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Continuous dependence of the solution of a stochastic differential equation with nonlocal conditions

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Abstract

In this paper we are concerned with a nonlocal problem of a stochastic differential equation that contains a Brownian motion. The solution contains both of mean square Riemann and mean square Riemann-Steltjes integrals, so we study an existence theorem for unique mean square continuous solution and its continuous dependence of the random data X_0 and the (non-random data) coefficients of the nonlocal condition a_k. Also, a stochastic differential equation with the integral condition will be considered.

Keywords: Integral condition, Brownian motion, unique mean square solution, continuous dependence, random data, non-random data, integral condition.

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1 Introduction

Many authors in the last decades studied a nonlocal problems of ordinary differential equations, the reader is referred to ([3]-[8]), and references therein.
Also the theory of stochastic differential equations, random fixed point theory, existence of solutions of stochastic differential equations by using successive approximation method and properties of these solutions have been extensively studied by several authors, especially those contain the Brownian motion as a formal derivative of the Gausian white noise, the Brownian motion $W(t), t \in R$, is defined as a stochastic process such that

$$W(0) = 0,\ E(W(t)) = 0,\ E(W(t))^2 = t$$

and $[W(t_1) - W(t_2)]$ is a Gaussian random variable for all $t_1, t_2 \in R$. The reader is referred to ([1]-[2]) and ([9]-[13]) and references therein.
Here we are concerned with the stochastic differential equation

$$dX(t) = f(t, X(t))dt + g(t)dW(t), \quad t \in (0, T] \tag{1.1}$$

with the nonlocal random initial condition

$$X(0) + \sum_{k=1}^{n} a_k X(\tau_k) = X_0, \quad a_k > 0,\ \tau_k \in (0, T), \tag{1.2}$$

where X_0 is a second order random variable independent of the Brownian motion $W(t)$ and a_k are positive real integers.

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The existence of a unique mean square solution will be studied. The continuous dependence on the random data X_0 and the non-random data a_k will be established. The problem (1.1) with the integral condition

$$X(0) + \int_0^T X(s)dv(s) = X_0. \tag{1.3}$$

will be considered.

2 Integral representation

Let $I = [0, T]$ and $C = C(I, L_2(\Omega))$ be the class of all mean square continuous second order stochastic process with the norm

$$\| X \|_C = \sup_{t\in[0,T]} \| X(t) \|_2 = \sup_{t\in[0,T]} \sqrt{E(X(t))^2}.$$

Throughout the paper we assume that the following assumptions hold

(H1) The function $f : [0, T] \times L_2(\Omega) \to L_2(\Omega)$ is mean square continuous.

(H2) There exists an integrable function $k : [0, T] \to R^+$, where

$$\sup_{t\in[0,T]} \int_0^t k(s)ds \leq m$$

such that the function f satisfies the mean square Lipschitz condition

$$\| f(t, X_1(t)) - f(t, X_2(t)) \|_2 \leq k(t) \| X_1(t) - X_2(t) \|_2 .$$

(H3) There exists a positive real number m_1 such that

$$\sup_{t\in[0,T]} | f(t, 0) | \leq m_1.$$

Now we have the following lemmas.

Lemma 2.1. *For a deterministic function $g(t) : I \to \Re^+$ and a Brownian motion $W(t)$*

$$\left\| \int_0^t g(s)dW(s) \right\|^2 = \int_0^t g^2(s)ds$$

Proof.

$$\begin{aligned}
\left\| \int_0^t g(s)dW(s) \right\|^2 &= E\left(\int_0^t g(s)dW(s)\right)^2 \\
&= E\left(\int_0^t g(s)dW(s)\right)\left(\int_0^t g(s)dW(s)\right) \\
&= E\left(\lim_{n\to\infty}\sum_{k=0}^{n-1} g(t_k)\Delta W(t_k)\right)\left(\lim_{n\to\infty}\sum_{k=0}^{n-1} g(t_k)\Delta W(t_k)\right) \\
&= \left(\lim_{n\to\infty}\sum_{k=0}^{n-1} g^2(t_k)E(\Delta W(t_k))^2\right) \\
&= \left(\lim_{n\to\infty}\sum_{k=0}^{n-1} g^2(t_k)(\Delta t_k)\right) \\
&= \int_0^t g^2(s)ds
\end{aligned}$$

This complete the proof. □

Lemma 2.2. *The solution of the problem (1.1)and(1.2) can be expressed by the integral equation*

$$X(t) = a\left(X_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f(s, X(s))ds - \sum_{k=1}^{m} a_k \int_0^{\tau_k} g(s)dW(s)\right) + \int_0^t f(s, X(s))ds + \int_0^t g(s)dW(s), \tag{2.1}$$

where $a = \left(1 + \sum_{k=1}^{n} a_k\right)^{-1}$.

Proof. . Integrating equation (1.1), we obtain

$$X(t) = X(0) + \int_0^t f(s, X(s))ds + \int_0^t g(s)dW(s)$$

and

$$X(\tau_k) = X(0) + \int_0^{\tau_k} f(s, X(s))ds + \int_0^{\tau_k} g(s)dW(s),$$

then

$$\sum_{k=1}^{n} a_k X(\tau_k) = \sum_{k=1}^{n} a_k X(0) + \sum_{k=1}^{n} a_k \int_0^{\tau_k} f(s, X(s))ds + \sum_{k=1}^{n} a_k \int_0^{\tau_k} g(s)dW(s)$$

$$X_0 - X(0) = \sum_{k=1}^{n} a_k X(0) + \sum_{k=1}^{n} a_k \int_0^{\tau_k} f(s, X(s))ds + \sum_{k=1}^{n} a_k \int_0^{\tau_k} g(s)dW(s)$$

and

$$\left(1 + \sum_{k=1}^{n} a_k\right) X(0) = X_0 - \sum_{k=1}^{n} a_k \int_0^{\tau_k} f(s, X(s))ds - \sum_{k=1}^{n} a_k \int_0^{\tau_k} g(s)dW(s),$$

then

$$X(0) = \left(1 + \sum_{k=1}^{n} a_k\right)^{-1} \left(X_0 - \sum_{k=1}^{n} a_k \int_0^{\tau_k} f(s, X(s))ds - \sum_{k=1}^{n} \int_0^{\tau_k} g(s)dW(s)\right).$$

Hence

$$X(t) = a\left(X_0 - \sum_{k=1}^{n} a_k \int_0^{\tau_k} f(s, X(s))ds - \sum_{k=1}^{n} a_k \int_0^{\tau_k} g(s)dW(s)\right) + \int_0^t f(s, X(s))ds + \int_0^t g(s)dW(s).$$

where $a = \left(1 + \sum_{k=1}^{n} a_k\right)^{-1}$. □

Now define the mapping

$$FX(t) = a\left(X_0 - \sum_{k=1}^{n} a_k \int_0^{\tau_k} f(s, X(s))ds - \sum_{k=1}^{n} a_k \int_0^{\tau_k} g(s)dW(s)\right) + \int_0^t f(s, X(s))ds + \int_0^t g(s)dW(s). \tag{2.2}$$

Then we can prove the following lemma.

Lemma 2.3. $F : C \to C$.

Proof. . Let $X \in C$, t_1 , $t_2 \in [0, T]$ such that $\mid t_2 - t_1 \mid < \delta$, then

$$FX(t_2) - FX(t_1) = \int_{t_1}^{t_2} f(s, X(s))ds + \int_{t_1}^{t_2} g(s)dW(s).$$

From assumption (H2) we have

$$\| f(t, X(t)) \|_2 - | f(t, 0) | \leq \| f(t, X(t)) - f(t, 0) \|_2 \leq k(t) \| X(t) \|_2,$$

then we have

$$\| f(t, X(t)) \|_2 \leq k(t) \| X(t) \|_2 + | f(t, 0) | \leq k(t) \| X \|_C + m_1.$$

So,

$$\| F X(t_2) - F X(t_1) \|_2 \leq \int_{t_1}^{t_2} ||f(s, X(s))||_2 \, ds + \| \int_{t_1}^{t_2} g(s) dW(s) \|_2,$$

using assumptions and lemma 2.1, we get

$$\| F X(t_2) - F X(t_1) \|_2 \leq \| X \|_C \int_{t_1}^{t_2} k(s) ds + m_1 (t_2 - t_1) + \sqrt{\int_{t_1}^{t_2} g^2(s) ds}$$

which proves that $F : C \to C$. □

3 Existence and uniqueness

For the existence of a unique continuous solution $X \in C$ of the problem (1.1)-(1.2), we have the following theorem.

Theorem 3.1. *Let the assumptions $(H1) - (H3)$ be satisfied. If $2m < 1$, then the problem (1.1)-(1.2) has a unique solution $X \in C$.*

Proof. Let X and $X^* \in C$, then

$$\begin{aligned}
& \| FX(t) - FX^*(t) \|_2 \\
= & \left\| \int_0^t [f(s, X(s)) - f(s, X^*(s))] ds - a \sum_{k=1}^{n} a_k \int_0^{\tau_k} [f(s, X(s)) - f(s, X^*(s))] ds \right\|_2 \\
\leq & \int_0^t ||f(s, X(s)) - f(s, X^*(s))||_2 \, ds + a \sum_{k=1}^{n} a_k \int_0^{\tau_k} ||f(s, X(s)) - f(s, X^*(s))||_2 \, ds \\
\leq & \; m \| X - X^* \|_C + \left[a \sum_{k=1}^{n} a_k \right] m \| X - X^* \|_C, \\
\leq & \left[1 + a \sum_{k=1}^{n} a_k \right] m \| X - X^* \|_C \\
\leq & \; 2m \| X - X^* \|_C .
\end{aligned}$$

Hence

$$\| FX - FX^* \|_C \leq 2m \| X - X^* \|_C .$$

If $2m < 1$, then F is contraction and there exists a unique solution $X \in C$ of the nonlocal stochastic problem (1.1)-(1.2), [2]. This solution is given by (2.1). □

4 Continuous dependence

Consider the stochastic differential equation (1.1) with the nonlocal condition

$$X(0) + \sum_{k=1}^{n} a_k X(\tau_k) = \tilde{X}_0 \quad , \tau_k \in (0, T) \tag{4.1}$$

Definition 4.1. *The solution* $X \in C$ *of the nonlocal problem (1.1)-(1.2) is continuously dependent (on the data* X_0*) if* $\forall \epsilon > 0\,,\ \exists \delta > 0$ *such that* $\parallel X_0 - \tilde{X}_0 \parallel_2 \leq \delta$ *implies that* $\parallel X - \tilde{X} \parallel_C \leq \epsilon$

Here, we study the continuous dependence (on the random data X_0) of the solution of the stochastic differential equation (1.1) and (1.2).

Theorem 4.2. *Let the assumptions* $(H1) - (H3)$ *be satisfied. Then the solution of the nonlocal problem (1.1)-(1.2) is continuously dependent on the random data* X_0*.*

Proof. Let

$$X(t) = a\left(X_0 - \sum_{k=1}^{n} a_k \int_0^{\tau_k} f(s, X(s))ds - \sum_{k=1}^{n} a_k \int_0^{\tau_k} g(s)dW(s)\right) + \int_0^t f(s, X(s))ds + \int_0^t g(s)dW(s)$$

be the solution of the nonlocal problem (1.1)-(1.2) and

$$\tilde{X}(t) = a\left(\tilde{X}_0 - \sum_{k=1}^{n} a_k \int_0^{\tau_k} f(s, \tilde{X}(s))ds - \sum_{k=1}^{n} a_k \int_0^{\tau_k} g(s)dW(s)\right) + \int_0^t f(s, \tilde{X}(s))ds + \int_0^t g(s)dW(s)$$

be the solution of the nonlocal problem (1.1) and (4.1). Then

$$\begin{aligned} X(t) - \tilde{X}(t) &= a[X_0 - \tilde{X}_0] - a\sum_{k=1}^{n} a_k \int_0^{\tau_k} [f(s, X(s)) - f(s, \tilde{X}(s))]ds \\ &+ \int_0^t [f(s, X(s)) - f(s, \tilde{X}(s))]ds. \end{aligned}$$

Using our assumptions, we get

$$\begin{aligned} \parallel X(t) - \tilde{X}(t) \parallel_2 &\leq a \parallel X_0 - \tilde{X}_0 \parallel_2 + a\sum_{k=1}^{n} a_k \int_0^{\tau_k} \parallel f(s, X(s)) - f(s, \tilde{X}(s)) \parallel_2 ds \\ &+ \int_0^t \parallel f(s, X(s)) - f(s, \tilde{X}(s)) \parallel_2 ds \\ &\leq a\delta + 2m \parallel X - \tilde{X} \parallel_2, \end{aligned}$$

then

$$\parallel X - \tilde{X} \parallel_C \ \leq \frac{a\delta}{1 - 2m} = \epsilon$$

This complete the proof. □

Now consider the stochastic differential equation (1.1) with the nonlocal condition

$$X(0) + \sum_{k=1}^{n} \tilde{a}_k X(\tau_k) = X_0 \quad , \tau_k \in (0, T) \tag{4.2}$$

Definition 4.2. *The solution* $X \in C$ *of the nonlocal problem (1.1)-(1.2) is continuously dependent (on the coefficient* a_k *of the nonlocal condition) if* $\forall \epsilon > 0\,,\ \exists \delta > 0$ *such that* $\mid a_k - \tilde{a}_k \mid \leq \delta$ *implies that* $\parallel X - \tilde{X} \parallel_C \leq \epsilon$

Here, we study the continuous dependence (on the coefficient a_k of the nonlocal condition) of the solution of the stochastic differential equation (1.1) and (1.2).

Theorem 4.3. *Let the assumptions* $(H1) - (H3)$ *be satisfied. Then the solution of the nonlocal problem (1.1)-(1.2) is continuously dependent on the coefficient* a_k *of the nonlocal condition.*

Proof. Let

$$X(t) = a\left(X_0 - \sum_{k=1}^{n} a_k \int_0^{\tau_k} f(s, X(s))ds - \sum_{k=1}^{n} a_k \int_0^{\tau_k} g(s)dW(s)\right) + \int_0^t f(s, X(s))ds + \int_0^t g(s)dW(s)$$

be the solution of the nonlocal problem (1.1)-(1.2) and

$$\tilde{X}(t) = \tilde{a}\left(X_0 - \sum_{k=1}^{n} \tilde{a}_k \int_0^{\tau_k} f(s, \tilde{X}(s))ds - \sum_{k=1}^{n} \tilde{a}_k \int_0^{\tau_k} g(s)dW(s)\right) + \int_0^t f(s, \tilde{X}(s))ds + \int_0^t g(s)dW(s)$$

be the solution of the nonlocal problem (1.1) and (4.2).

Then

$$\begin{aligned} X(t) - \tilde{X}(t) &= [a - \tilde{a}]X_0 + \int_0^t [f(s, X(s)) - f(s, \tilde{X}(s))]ds - \left[\sum_{k=1}^{n} a_k - \sum_{k=1}^{n} \tilde{a}_k\right] \int_0^{\tau_k} g(s)dW(s) \\ &- a\sum_{k=1}^{n} a_k \int_0^{\tau_k} f(s, X(s))ds + \tilde{a}\sum_{k=1}^{n} \tilde{a}_k \int_0^{\tau_k} f(s, \tilde{X}(s))ds. \end{aligned}$$

Now

$$|\, a - \tilde{a} \,| = \left|\frac{1}{1 + \sum_{k=1}^{n} a_k} - \frac{1}{1 + \sum_{k=1}^{n} \tilde{a}_k}\right| = \left|\frac{\sum_{k=1}^{n} (\tilde{a}_k - a_k)}{\left(1 + \sum_{k=1}^{n} a_k\right)\left(1 + \sum_{k=1}^{n} \tilde{a}_k\right)}\right| \leq \left|\sum_{k=1}^{n} (\tilde{a}_k - a_k)\right| \leq n\delta$$

and

$$\begin{aligned} &\tilde{a}\sum_{k=1}^{n} \tilde{a}_k \int_0^{\tau_k} f(s, \tilde{X}(s))ds - a\sum_{k=1}^{n} a_k \int_0^{\tau_k} f(s, X(s))ds \\ =\ & \tilde{a}\left(1 + \sum_{k=1}^{n} \tilde{a}_k\right) \int_0^{\tau_k} f(s, \tilde{X}(s))ds - a\left(1 + \sum_{k=1}^{n} a_k\right) \int_0^{\tau_k} f(s, X(s))ds \\ -\ & \tilde{a}\int_0^{\tau_k} f(s, \tilde{X}(s))ds + a\int_0^{\tau_k} f(s, X(s))ds \\ =\ & \tilde{a}(\tilde{a}^{-1}) \int_0^{\tau_k} f(s, \tilde{X}(s))ds - a(a^{-1}) \int_0^{\tau_k} f(s, X(s))ds \\ -\ & \tilde{a}\int_0^{\tau_k} f(s, \tilde{X}(s))ds + a\int_0^{\tau_k} f(s, X(s))ds \\ =\ & -\int_0^{\tau_k} [f(s, X(s)) - f(s, \tilde{X}(s))]ds + a\int_0^{\tau_k} f(s, X(s))ds - \tilde{a}\int_0^{\tau_k} f(s, \tilde{X}(s))ds \\ -\ & \tilde{a}\int_0^{\tau_k} f(s, X(s))ds + \tilde{a}\int_0^{\tau_k} f(s, X(s))ds \\ =\ & -\int_0^{\tau_k} [f(s, X(s)) - f(s, \tilde{X}(s))]ds + [a - \tilde{a}] \int_0^{\tau_k} f(s, X(s))ds \\ +\ & \tilde{a}\int_0^{\tau_k} [f(s, X(s)) - f(s, \tilde{X}(s))]ds \end{aligned}$$

and

$$\begin{aligned}\left[a\sum_{k=1}^{n}a_k-\tilde{a}\sum_{k=1}^{n}\tilde{a}_k\right]\int_0^{\tau_k}g(s)dW(s) &= \left[a\left(1+\sum_{k=1}^{n}a_k\right)-\tilde{a}\left(1+\sum_{k=1}^{n}\tilde{a}_k\right)\right]\int_0^{\tau_k}g(s)dW(s)\\ &- [a-\tilde{a}]\int_0^{\tau_k}g(s)dW(s)\\ &= \left[aa^{-1}-\tilde{a}\tilde{a}^{-1}\right]\int_0^{\tau_k}g(s)dW(s)-[a-\tilde{a}]\int_0^{\tau_k}g(s)dW(s)\\ &= -[a-\tilde{a}]\int_0^{\tau_k}g(s)dW(s).\end{aligned}$$

Then

$$\begin{aligned}\| X(t)-\tilde{X}(t)\|_2 &\leq n\delta \| X_0\|_2+\int_{\tau_k}^{t}\| f(s,X(s))-f(s,\tilde{X}(s))\|_2\, ds+n\delta\left\|\int_0^{\tau_k}g(s)dW(s)\right\|_2\\ &+ n\delta\,[m\| X\|_C+m_1T]+\tilde{a}\int_0^{\tau_k}\| f(s,X(s))-f(s,\tilde{X}(s))\|_2\, ds.\end{aligned}$$

Using our assumptions we get

$$\begin{aligned}\| X-\tilde{X}\|_C &\leq n\delta\| X_0\|_2+m\| X-\tilde{X}\|_C+n\delta\sqrt{\int_0^{\tau_k}g^2(s)ds}+n\delta\,[m\| X\|_C+m_1T]\\ &+ \tilde{a}m\| X-\tilde{X}\|_C,\end{aligned}$$

then

$$\begin{aligned}\| X-\widetilde{X}\|_C &\leq n\delta\left[\| X_0\|_2+m\| X\|_C+m_1T+\sqrt{\int_0^{\tau_k}g^2(s)ds}\right]+(1+\tilde{a})m\| X-\tilde{X}\|_C\\ &\leq n\delta\left[\| X_0\|_2+m\| X\|_C+m_1T+\sqrt{\int_0^{\tau_k}g^2(s)ds}\right]+2m\| X-\tilde{X}\|_C\,.\end{aligned}$$

Hence

$$\| X-\widetilde{X}\|_C \leq \frac{n\delta\left[\| X_0\|_2+m\| X\|_C+m_1T+\sqrt{\int_0^{\tau_k}g^2(s)ds}\right]}{1-2m}=\epsilon.$$

This complete the proof. □

5 Nonlocal Integral Condition

Let

$$a_k=v(t_k)-v(t_{k-1}),\ \tau_k\in\ (t_{k-1},t_k),$$

where

$$0\ <\ t_1\ <\ t_2\ <\ t_3\ <\ ....\ <\ T.$$

Then, the nonlocal condition (1.2) will be in the form

$$X(0) \; + \; \sum_{k=1}^{n} \; X(\tau_k) \; (v(t_k) - v(t_{k-1})) = \; X_0.$$

From the mean square continuity of the solution of the nonlocal problem (1.1)-(1.2), we obtain from [13]

$$\underset{n\to\infty}{l.i.m} \sum_{k=1}^{n} X(\tau_k) \; (v(t_k) - v(t_{k-1})) = \int_0^T \; X(s)dv(s),$$

that is, the nonlocal conditions (1.2) is transformed to the mean square Riemann-Steltjes integral condition

$$X(0) \; + \; \int_0^T \; X(s)dv(s) \; = \; X_0.$$

Now, we have the following theorem.

Theorem 5.4. *Let the assumptions (H1)-(H3) be satisfied, then the stochastic differential equation (1.1) with the nonlocal integral condition (1.3) has a unique mean square continuous solution represented in the form*

$$X(t) = a^\star \left(X_0 - \int_0^T \int_0^s f(\theta, X(\theta))d\theta dv(s) - \int_0^T \int_0^s g(\theta)dW(\theta)dv(s) \right) + \int_0^t f(\theta, X(\theta))d\theta + \int_0^t g(\theta)dW(\theta),$$

where $a^\star \; = (1 + v(T) - v(0))^{-1}$.

Proof. Taking the limit of equation (2.1) we get the proof. □

6 Conclusion

Here we defined the mean square continuous solution for the stochastic differential equation and proved the existence of unique solution of the problem (1.1)-(1.2), then we studied the continuous dependence of the solution of (1.1)-(1.2) on the initial random data and the nonrandom coefficient of the nonlocal condition .

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