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On the Biordered set of Rings

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Abstract

In [4] K.S.S. Nambooripad introduced biordered sets as a partial algebra (E, ω^r, ω^l) where ω^r and ω^l are two quasiorders on the set E satisfying biorder axioms; to study the structure of a regular semigroup. Later in [2] David Esdown showed that the set of idempotents of a regular semigroup forms a regular biordered set. Here we extend the idea of biordered sets into rings and discussed some of its properties.

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1 Introduction

The set of idempotent elements in a semigroup S usually denoted as $E(S)$ and is important structural objects which can be used effectively in analyzing the structure of the semigroup. The concept of biordered set was originally introduced by Nambooripad[1972, 1979] to describe the structure of the set of idempotents of a semigroup in general and that of a regular semigroup in particular. A biordered set is a partial algebra (partial semigroup) together with two quasi orders on the domain of definition of the partial binary operation. Nambooripad identified a partial binary operation on the set of idempotents $E(S)$ of a semigroup S arising from the binary operation in S, defined two quasi orders on $E(S)$ and the resulting structure is abstracted as a biordered set. later on david Esdown showed that any biordeed set arises as the set of idempotents of a semigroup (see[2]).

In this paper we discuss the biordered sets which are the set of idempotents of a ring and we provide certain examples of such biordered sets.

2 Preliminaries

First we recall some basic definitions regarding semigroups, biorderede sets and rings needed in the sequel. A set S in which for every pair of elements $a, b \in S$ there is an element $a \cdot b \in S$ which is called the product of a by b is called a groupoid. A groupoid S is a semigroup if the binary operation on S is associative. An element $a \in S$ is called regular if there exists an element $a' \in S$ such that $aa'a = a$, if every element of S is regular then S is a regular semigroup. An element $e \in S$ such that $e \cdot e = e$ is called an idempotent and the set of all idempotents in S will be denoted by $E(S)$.

2.1 Biordered Sets

By a partial algebra E we mean a set together with a partial binary operation on E. Then $(e, f) \in D_E$ if and only if the product ef exists in the partial algebra E. If E is a partial algebra, we shall often denote the underlying set by E itself; and the domain of the partial binary operation on E will then be denoted by

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D_E. Also, for brevity, we write $ef = g$, to mean $(e, f) \in D_E$ and $ef = g$. The dual of a statement T about a partial algebra E is the statement T^* obtained by replacing all products ef by its left-right dual fe. When D_E is symmetric, T^* is meaningful whenever T is. On E we define

$$\omega^r = \{(e, f) : fe = e\} \; \omega^l = \{(e, f) : ef = e\}$$

and $\mathcal{R} = \omega^r \cap (\omega^r)^{-1}$, $\mathcal{L} = \omega^l \cap (\omega^l)^{-1}$, $\omega = \omega^r \cap \omega^l$. The data required to specify a biordered set E consists of a pair of quasiorders ω^r and ω^l. We will refer to ω^r as the right quasiorder of E and, ω^l as the left quasiorder of E.

Definition 2.1. *Let E be a partial algebra. Then E is a biordered set if the following axioms and their duals hold:*

1. *ω^r and ω^l are quasi orders on E and*

$$D_E = (\omega^r \cup \omega^l) \cup (\omega^r \cup \omega^l)^{-1}$$

2. *$f \in \omega^r(e) \Rightarrow f\mathcal{R}fe\omega e$*
3. *$g\omega^l f$ and $\quad f, g \in \omega^r(e) \Rightarrow ge\omega^l fe$.*
4. *$g\omega^r f\omega^r e \Rightarrow gf = (ge)f$*
5. *$g\omega^l f$ and $f, g \in \omega^r(e) \Rightarrow (fg)e = (fe)(ge)$.*

We shall often write $E =< E, \omega^l, \omega^r >$ to mean that E is a biordered set with quasiorders ω^l, ω^r. The relation ω defined is a partial order and

$$\omega \cap (\omega)^{-1} \subset \omega^r \cap (\omega^l)^{-1} = 1_E.$$

Definition 2.2. *Let $\mathcal{M}(e, f)$ denote the quasi ordered set $(\omega^l(e) \cap \omega^r(f), <)$ where $<$ is defined by $g < h \Leftrightarrow$ $eg\omega^r eh$, and $gf\omega^l hf$. Then the set*

$$S(e, f) = \{h \in M(e, f) : g < h \text{ forall } g \in M(e, f)\}$$

is called the sandwich set of e and f.

1. *$f, g \in \omega^r(e) \Rightarrow S(f, g)e = S(fe, ge)$*

The biordered set E is said to be regular if $S(e, f) \neq \emptyset \; \forall e, f \in E$

A ring is a set R together with two binary operations $'+', '\cdot'$ with the following properties.

1. The set $(R, +)$ is an abelian group.
2. The set $(R, \cdot)$ is a semigroup.
3. The operation $\cdot$ is distributive over $+$.

3 Biordered set of a Ring

Let $(R, +, .)$ be a ring. An element $e \in R$ is a multiplicative idempotent if $e \cdot e = e$ and an additive idempotent if $e + e = e$ and e is an idempotent in the ring R if and only if e is both an additive and a multiplicative idempotent. Denote E as the set of all multiplicative idempotents in R. In $(R, +, \cdot)$ define

$$a \oplus b = a + b - ab.$$

It is easy to see that $\oplus$ is an associative binary operation on R and both the additive reduct $(R, \oplus)$ and the multiplicative reduct $(R, \cdot)$ are semigroups. Further it can be seen that every multiplicative idempotent in $(R, \cdot)$ is an additive idempotent in $(R, \oplus)$ and hence the set of multiplicative idempotents E of $(R, \cdot)$ coinsides with the set of additive idempotents of $E^{\oplus}$ $(R, \oplus)$.

Lemma 3.1. *Let e, f be idempotents in R then,*

$$e \oplus f = e \Longleftrightarrow f\omega^r e$$
$$e \oplus f = f \Longleftrightarrow e\omega^l f$$

Proof. Suppose $e \oplus f = e$, then

$$e + f - ef = e \Rightarrow f - ef = 0 \Rightarrow f = ef \Rightarrow f\omega^r e.$$

Conversely, let $f\omega^r e$ then, $ef = f$. Consider $e \oplus f$, we have

$$e \oplus f = e + f - ef = e + f - f = e.$$

Similarly, let $e \oplus f = f$ then by definition,

$$e + f - ef = f \Rightarrow e - ef = 0 \Rightarrow ef = e \Rightarrow e\omega^l f.$$

Conversely, assume that $e\omega^l F$ then $ef = e$. Therefore,

$$e \oplus f = e + f - ef = e + f - e = f$$

□

It is easy to observe that the domain of both the binary operations $\cdot$ and $\oplus$ coincides and we denote this domain by D, for $(e, f) \in D$ either $(e, f) \in \omega^r \cup \omega^l$ or $(f, e) \in \omega^r \cup \omega^l$. In the first case either $f \oplus e = e$ or $e \oplus f = e$. If $f \oplus e = e, (e \oplus f)^2 = (e \oplus f) \oplus (e \oplus f) = e \oplus (f \oplus e) \oplus f = e \oplus e \oplus f = e \oplus f$ and so $e \oplus f \in E^{\oplus}$. Thus $e \oplus f \in E^{\oplus}$ whenever $(e, f) \in \omega^r \cup \omega^l$. Similarly, it can be seen that $e \oplus f \in E^{\oplus}$ when $(f, e) \in \omega^r \cup \omega^l$. Thus, by restricting the operation in $(R, \oplus, \cdot)$ to D we obtain the partial algebra $(D, \oplus)$ defining the operations in the ring R to $(D, \oplus)$, we obtain a partial algebra on $E^{\oplus}$. Now in the light of the biorder axioms we have the following Proposition.

Proposition 3.1. *Let e, f, g be idempotents in R. Then*

1. $e\omega^l f \Rightarrow e\omega f \oplus e\mathcal{L}f$
2. $g\omega^l f,\ e \in \omega^l(f) \cap \omega^l(g) \Rightarrow e \oplus g\omega^l e \oplus f$
3. $e\omega^l f\omega^l g \Rightarrow (f \oplus e) \oplus g = f \oplus g$
4. $f\omega^r g,\ e \in \omega^l(f) \cap \omega^l(g) \Rightarrow e \oplus (f \oplus g) = (e \oplus f) \oplus (e \oplus g)$

Proof. (1) $e\omega^l f$, so $e(f \oplus e) = e(f + e - fe) = e$ and $(f \oplus e)e = (f + e - fe)e = e$ that is $e\omega(f \oplus e)$. Also $(f \oplus e)f = (f + e - fe)f = f + ef - fef = f + e - fe = f \oplus e$ and $f(f \oplus e) = f(f + e - fe) = f$ that is $f \oplus e\mathcal{L}f$.

(2) $g\omega^l f$ and $e \in \omega^l(f) \cap \omega^l(g)$. Therefore,

$$(e \oplus g) \cdot (e \oplus f) = (e + g - eg) \cdot f = e \oplus g$$

Thus, $(e \oplus g)\omega^l(e \oplus f)$.

(3) $e\omega^l f\omega^l g$, we have $e \oplus f = f$, $f \oplus g = g$ and $e \oplus g = g$. Therefore,

$$f \oplus g = f \oplus (e \oplus g) = (f \oplus e) \oplus g.$$

(4) Since $f\omega^r g$, $e \in \omega^l(f) \cap \omega^l(g)$ we have, $f \oplus g = g$, $e \oplus f = f$ and $e \oplus g = g$. Therefore,

$$e \oplus (f \oplus g) = (e \oplus f) \oplus g = (e \oplus f) \oplus (e \oplus g).$$

□

Next we proceed to define the addictive sandwich set of the biordered set $E^{\oplus}$.

Proposition 3.2. *For $e, f \in E^{\oplus}$, let*

$$\tilde{M}(e,f) = \{g \in E_{\mathscr{R}} \,:\, e \in \omega^r(g) \text{ and } f \in \omega^l(g), \prec\}$$

where $\prec$ is defined by $h \prec g \Longleftrightarrow hg = gh = h$. Then $\tilde{M}(e,f)$ is a quasiordered set and the set

$$\tilde{S}(e,f) = \left\{h \in \tilde{M}(e,f) : h \prec g \text{ for all } g \in \tilde{M}(e,f)\right\}$$

is called the addictive sandwich set of e and f (in that order).

Proof. For $g, h \in \tilde{M}(e,f)$, then both gh and hg in $\tilde{M}(e,f)$ also $h \prec h$ and if $h \prec g$, $g \prec k$ then $h \prec k$. Thus $\tilde{M}(e,f)$ is a quasiordered set and $\tilde{S}(e,f)$ are minimal elements of $\tilde{M}(e,f)$. □

Lemma 3.2. *For any idempotents $e, f \in R$ and $h \in \tilde{S}(e,f)$ then $f \oplus h \oplus e = h$.*

Proof. Since $h \in \tilde{S}(e,f)$, we have $he = e$ and $fh = f$ thus

$$\begin{aligned} f \oplus h \oplus e &= (f \oplus h) + e - (f \oplus h)e \\ &= f + h - f + e - (f + h - fh)e \\ &= h. \end{aligned}$$

□

Remark 3.1. *For any two idempotents $e, f \in R$ and $e \neq f$ then $\tilde{S}(e,f)$ and $S(e,f)$ are disjoint.*

Example 3.1. *A complemented distributive lattice is called a Boolean lattice. Let $(L, \vee, \wedge)$ be a Boolean lattice. Then $(L, +, \cdot)$ where $e + f = e \vee f$ and $e \cdot f = e \wedge f$ is a ring. Now define $\oplus$ on $(L, +, \cdot)$ as follows*

$$e \oplus f = (e \wedge f') \vee (e' \wedge f)$$

so $e \oplus f = (e + f) - ef$ and $\mathscr{L} = (L, \oplus)$ is a semigroup and we denote the addictive idempotent set by $E^{\oplus}$. It should be noted that the set of multiplicative idempotents E and the set of all addictive idempotent set $E^{\oplus}$ coincides with L and $\mathscr{L}$ (ie., the lattice is a band with respect to both $\cdot$ and $\oplus$. Let us now describe the biordered set E as follows:
ω^r and ω^l, defined by $e\omega^r f \Rightarrow f \wedge e = e$ and $e\omega^l f \Rightarrow e \wedge f = e$ are quasiorders and $\omega = \omega^r \cap \omega^l$ is a partial order. Since $e \wedge f = f \wedge e$ we have $\omega^r = \omega^l = \omega$ on E. Also $M(e,f) = (\omega^l(e) \cap \omega^r(f), <)$ where $g < h \Leftrightarrow eg\omega^r eh$, $gf\omega^l hf$, and $S(e,f)$ the maximal elements of $M(e,f)$, thus $S(e,f) = \{e \wedge f\}$.
Next we define the addictive sandwitch set $E^{\oplus}$ as follows

$$\tilde{M}(e,f) = \{g \,:\, e\omega^r g \text{ and } f\omega^l g, \prec\}$$

where $h \prec g$ means $hg = gh = h$, thus we have $\tilde{M}(e,f) = \{e \vee f\}$ and

$$\tilde{S}(e,f) = \{e \vee f\}\,.$$

Example 3.2. *Consider the real quarternions $Q = \{q = q_0 + q_1 i + q_2 j + q_3 j \mid q_i \in R\}$. It is well known that with respect to the usual additin and multplication defined by the rule $i^2 = j^2 = k^2 = -1$ and $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$ is a noncommutative skewfield. The idempotent set is*

$$E_Q = \{e = (0,0,0,0), f = (1,0,0,0)\}$$

then $\omega^l(e) = \{e\}$ and $\omega^r(f) = \{e, f\}$, so $M(e,f) = \{e\} = S(e,f)$.
Now for $q, r \in Q$ define $q \oplus r = q + r - qr$, it is easy to obseve that $\mathscr{Q} = (Q, \oplus)$ is a semigroup and $E_{\mathscr{Q}} = E_Q$. The additive sandwitch set of $\mathscr{Q}$ is described as follows.

$$\tilde{M}(e,f) = \{g \in E_{\mathscr{R}} \,:\, e \in \omega^r(g) \text{ and } f \in \omega^l(g), \prec\}$$

since $e \in \omega^l(f)$ and $f \in \omega^r(f)$, we have $\tilde{M}(e,f) = \{f\}$ Also since

$$\tilde{S}(e,f) = \left\{h \in \tilde{M}(e,f) : h \prec g \text{ for all } g \in \tilde{M}(e,f)\right\}$$

we have $\tilde{S}(e,f) = \{f\}$.

Example 3.3. *Consider the set* $\mathscr{M}_2(\mathcal{Z})$ *of* 2×2 *matrices with integer entries. This is a non-commutative ring with usual addition and multiplication of matrices. The possible idempotents* $E_{\mathscr{R}}$ *in this ring are*

$$\begin{bmatrix} 0 & 0 \\ 0 & 0, \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

Let $e = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $f = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \in (E_{\mathscr{R}}, \cdot)$. *then*

$$\omega^l(e) = \{\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\} \text{ and } \omega^r(f) = \{\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}$$

Thus $M(e,f) = \{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, <\}$ *and so* $S(e,f) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$,

Now we proceed to describe the addictive sandwitch set, we have

$$\tilde{M}(e,f) = \{g : e \in \omega^r(g), f \in \omega^l(g), \prec\}$$

where $h \prec g$ *means* $hg = gh = h$. *Thus* $\tilde{M}(e,f) = \{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \prec\}$.

Thus

$$\tilde{S}(e,f) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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