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On the maximal and minimal solutions of a nonlocal problem of a delay stochastic differential equation

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Abstract

In this paper we are concerned with a problem of of a delay stochastic differential equation with nonlocal condition, the solution is represented as stochastic integral equation that contain mean square Riemann integral. We study the existence of at least mean square continuous solution for this problem. The existence of the maximal and minimal solutions will be proved.

Keywords: Nonlocal condition, delay equation, random Caratheodory function, stochastic Lebesgue dominated convergence theorem, at least mean square continuous solution, maximal solution, minimal solution.

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1 Introduction

The problems of differential equation with nonlocal condition studied recently by some authors, see ([3]-[5]) and ([7]-[8]) and references therein. Problems of the stochastic differential equations have been extensively studied by several authors in the last decades The reader is referred to ([1]-[2]), ([6]) and ([9]-[14]) and references therein.
Let $\phi : [0,T] \to [0,T]$ be continuous real-valued function such that $\phi(t) \leq t,\ t \in [0,T]$.
Here we are concerned with the delay stochastic differential equation

$$\frac{dX(t)}{dt} = f(t, X(\phi(t))), \quad t \in (0,T] \tag{1.1}$$

with the random nonlocal initial condition

$$X(0) + \sum_{k=1}^{m} a_k X(\tau_k) = X_0, \quad \tau_k \in (0,T), \tag{1.2}$$

where X_0 is a second order random variable and a_k are positive real numbers.
Our aim is to study the existence of at least mean square continuous solution of the problem (1.1)-(1.2). Also we define the maximal and minimal solution of the stochastic differential equation. Hence we study the existence of maximal and minimal solution of the problem (1.1)-(1.2).

2 Preliminaries

Here we give some preliminaries which will be needed in our work.

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Definition 2.1. *[13][Random Caratheodory function]*
Let X be a stochastic process and let $t \in I = [a, b]$, a and b are real numbers. A stochastic function $f(t, X(\omega))$ is called a Caratheodory function if it satisfies the following conditions

1. *$f(t, X(.))$ is measurable for every t,*

2. *$f(., X(\omega))$ is continuous for a.e. stochastic process X.*

Theorem 2.1. *[12][Schauder and Tychonoff theorem]*
Let Q be a closed bounded convex set in a Banach space and Let T be a completely continuous operator on Q such that $T(Q) \subset Q$. Then T has at least one fixed point in Q. That is, there is at least one $x^* \in Q$ such that $T(x^*) = x^*$.

Definition 2.2. *[10] A family of real random functions $(X_1(t), X_2(t), ..., X_k(t))$ is uniformly bounded in mean square sense if there exist a $\beta \in R$ (β is finite) such that $E(X_n^2(t)) < \beta$ for all $n \geq 1$ and all $t \in I = [a, b]$, where a, b are real numbers.*

Definition 2.3. *[10] A family of real random functions $(X_1(t), X_2(t), ..., X_k(t))$ is equicontinuous in mean square sense if for each $t \in I = [a, b]$, where a, b are real numbers and $\epsilon > 0$, there exist a $\delta > 0$ such that*

$$E([X_n(t_2) - X_n(t_1)]^2) < \epsilon, \ \forall \, n \geq 1 \quad \textit{when ever} \quad | \, t_2 - t_1 \, | < \delta.$$

Theorem 2.2. *[10][Arzela theorem]*
Every uniformly bounded equicontinuous family (sequence) of functions $(f_1(x), f_2(x), ..., f_k(x))$ has at least one subsequence which converges uniformly on the $I = [a, b]$, where a, b are real numbers

Theorem 2.3. *[11][Stochastic Lebesgue dominated convergence theorem]*
Let $X_n(t)$ be a sequence of random vectors (or functions) is converging to $X(t)$ such that

$$X(t) = \lim_{n\to\infty} X_n(t), \quad \textit{a.s.},$$

and $X_n(t)$ is dominated by an integrable function $a(t)$ such that $\| X_n(t) \|_2 \leq a(t)$. Then

1. *$E[\lim_{n\to\infty} X_n] = \lim_{n\to\infty} E[X_n]$ and*

2. *$E[X_n(t) - X(t)] \to 0 \quad as \quad n \to \infty$*

where a.s. means that it happens with probability one.

3 Integral representation

Let $I = [0, T]$ and $C = C(I, L_2(\Omega))$ be the class of all mean square continuous second order stochastic process with the norm

$$\| X \|_C = \sup_{t\in[0,T]} \| X(t) \|_2 = \sup_{t\in[0,T]} \sqrt{E\,(X(t))^2}.$$

Throughout the paper we assume that the following assumptions hold

i- The functions $f : [0, T] \times L_2(\Omega) \to L_2(\Omega)$ is Caratheodory function in mean square sense.

ii- There exists an integrable function $l(t) \in L^1$ such that

$$\| f(t, X(t)) \|_2 \leq l(t)\,, \quad \forall (t, X) \in I \times L_2(\Omega)$$

with $\left[\sup_{t\in[0,T]} \int_0^t l(s)ds \leq M \right]$, where M is a positive real number.

Now we have the following lemma.

Lemma 3.1. *The solution of the nonlocal stochastic problem (1.1) and (1.2) can be expressed by the stochastic integral equation*

$$X(t) = a\left(X_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f(s, X(\phi(s)))ds\right) + \int_0^t f(s, X(\phi(s)))ds \tag{3.1}$$

where $a = \left(1 + \sum_{k=1}^{m} a_k\right)^{-1}$.

Proof. . Integrating equation (1.1), we obtain

$$X(t) = X(0) + \int_0^t f(s, X(\phi(s)))ds$$

and

$$X(\tau_k) = X(0) + \int_0^{\tau_k} f(s, X(\phi(s)))ds,$$

then

$$\sum_{k=1}^{m} a_k X(\tau_k) = \sum_{k=1}^{m} a_k X(0) + \sum_{k=1}^{m} a_k \int_0^{\tau_k} f(s, X(\phi(s)))ds,$$

$$X_0 - X(0) = \sum_{k=1}^{m} a_k X(0) + \sum_{k=1}^{m} a_k \int_0^{\tau_k} f(s, X(\phi(s)))ds$$

and

$$\left(1 + \sum_{k=1}^{m} a_k\right) X(0) = X_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f(s, X(\phi(s)))ds,$$

then

$$X(0) = \left(1 + \sum_{k=1}^{m} a_k\right)^{-1} \left(X_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f(s, X(\phi(s)))ds\right).$$

Hence

$$X(t) = a\left(X_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f(s, X(\phi(s)))ds\right) + \int_0^t f(s, X(\phi(s)))ds,$$

where $a = \left(1 + \sum_{k=1}^{m} a_k\right)^{-1}$. □

4 Existence of at least mean square continuous solution

For the existence of at least continuous solution $X \in C$ of the stochastic problem (1.1) and (1.2), we have the following theorem.

Theorem 4.4. *Let the assumptions (i)-(ii) be satisfied, then the problem (1.1)-(1.2) has at least a solution* $X \in C$ *given by the stochastic integral equation (3.1).*

Proof. . Consider in the space C, the set Q such that

$$Q \;=\; \{X \in C : \| X \|_C \leq \beta;\ \ \beta \text{ is a positive real number}\}$$

Now for each $X(t) \in Q$ we can define the operator H by

$$HX(t) = a\left(X_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f(s, X(\phi(s)))ds\right) + \int_0^t f(s, X(\phi(s)))ds$$

we shall prove that $HX(t) \in Q$. For that let $X(t) \in Q$, then

$$\begin{aligned}
\parallel HX(t) \parallel_2 &\leq a \parallel X_0 \parallel_2 + a\sum_{k=1}^{m} a_k \int_0^{\tau_k} \parallel f(s, X(\phi(s))) \parallel_2 ds + \int_0^t \parallel f(s, X(\phi(s))) \parallel_2 ds \\
&\leq a \parallel X_0 \parallel_2 + a\sum_{k=1}^{m} a_k \int_0^{\tau_k} l(\phi(s))ds + \int_0^t l(\phi(s))ds \\
&\leq a \parallel X_0 \parallel_2 + a\sum_{k=1}^{m} a_k \int_0^{\tau_k} l(s)ds + \int_0^t l(s)ds \\
&\leq a \parallel X_0 \parallel_2 + a\sum_{k=1}^{m} a_k M + M.
\end{aligned}$$

Let $a \parallel X_0 \parallel_2 + a\sum_{k=1}^{m} a_k M + M = \beta$, β is clearly a positive real number, then ($\parallel HX \parallel_C \leq \beta$), so $HX \in Q$ and hence $HQ \subset Q$ and is also uniformly bounded.
For $t_1, t_2 \in R^+$, $t_1 < t_2$, let $\mid t_2 - t_1 \mid < \delta$, then

$$\parallel HX(t_2) - HX(t_1) \parallel_2 \leq \int_{t_1}^{t_2} \parallel f(s, X(\phi(s))) \parallel_2 ds \leq \int_{t_1}^{t_2} l(s)ds \leq M.$$

Then $\{HX\}$ is a class of equicontinuous functions. Therefore the operator H is equicontinuous and uniformly bounded.
Suppose that $\{X_n\} \in C$ such that $X_n \to X$ in mean square sense.
So,

$$\begin{aligned}
\underset{n\to\infty}{l.i.m}\, HX_n(t) &= \underset{n\to\infty}{l.i.m} \left[aX_0 - a\sum_{k=1}^{m} a_k \int_0^{\tau_k} f(s, X_n(\phi(s)))ds \right] + \underset{n\to\infty}{l.i.m} \left[\int_0^t f(s, X_n(\phi(s)))ds \right] \\
&= aX_0 - \left(a\sum_{k=1}^{m} a_k \right) \underset{n\to\infty}{l.i.m} \left[\int_0^{\tau_k} f(s, X_n(\phi(s)))ds \right] + \underset{n\to\infty}{l.i.m} \left[\int_0^t f(s, X_n(\phi(s)))ds \right].
\end{aligned}$$

Using our assumptions and then applying stochastic Lebesgue dominated convergence theorem, we get

$$\begin{aligned}
\underset{n\to\infty}{l.i.m}\, HX_n(t) &= aX_0 - a\sum_{k=1}^{m} a_k \int_0^{\tau_k} \underset{n\to\infty}{l.i.m} [f(s, X_n(s))]ds + \int_0^t \underset{n\to\infty}{l.i.m} [f(s, X_n(\phi(s)))]ds \\
&= aX_0 - a\sum_{k=1}^{m} a_k \int_0^{\tau_k} [f(s, \underset{n\to\infty}{l.i.m} X_n(\phi(s)))]ds + \int_0^t [f(s, \underset{n\to\infty}{l.i.m} X_n(\phi(s)))]ds \\
&= aX_0 - a\sum_{k=1}^{m} a_k \int_0^{\tau_k} f(s, X(\phi(s)))ds + \int_0^t f(s, x(\phi(s)))ds \\
&= HX(t)
\end{aligned}$$

This proves that H is continuous operator, then H is continuous and compact.

Then H has a fixed point $X \in C$ which proves that there exists at least one solution of the stochastic differential equation (1.1)-(1.2) given by (3.1). □

5 Maximal and minimal solution

Now we give the following definition.

Definition 5.4. *Let $q(t)$ be a solution of the problem (1.1)-(1.2), then $q(t)$ is said to be a maximal solution of (1.1)-(1.2) if every solution $X(t)$ of (1.1)-(1.2) satisfies the inequality*

$$\| X(t) \|_2 < \| q(t)) \|_2 .$$

A minimal solution $s(t)$ can be defined by similar way by reversing the above inequality i.e.

$$\| X(t) \|_2 > \| s(t) \|_2 .$$

In this section f assumed to satisfy the following definition.

Definition 5.5. *The functions $f : [0,T] \times L_2(\Omega) \to L_2(\Omega)$ is said to be stochastically decreasing if for any $X, Y \in L_2(\Omega)$ satisfying*

$$\| X(t) \|_2 < \| Y(t) \|_2$$

implies that

$$\| f(t, X(t)) \|_2 < \| f(t, Y(t)) \|_2 .$$

Now we have the following lemma.

Lemma 5.2. *Let the assumptions (i)-(ii) be satisfied and let $X, Y \in L_2(\Omega)$ satisfying*

$$\| X(t) \|_2 \leq a \left(\| X_0 \|_2 + \sum_{k=1}^{m} a_k \int_0^{\tau_k} \| f(s, X(\phi(s))) \|_2 \, ds \right) + \int_0^t ||f(s, X(\phi(s)))||_2 \, ds$$

and

$$\| Y(t) \|_2 \geq a \left(\| X_0 \|_2 + \sum_{k=1}^{m} a_k \int_0^{\tau_k} \| f(s, Y(\phi(s))) \|_2 \, ds \right) + \int_0^t ||f(s, Y(\phi(s)))||_2 \, ds.$$

If $f(t;x)$ is stochastically decreasing function . Then

$$\| X(t) \|_2 < \| Y(t) \|_2 \tag{5.1}$$

Proof. . Let the conclusion (5.1) be false, then there exists t_1 such that

$$\| X(t_1) \|_2 = \| Y(t_1) \|_2, \ \ t_1 > 0 \tag{5.2}$$

and

$$\| X(t) \|_2 < \| Y(t) \|_2, \ \ 0 < t < t_1 \tag{5.3}$$

since $f(t;x)$ satisfies the definition (5.5)and using equation (5.3), we get

$$\begin{aligned} \| X(t_1) \|_2 &\leq a \left(\| X_0 \|_2 + \sum_{k=1}^{m} a_k \int_0^{\tau_k} \| f(s, X(\phi(s))) \|_2 \, ds \right) + \int_0^{t_1} \| f(s, X(\phi(s))) \|_2 \, ds \\ &< a \left(\| X_0 \|_2 + \sum_{k=1}^{m} a_k \int_0^{\tau_k} \| f(s, Y(\phi(s))) \|_2 \, ds \right) + \int_0^{t_1} \| f(s, Y(\phi(s))) \|_2 \, ds \\ &< \| Y(t) \|_2, \ \ 0 < t < t_1, \end{aligned}$$

which contradicts equation (5.2), then

$$\| X(t) \|_2 < \| Y(t) \|_2 .$$

□

Now we have the following theorem.

Theorem 5.5. *Let the assumptions (i)-(ii) be satisfied. If $f(t, X(t))$ satisfies the definition (5.5), then there exist a maximal solution of the problem (1.1)-(1.2).*

Proof. . Firstly we shall prove the existence of the maximal solution of the problem.
Let $\epsilon > 0$ be given. Now consider the integral equation

$$X_\epsilon(t) = a\left(X_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f_\epsilon(s, X_\epsilon(\phi(s)))ds\right) + \int_0^t f_\epsilon(s, X_\epsilon(\phi(s)))ds, \tag{5.4}$$

where

$$f_\epsilon(t, X_\epsilon(t)) = f(s, X_\epsilon(t)) + \epsilon$$

Clearly the function $f_\epsilon(t, X_\epsilon(t))$ satisfies the conditions (i)-(ii) and

$$\| f_\epsilon(t, X_\epsilon(t)) \|_2 \leq l(t) + \epsilon = \grave{l}(t),$$

then equation (5.4) is a solution of the problem (1.1)-(1.2) according to Theorem (4.4).
Now let ϵ_1 and ϵ_2 be such that $0 < \epsilon_2 < \epsilon_1 < \epsilon$ Then

$$\begin{aligned} X_{\epsilon_1}(t) &= a\left(X_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f_{\epsilon_1}(s, X_{\epsilon_1}(\phi(s)))ds\right) + \int_0^t f_{\epsilon_1}(s, X_{\epsilon_1}(\phi(s)))ds, \\ &= a\left(X_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} (f(s, X_{\epsilon_1}(\phi(s))) + \epsilon_1)\, ds\right) + \int_0^t (f(s, X_{\epsilon_1}(\phi(s))) + \epsilon_1)\, ds, \end{aligned}$$

this implies that

$$\begin{aligned} \| X_{\epsilon_1}(t) \|_2 &\geq a \| X_0 \|_2 + a\sum_{k=1}^{m} a_k \int_0^{\tau_k} ||f(s, X_{\epsilon_1}(\phi(s))) + \epsilon_1||_2\, ds + \int_0^t ||f(s, X_{\epsilon_1}(\phi(s))) + \epsilon_2||_2\, ds \\ &\geq a \| X_0 \|_2 + a\sum_{k=1}^{m} a_k \int_0^{\tau_k} ||f(s, X_{\epsilon_1}(\phi(s))) + \epsilon_2||_2\, ds + \int_0^t ||f(s, X_{\epsilon_1}(\phi(s))) + \epsilon_2||_2\, ds, \ \ \epsilon_2 < \epsilon_1 \end{aligned} \tag{5.5}$$

and

$$\| X_{\epsilon_2}(t) \|_2 \leq a\left(\| X_0 \|_2 + \sum_{k=1}^{m} a_k \int_0^{\tau_k} ||(f(s, X_{\epsilon_2}(\phi(s))) + \epsilon_2)||_2\, ds\right) + \int_0^t ||(f(s, X_{\epsilon_2}(\phi(s))) + \epsilon_2)||_2\, ds. \tag{5.6}$$

Using Lemma (5.2), then equations (5.5) and (5.6) implies

$$\| X_{\epsilon_2}(t) \|_2 < \| X_{\epsilon_1}(t) \|_2$$

As shown before in the proof of Theorem (4.4) the family of functions $x_\epsilon(t)$ defined by equation (3.1) is uniformly bounded and equicontinuous functions. Hence by Arzela Theorem, there exists a decreasing sequence ϵ_n such that $\epsilon \to 0$ as $n \to \infty$ and $\lim_{n\to\infty} X_{\epsilon_n}(t)$ exists uniformly in C and denote this limit by $q(t)$, then from the continuity of the function f_{ϵ_n} in the second argument and applying Lebesgue dominated convergence Theorem, we get

$$q(t) = \lim_{n\to\infty} X_{\epsilon_n}(t)$$

which proves that $q(t)$ is a solution of the problem (1.1)-(1.2)
Finally, we shall show that $q(t)$ is the maximal solution of the problem (1.1)-(1.2). To do this, let $X(t)$ be any solution of the problem (1.1)-(1.2).
Then

$$\| X_\epsilon(t) \|_2 \geq a \| X_0 \|_2 + a\sum_{k=1}^{m} a_k \int_0^{\tau_k} ||f(s, X_\epsilon(\phi(s))) + \epsilon||_2\, ds + \int_0^t ||f(s, X_\epsilon(\phi(s))) + \epsilon||_2\, ds$$

and

$$\| X(t) \|_2 \leq a \left(||X_0||_2 + \sum_{k=1}^{m} a_k \int_0^{\tau_k} \| f(s, X_\epsilon(\phi(s))) \|_2 \, ds \right) + \int_0^t \| f(s, X(\phi(s))) \|_2 \, ds.$$

Applying Lemma (5.2), we get

$$\| X_\epsilon(t) \|_2 > \| X(t) \|_2$$

from the uniqueness of the maximal solution (see [6]), it is clear that $X_\epsilon(t)$ tends to $q(t)$ uniformly as $\epsilon \to 0$. □

By similar way as done above we can prove that $s(t)$ is the minimal solution of the problem (1.1)-(1.2). The maximal and minimal solutions of the problem (1.1)-(1.2) can be defined in the same fashion as done above. If the function f assumed to satisfy the following definition.

Definition 5.6. *The functions $f : [0, T] \times L_2(\Omega) \to L_2(\Omega)$ is said to be stochastically increasing if for any $X, Y \in L_2(\Omega)$ satisfying*

$$\| X(t) \|_2 < \| Y(t) \|_2$$

implies that

$$\| f(t, X(t)) \|_2 > \| f(t, Y(t)) \|_2 .$$

Now we have the following theorem.

Theorem 5.6. *Let the assumptions (i)-(ii) be satisfied. If $f(t, X)$ satisfies the definition (5.6), then there exist a minimal solution of the problem (1.1)-(1.2).*

6 Examples

Here, as an application of our results, we give the following two examples.

Example 6.1. *Let $\beta \in (0, 1]$. As ϕ, one can take, for example $\phi(t) = \beta t$.*

Let the assumptions of Theorem (4.4) be satisfied. Then the problem

$$\frac{dX(t)}{dt} = f(t, X(\beta t)), \quad t \in (0, T]$$

with the nonlocal random initial condition

$$X(0) + \sum_{k=1}^{n} a_k X(\tau_k) = X_0, \;\; \tau_k \in (0, T),$$

has at least one solution $X \in C([0, T], L^2(\Omega))$.

Example 6.2. *Let the assumptions of Theorem (4.4) be satisfied, let $\gamma \geq 1$. As ϕ, one can tack, for example $\phi(t) = t^\gamma$. Then the problem*

$$\frac{dX(t)}{dt} = f(t, X(t^\gamma)), \quad t \in (0, 1]$$

with the nonlocal random initial condition

$$X(0) + \sum_{k=1}^{n} a_k X(\tau_k) = X_0, \;\; \tau_k \in (0, 1),$$

has at least one solution $X \in C([0, 1], L^2(\Omega))$.

7 Conclusion

Here we defined the mean square solution for the stochastic differential equation and proved the existence of at least one solution of the problem (1.1)-(1.2), then we proved the existence of the maximal and minimal solution of (1.1)-(1.2).

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