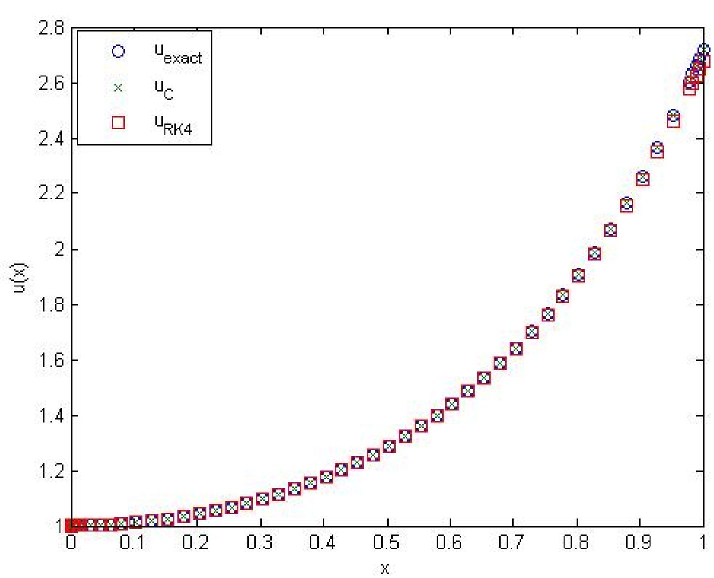

Figure 3: Comparison between the exact solution $u(x)$, u_{RK4} and the approximate solution of the proposed method $u_C(x)$.

Model problem 2

Consider the following non-linear ordinary differential equation

$$u'' + u\,u' = f(x), \quad x \in [0,1], \tag{4.21}$$

where $f(x) = x\sin(2x^2) - 4x^2\sin(x^2) + 2\cos(x^2)$ with the following initial conditions

$$u(0) = 0, \quad u'(0) = 0. \tag{4.22}$$

The exact solution of this problem is $u(x) = \sin(x^2)$.
The procedure of the solution follows the following two steps:
Step 1. Expand the function $f(x)$ using shifted Chebyshev polynomials:
Using the above consideration, the function $f(x)$ can be approximated by eight terms $(m = 8)$ of the expansion (4.12) as follows

$$f_C(x) \approx 2 - 0.0003\,x + 0.008\,x^2 + 1.892\,x^3 - 4.308\,x^4 - 2.399\,x^5 + 4.682\,x^6 - 6.276\,x^7 + 3.025\,x^8.$$

Step 2. Implementation of PIM:
To solve Eq.(4.21) by the PIM we reduce this equation to the following system of ODEs

$$u'(x) = v(x), \tag{4.23}$$

$$v'(x) = -u(x)\,v(x) + f(x), \tag{4.24}$$

with the following initial conditions $u(0) = 0,\ v(0) = 0$.
According to PIM we can construct the following iteration formula

$$u_{n+1}(x) = u_0 + \int_0^x [v_n(\tau)]d\tau, \qquad n \geq 0. \tag{4.25}$$

$$v_{n+1}(x) = v_0 - \int_0^x [u_n(\tau)\,v_n(\tau) - f(\tau)\,]d\tau, \qquad n \geq 0. \tag{4.26}$$

Therefore, the first three components of the solution $u(x)$ of Eq.(4.21) using (4.25)-(4.26) are

$$\begin{aligned}
u_0(x) &= 0,\\
u_1(x) &= x^2 + 0.1\,x^5 - 0.166667\,x^6 - 0.0185185\,x^9 + 0.00833333\,x^{10} + ...,\\
u_2(x) &= x^2 - 0.166667\,x^6 - 0.012\,x^8 + 0.008333\,x^{10} - 0.0004545\,x^{11} + 0.002932\,x^{12} + ...,\\
u_3(x) &= x^2 - 0.1667\,x^6 + 0.0083\,x^{10} + 0.0011\,x^{11} - 0.0017\,x^{13} + 0.00003\,x^{14} - 0.0003\,x^{15} + ...,
\end{aligned}$$

Now, if we expand the function $f(x)$ by the Taylor series (4.20) with eight terms, we have

$$f_T(x) \approx 2 + 2\,x^3 - 5\,x^4 - 1.33333\,x^7 + 0.75\,x^8 + O(x^9).$$

So, the first three components of the solution $u(x)$ of Eq.(4.21) using (4.25)-(4.26) are

$$\begin{aligned}
u_0(x) &= 0,\\
u_1(x) &= x^2 - 0.00004\,x^3 + 0.0007\,x^4 + 0.0946\,x^5 - 0.1436\,x^6 - 0.0571\,x^7 + 0.0836\,x^8 + ...,\\
u_2(x) &= x^2 - 0.00004\,x^3 + 0.0007\,x^4 - 0.0054\,x^5 - 0.143585\,x^6 - 0.0572\,x^7 + 0.0718\,x^8 + ...,\\
u_3(x) &= x^2 - 0.00004\,x^3 + 0.0007\,x^4 - 0.0054\,x^5 - 0.1436\,x^6 - 0.0572\,x^7 + 0.0843\,x^8 + ...
\end{aligned}$$

Figure 4 presents the absolute error between the function $f(x)$ and its approximation by using the Taylor expansion (Top) and the Chebyshev expansion (Bottom).

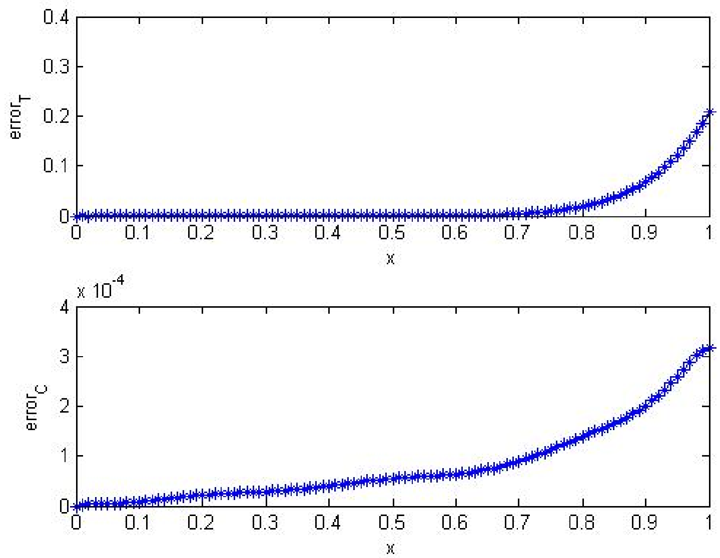

Figure 4: The absolute error: $|f(x) - f_T(x)|$ (Top) and $|f(x) - f_C(x)|$ (Bottom).

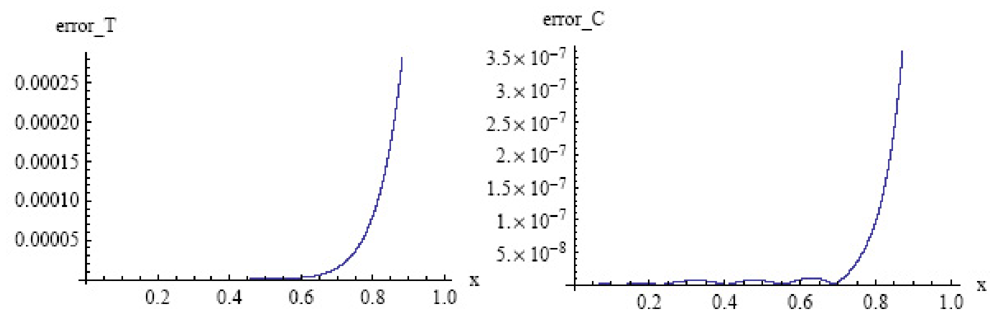

Figure 5: The absolute error: $|u(x) - u_T(x)|$ (Left) and $|u(x) - u_C(x)|$ (Right).

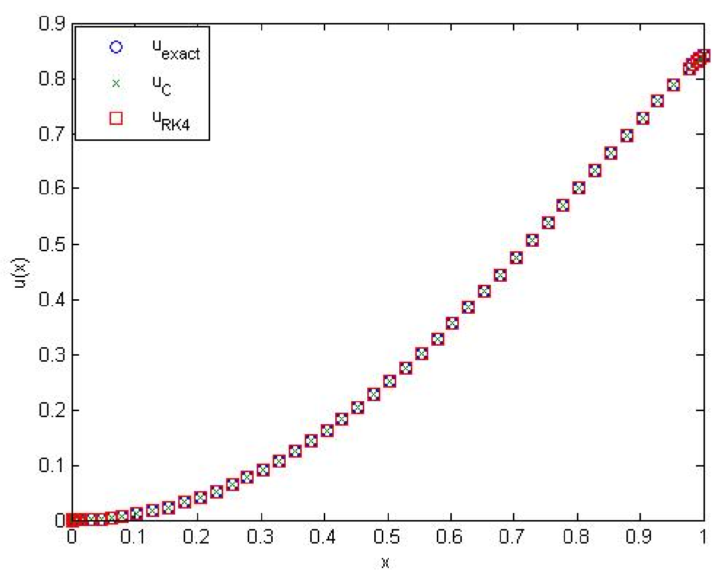

Figure 6: Comparison between the exact solution $u(x)$, u_{RK4} and the approximate solution of the proposed method $u_C(x)$.

The absolute error between the exact solution $u(x)$ and the approximate solution $u_C(x) \simeq u_4(x)$ (after four iterations) using the Chebyshev expansion for $f(x)$ with $m = 8$ is presented in figure 5(Right). Also, the absolute error between the exact solution $u(x)$ and the approximate solution $u_T(x) \simeq u_4(x)$ (after four iterations) using the Taylor expansion for $f(x)$ with eight terms is presented in figure 5(Left). Also, the figure 6 presents a comparison between exact solution $u(x)$, with the numerical solution u_{RK4} using fourth-order Runge-Kutta and the approximate solution of the proposed method $u_C(x)$. From these figures, we can conclude that the proposed method is in excellent agreement with the exact solution.

5 Conclusion

In this article, we used the properties of the shifted Chebyshev polynomials to introduce an efficient modification of PIM. Also, we presented comparative solutions with the proposed method and fourth-order Runge-Kutta method. From the introduced model problems, we can conclude that the proposed idea can be applied to solve the non-linear models of ordinary differential equations. Also, the obtained results demonstrate reliability and efficiency of the proposed method and achieve the convergence study of the method. From the resulting numerical solution we can conclude that the solution using this modification converges faster and is in excellent conformance with the exact solution. An interesting point about PIM is that only few iterations or, even in some special cases, one iteration, lead to exact solution or solution with high accuracy. Finally, all the obtained numerical results are done by using Matlab 8.

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