

Coefficient Estimates for Bazilevič Ma-Minda Functions in the Space of Sigmoid Function

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Abstract

In this work, the authors investigated the coefficient estimates for Bazilevič Ma-Minda Functions for the class $T_n^\alpha(\lambda, \beta, l, \Phi)$. The first few coefficient bounds for this class were obtained and also the relevant connection to Fekete-Szegö theorem and were briefly discussed. Our results serve as a new generalization in this direction and gives birth to many corollaries.

Keywords: Analytic Function, Univalent Function, Starlike Function, Convex Function, Bazilevič Function, Subordination, Sigmoid Function, Fekete-Szegö Inequality.

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1 Introduction

In the twentieth century, the theory of special functions was overshadowed by other fields like functional analysis, real analysis, algebra, topology, differential equations and so on. These functions do not have specific definitions but they constitute an information process that is inspired by the way biological nervous system such as the brain processes information. This information process contains large numbers of highly interconnected elements (neurons) working together to perform specific tasks.

Special functions can be categorized into three, namely ramp function, sigmoid function and threshold function. The most popular of the functions is the sigmoid function because of its gradient descent algorithm. It can be evaluated by truncated series expansion (see details in [5], [9] and [11]).

The sigmoid function of the form

$$g(z) = \frac{1}{1 + e^{-z}} \quad (1.1)$$

is differentiable and has the following properties:

- (i) it outputs real numbers between 0 and 1.
- (ii) it maps a very large input domain to a small range of outputs.
- (iii) it never loses information because it is a one-to-one function.
- (iv) it increases monotonically.

The four properties show that sigmoid function is very useful in geometric functions theory.

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Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in U) \tag{1.2}$$

which are analytic in the open disk $U = \{z : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$.

A domain $U \subset \mathbb{C}$ is convex if the line segment joining any two points in U lies entirely in U , while a domain is starlike with respect to a point $\omega_0 \in U$ if the line segment joining any point of U to ω_0 lies inside U . A function $f \in A$ is starlike if $f(U)$ is a starlike domain with respect to the origin and convex if $f(U)$ is convex.

Recall that starlike and convex functions are denoted by ST and CV respectively and analytically written as $Re \frac{zf'(z)}{f(z)} > 0$ and $Re \left(1 + \frac{zf''(z)}{f'(z)}\right) > 0$. Starlike and convex functions of type α are denoted by $ST(\alpha)$ and $CV(\alpha)$ respectively and characterized by $Re \frac{zf'(z)}{f(z)} > \alpha$ and $Re \left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha$ where $\alpha : 0 \leq \alpha < 1$ (see detail in [2]).

The two functions f and g are analytic in the open unit disk U . We say f is subordinate to g written as $f < g \in U$ if there exists a Schwarz function $w(z)$ which is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. It follows from Schwarz lemma that $f(z) < g(z) \quad (z \in U) \implies f(0) = g(0)$ and $f(U) \subset g(U)$ (see details in [8]).

Ma and Minda[7] unified various subclasses of starlike and convex functions for which either of the quantity $\frac{zf'(z)}{f(z)}$ or $1 + \frac{zf''(z)}{f'(z)}$ is subordinate to a more general superordinate function. For this purpose, they considered an analytic function φ with positive real part in the open unit disk U , $\varphi(0) = 1$ and $\varphi'(0) > 0$ and φ maps U onto a region starlike with respect to 1 and symmetric with respect to the real axis. The class of Ma-Minda starlike function consists of functions $f \in A$ satisfying the subordination $\frac{zf'(z)}{f(z)} < \varphi(z)$ and Ma-Minda convex function consists of functions $f \in A$ satisfying subordination $1 + \frac{zf''(z)}{f'(z)} < \varphi(z)$ (detail in [2]).

Lemma 1.1 (Pommerenke[13]). If a function $p \in P$ is given by

$$p(z) = 1 + p_1z + p_2z^2 + \dots \quad (z \in U) \tag{1.3}$$

then $|p_k| \leq 2 \quad (k \in \mathbb{N})$, where P is the class of Caratheodory function, analytic in U for which $p(0) = 1$ and $Re p(z) > 0 \quad (z \in U)$.

Let $\alpha > 0$ (α is real), then

$$f(z)^\alpha = \left(z + \sum_{k=2}^{\infty} a_k z^k \right)^\alpha \tag{1.4}$$

which gives

$$f(z)^\alpha = (z + a_2z^2 + a_3z^3 + a_4z^4 + \dots)^\alpha \tag{1.5}$$

Or, equivalently

$$f(z)^\alpha = (z(1 + a_2z + a_3z^2 + a_4z^3 + \dots))^\alpha \tag{1.6}$$

Using simple expansion for (1.6), we have

$$f(z)^\alpha = z^\alpha \left(1 + \alpha(a_2z + a_3z^2 + a_4z^3 + \dots) + \frac{\alpha(\alpha - 1)}{2!} (a_2z + a_3z^2 + a_4z^3 + \dots)^2 + \dots \right) \tag{1.7}$$

Since the expansion continues, then

$$f(z)^\alpha = z^\alpha \left(1 + \alpha(a_2z + a_3z^2 + a_4z^3 + \dots) \right)$$

which implies

$$f(z)^\alpha = z^\alpha + \alpha a_2 z^{\alpha+1} + \alpha a_3 z^{\alpha+2} + \alpha a_4 z^{\alpha+3} + \dots$$

This finally gives

$$f(z)^\alpha = z^\alpha + \sum_{k=2}^{\infty} a_k(\alpha) z^{\alpha+k-1} \quad (1.8)$$

Catas et al.[3] defined the Catas Operator as follows:

$$\begin{aligned} I^0(\lambda, l) &: A \rightarrow A \\ I^0(\lambda, l)f(z) &= f(z) \\ I^1(\lambda, l)f(z) &= (I(\lambda, l)f(z)) \left(\frac{1-\lambda+l}{1+l} \right) + (I(\lambda, l)f(z)) \left(\frac{\lambda z}{1+l} \right) = z + \sum_{k=2}^{\infty} \left(\frac{1+\lambda(k-1)+l}{1+l} \right) a_k z^k \end{aligned}$$

and

$$I^2(\lambda, l)f(z) = (I^1(\lambda, l)f(z)) \left(\frac{1-\lambda+l}{1+l} \right) + (I^1(\lambda, l)f(z)) \left(\frac{\lambda z}{1+l} \right) = z + \sum_{k=2}^{\infty} \left(\frac{1+\lambda(k-1)+l}{1+l} \right)^2 a_k z^k$$

In general,

$$I^n(\lambda, l)f(z) = I(\lambda, l)(I^{n-1}(\lambda, l)f(z)) = z + \sum_{k=2}^{\infty} \left(\frac{1+\lambda(k-1)+l}{1+l} \right)^n a_k z^k \quad (1.9)$$

Applying (1.9) in (1.8), we have

$$I^n(\lambda, l)f(z)^\alpha = \left(\frac{1+\lambda(\alpha-1)+l}{1+l} \right)^n z^\alpha + \sum_{k=2}^{\infty} \left(\frac{1+\lambda(\alpha+k-2)+l}{1+l} \right)^n a_k(\alpha) z^{\alpha+k-1} \quad (1.10)$$

where $n \in N_0$, $\alpha > 0$ (α is real), $\lambda \geq 0$, $l \geq 0$.

Oladipo and Olatunji[10] used (1.10) to define a class $T_n^\alpha(\lambda, \beta, l)$ with geometric condition satisfying

$$Re \frac{I^n(\lambda, l)f(z)^\alpha}{\left(\frac{1+\lambda(\alpha-1)+l}{1+l} \right)^n z^\alpha} > \beta \quad (1.11)$$

where $n \in N_0$, $\alpha > 0$ (α is real), $\lambda \geq 0$, $l \geq 0$ and $0 \leq \beta < 1$. The first few coefficient bounds for the class were obtained and the coefficient inequalities for the class were derived by employing Hayami's method [6]. By specializing the parameters involved in (1.11), we obtain various subclasses of analytic functions studied by [1], [12], [14], [15] and so on.

In this work, the authors defined a new class of functions denoted by $T_n^\alpha(\lambda, \beta, l, \Phi)$ as related to modified sigmoid function with geometric condition satisfying

$$\frac{Re \frac{I^n(\lambda, l)f(z)^\alpha}{\left(\frac{1+\lambda(\alpha-1)+l}{1+l} \right)^n z^\alpha} - \beta}{1 - \beta} < \varphi(z) \quad (1.12)$$

where $n \in N_0$, $\alpha > 0$ (α is real), $\lambda \geq 0$, $l \geq 0$ and $0 \leq \beta < 1$. The first few coefficient estimates for the class are obtained. Also, the relevant connection to Fekete-Szegő theorem are briefly discussed.

For the purpose of our results, we require the following lemmas.

Lemma 1.2 (Fadipe-Joseph et al.[5]). Let g be a sigmoid function and

$$\Phi(z) = 2g(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n \right)^m \quad (1.13)$$

then $\Phi(z) \in P$, $|z| < 1$ where $\Phi(z)$ is a modified sigmoid function.

Lemma 1.3 (Fadipe-Joseph et al.[5]). Let

$$\Phi_{m,n}(z) = 2g(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n \right)^m \quad (1.14)$$

then $|\Phi_{m,n}(z)| < 2$.

Lemma 1.4 (Fadipe-Joseph et al.[5]). If $\Phi(z) \in P$ and it is starlike, then f is a normalized univalent function of the form (1.2).

Setting $m = 1$, Fadipe-Joseph et al.[5] remarked that $\Phi(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ where $c_n = \frac{(-1)^{n+1}}{(2n)!}$. As such, $|c_n| \leq 2, n = 1, 2, 3, \dots$ and the result is sharp for each n .

2 Coefficient Estimates

In the sequel, it is assumed that φ is an analytic function with positive real part in the open unit disk U , with $\varphi(0) = 1, \varphi'(0) > 0$ and $\varphi(U)$ is symmetric with respect to the real axis. Such a function has a series expansion of the form

$$\varphi(z) = 1 + \beta_1 z + \beta_2 z^2 + \beta_3 z^3 + \beta_4 z^4 + \dots \quad (\beta_1 > 0) \tag{2.15}$$

For functions in the class $T_n^\alpha(\lambda, \beta, l, \Phi)$, the following results are obtained.

Theorem 2.1. *If $f(z)^\alpha \in T_n^\alpha(\lambda, \beta, l, \Phi)$ is given by (1.12), then*

$$|a_2(\alpha)| \leq \frac{(1 - \beta)B_1}{4\alpha \left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^n} \tag{2.16}$$

$$|a_3(\alpha)| \leq \frac{(1 - \beta) \left[2\alpha(B_2 - B_1) \left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^{2n} - (\alpha - 1)(1 - \beta)B_1^2 \left(\frac{1+\lambda(\alpha+1)+l}{1+\lambda(\alpha-1)+l}\right)^n \right]}{32\alpha^2 \left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^{2n} \left(\frac{1+\lambda(\alpha+1)+l}{1+\lambda(\alpha-1)+l}\right)^n} \tag{2.17}$$

$$|a_4(\alpha)| \leq \frac{2(1 - \beta)(3B_3 - 6B_2 - B_1)}{384\alpha \left(\frac{1+\lambda(\alpha+2)+l}{1+\lambda(\alpha-1)+l}\right)^n} - \frac{(\alpha - 1)(1 - \beta)^3 B_1}{384\alpha^3 \left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^{3n}} \left\{ \frac{3 \left[2\alpha(B_2 - B_1) \left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^{2n} - (\alpha - 1)(1 - \beta)B_1^2 \left(\frac{1+\lambda(\alpha+1)+l}{1+\lambda(\alpha-1)+l}\right)^n \right]}{\left(\frac{1+\lambda(\alpha+1)+l}{1+\lambda(\alpha-1)+l}\right)^n} \right\} + \frac{(\alpha - 2)(1 - \beta)^2 B_1^3}{384\alpha} \tag{2.18}$$

Proof. Let $f(z)^\alpha \in T_n^\alpha(\lambda, \beta, l, \Phi)$. Then there are analytic functions $u : U \rightarrow U$ with $u(0) = 0$ satisfying

$$\frac{I^n(\lambda, l)f(z)^\alpha}{\left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^n z^\alpha} - \beta}{1 - \beta} = \varphi(u(z)) \tag{2.19}$$

Define the function $\Phi(z)$ by

$$\Phi(z) = \frac{1 + u(z)}{1 - u(z)} = 1 + \frac{1}{2}z - \frac{1}{24}z^3 + \frac{1}{240}z^5 - \frac{1}{64}z^6 + \frac{779}{20160}z^7 - \dots \tag{2.20}$$

or, equivalently

$$u(z) = \frac{\Phi(z) - 1}{\Phi(z) + 1} = \frac{1}{4}z - \frac{1}{16}z^2 - \frac{1}{192}z^3 - \frac{5}{768}z^4 - \frac{13}{15360}z^5 + \dots \tag{2.21}$$

In view of (2.19), (2.20) and (2.21), clearly

$$\frac{I^n(\lambda, l)f(z)^\alpha}{\left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^n z^\alpha} - \beta}{1 - \beta} = \varphi\left(\frac{\Phi(z) - 1}{\Phi(z) + 1}\right) \tag{2.22}$$

Using (2.21) together with (2.15), it is evident that

$$\varphi\left(\frac{\Phi(z) - 1}{\Phi(z) + 1}\right) = 1 + \frac{B_1}{4}z + \frac{B_2 - B_1}{16}z^2 - \frac{B_1 + 6B_2 - 3B_3}{192}z^3 + \frac{5B_1 + B_2 - 9B_3 + 3B_4}{768}z^4 + \dots \tag{2.23}$$

Recall that

$$\frac{I^n(\lambda, l)f(z)^\alpha}{\left(\frac{1+\lambda(\alpha-1)+l}{1+l}\right)^n z^\alpha} = 1 + \sum_{k=2}^\infty \left(\frac{1+\lambda(\alpha+k-2)+l}{1+\lambda(\alpha-1)+l}\right)^n a_k(\alpha)z^{k-1}$$

which has the expansion

$$\begin{aligned} &1 + \alpha \left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^n a_2z + \left(\alpha a_3 + \frac{\alpha(\alpha-1)}{2} a_2^2\right) \left(\frac{1+\lambda(\alpha+1)+l}{1+\lambda(\alpha-1)+l}\right)^n z^2 \\ &+ \left(\alpha a_4 + \alpha(\alpha-1)a_2a_3 + \frac{\alpha(\alpha-1)(\alpha-2)}{6} a_2^3\right) \left(\frac{1+\lambda(\alpha+2)+l}{1+\lambda(\alpha-1)+l}\right)^n z^3 \\ &+ \left(\alpha a_5 + \frac{\alpha(\alpha-1)}{2!}(2a_2a_4 + a_3^2) + \frac{\alpha(\alpha-1)(\alpha-2)}{3} a_2^2a_3 + \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{24} a_2^4\right) \left(\frac{1+\lambda(\alpha+3)+l}{1+\lambda(\alpha-1)+l}\right)^n z^4 \\ &+ \dots \end{aligned} \tag{2.24}$$

Therefore (2.22) yields

$$\begin{aligned} &1 + \alpha \left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^n a_2z + \left(\alpha a_3 + \frac{\alpha(\alpha-1)}{2} a_2^2\right) \left(\frac{1+\lambda(\alpha+1)+l}{1+\lambda(\alpha-1)+l}\right)^n z^2 \\ &+ \left(\alpha a_4 + \alpha(\alpha-1)a_2a_3 + \frac{\alpha(\alpha-1)(\alpha-2)}{6} a_2^3\right) \left(\frac{1+\lambda(\alpha+2)+l}{1+\lambda(\alpha-1)+l}\right)^n z^3 \\ &+ \left(\alpha a_5 + \frac{\alpha(\alpha-1)}{2!}(2a_2a_4 + a_3^2) + \frac{\alpha(\alpha-1)(\alpha-2)}{3} a_2^2a_3 + \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{24} a_2^4\right) \left(\frac{1+\lambda(\alpha+3)+l}{1+\lambda(\alpha-1)+l}\right)^n z^4 \\ &+ \dots = \beta + (1-\beta) \left[1 + \frac{B_1}{4}z + \frac{B_2 - \beta_1}{16}z^2 - \frac{B_1 + 6B_2 - 3B_3}{192}z^3 + \frac{5B_1 + B_2 - 9B_3 + 3B_4}{768}z^4 + \dots\right] \end{aligned} \tag{2.25}$$

Comparing the L.H.S. and R.H.S. of (2.25), it gives

$$\alpha \left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^n a_2(\alpha) = \frac{(1-\beta)B_1}{4} \tag{2.26}$$

$$\left(\alpha a_3(\alpha) + \frac{\alpha(\alpha-1)}{2} a_2^2(\alpha)\right) \left(\frac{1+\lambda(\alpha+1)+l}{1+\lambda(\alpha-1)+l}\right)^n = \frac{(1-\beta)(B_2 - B_1)}{16} \tag{2.27}$$

$$\left(\alpha a_4(\alpha) + \alpha(\alpha-1)a_2(\alpha)a_3(\alpha) + \frac{\alpha(\alpha-1)(\alpha-2)}{6} a_2^3(\alpha)\right) \left(\frac{1+\lambda(\alpha+2)+l}{1+\lambda(\alpha-1)+l}\right)^n = -\frac{(1-\beta)(B_1 + 6B_2 - 3B_3)}{192} \tag{2.28}$$

So, by simple computation, we obtain

$$|a_2(\alpha)| \leq \frac{(1-\beta)B_1}{4\alpha \left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^n} \tag{2.29}$$

$$|a_3(\alpha)| \leq \frac{(1-\beta) \left[2\alpha(B_2 - B_1) \left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^{2n} - (\alpha-1)(1-\beta)B_1^2 \left(\frac{1+\lambda(\alpha+1)+l}{1+\lambda(\alpha-1)+l}\right)^n\right]}{32\alpha^2 \left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^{2n} \left(\frac{1+\lambda(\alpha+1)+l}{1+\lambda(\alpha-1)+l}\right)^n} \tag{2.30}$$

$$\begin{aligned} |a_4(\alpha)| &\leq \frac{2(1-\beta)(3B_3 - 6B_2 - B_1)}{384\alpha \left(\frac{1+\lambda(\alpha+2)+l}{1+\lambda(\alpha-1)+l}\right)^n} \\ &- \frac{(\alpha-1)(1-\beta)^3 B_1}{384\alpha^3 \left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^{3n}} \left\{ \frac{3 \left[2\alpha(B_2 - B_1) \left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^{2n} - (\alpha-1)(1-\beta)B_1^2 \left(\frac{1+\lambda(\alpha+1)+l}{1+\lambda(\alpha-1)+l}\right)^n\right]}{\left(\frac{1+\lambda(\alpha+1)+l}{1+\lambda(\alpha-1)+l}\right)^n} \right\} \\ &+ \frac{(\alpha-2)(1-\beta)^2 B_1^3}{384\alpha} \end{aligned} \tag{2.31}$$

□

and this completes the proof of Theorem (2.1).

By specializing some parameters that are involved, we obtain some corollaries.

Setting $\beta = 0$, it gives the following corollary

Corollary 2.1. If $f(z)^\alpha \in T_n^\alpha(\lambda, 0, l, \Phi)$ is given by (1.12), then

$$|a_2(\alpha)| \leq \frac{B_1}{4\alpha \left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^n} \tag{2.32}$$

$$|a_3(\alpha)| \leq \frac{\left[2\alpha(B_2 - B_1) \left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^{2n} - (\alpha - 1)B_1^2 \left(\frac{1+\lambda(\alpha+1)+l}{1+\lambda(\alpha-1)+l}\right)^n\right]}{32\alpha^2 \left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^{2n} \left(\frac{1+\lambda(\alpha+1)+l}{1+\lambda(\alpha-1)+l}\right)^n} \tag{2.33}$$

$$|a_4(\alpha)| \leq \frac{2(3B_3 - 6B_2 - B_1)}{384\alpha \left(\frac{1+\lambda(\alpha+2)+l}{1+\lambda(\alpha-1)+l}\right)^n} - \frac{(\alpha - 1)B_1}{384\alpha^3 \left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^{3n}} \left\{ \frac{3 \left[2\alpha(B_2 - B_1) \left(\frac{1+\lambda\alpha+l}{1+\lambda(\alpha-1)+l}\right)^{2n} - (\alpha - 1)B_1^2 \left(\frac{1+\lambda(\alpha+1)+l}{1+\lambda(\alpha-1)+l}\right)^n\right]}{\left(\frac{1+\lambda(\alpha+1)+l}{1+\lambda(\alpha-1)+l}\right)^n} \right\} + \frac{(\alpha - 2)B_1^3}{384\alpha} \tag{2.34}$$

Setting $\alpha = 1$ in Corollary (2.1) gives

Corollary 2.2. If $f(z) \in T_n^1(\lambda, 0, l, \Phi)$ is given by (1.12), then

$$|a_2(1)| \leq \frac{B_1}{4 \left(\frac{1+\lambda+l}{1+l}\right)^n} \tag{2.35}$$

$$|a_3(1)| \leq \frac{\left[2(B_2 - B_1) \left(\frac{1+\lambda+l}{1+l}\right)^{2n}\right]}{32 \left(\frac{1+\lambda+l}{1+l}\right)^{2n} \left(\frac{1+2\lambda+l}{1+l}\right)^n} \tag{2.36}$$

$$|a_4(1)| \leq \frac{2(3B_3 - 6B_2 - B_1)}{384 \left(\frac{1+3\lambda+l}{1+l}\right)^n} - \frac{B_1^3}{384} \tag{2.37}$$

Putting $\lambda = 1$ in Corollary (2.2) yields

Corollary 2.3. If $f(z) \in T_n^1(1, 0, l, \Phi)$ is given by (1.12), then

$$|a_2(1)| \leq \frac{B_1}{4 \left(\frac{2+l}{1+l}\right)^n} \tag{2.38}$$

$$|a_3(1)| \leq \frac{\left[2(B_2 - B_1) \left(\frac{2+l}{1+l}\right)^{2n}\right]}{32 \left(\frac{2+l}{1+l}\right)^{2n} \left(\frac{3+l}{1+l}\right)^n} \tag{2.39}$$

$$|a_4(1)| \leq \frac{2(3B_3 - 6B_2 - B_1)}{384 \left(\frac{4+l}{1+l}\right)^n} - \frac{B_1^3}{384} \tag{2.40}$$

Taking $l = 0$ in Corollary (2.3) it is seen that

Corollary 2.4. If $f(z) \in T_n^1(1, 0, 0, \Phi)$ is given by (1.12), then

$$|a_2(1)| \leq \frac{B_1}{4(2)^n} \quad (2.41)$$

$$|a_3(1)| \leq \frac{[2(B_2 - B_1)(2)^{2n}]}{32(2)^{2n}3^n} \quad (2.42)$$

$$|a_4(1)| \leq \frac{2(3B_3 - 6B_2 - B_1)}{384(4)^n} - \frac{B_1^3}{384}. \quad (2.43)$$

If $n = 0$ in Corollary (2.4) we get

Corollary 2.5. If $f(z) \in T_0^1(1, 0, 0, \Phi)$ is given by (1.12), then

$$|a_2(1)| \leq \frac{B_1}{4} \quad (2.44)$$

$$|a_3(1)| \leq \frac{(B_2 - B_1)}{16} \quad (2.45)$$

$$|a_4(1)| \leq \frac{(3B_3 - 6B_2 - B_1)}{192} - \frac{B_1^3}{384}. \quad (2.46)$$

3 The Fekete-Szegő Inequality

In order to obtain the Fekete-Szegő Inequalities, we shall employ the Deniz and Orhan[4] and Ma and Minda[7] approach.

Theorem 3.1. If $f(z)^\alpha \in T_n^\alpha(\lambda, \beta, l, \Phi)$ is given by (1.12), then

$$|a_3 - \mu a_2^2| \leq \frac{1 - \beta}{32} \left| \frac{B_1^2(\beta - 1)(\alpha + 2\mu - 1) \left(\frac{1 + \lambda(\alpha + 1) + l}{1 + \lambda(\alpha - 1) + l} \right)^n - 2\alpha(B_1 - B_2) \left(\frac{1 + \lambda\alpha + l}{1 + \lambda(\alpha - 1) + l} \right)^{2n}}{\alpha^2 \left(\frac{1 + \lambda(\alpha + 1) + l}{1 + \lambda(\alpha - 1) + l} \right)^n \left(\frac{1 + \lambda\alpha + l}{1 + \lambda(\alpha - 1) + l} \right)^{2n}} \right|. \quad (3.47)$$

Proof. From (2.29) and (2.30), we have

$$a_3 - \mu a_2^2 = \frac{(1 - \beta) \left[2\alpha(B_2 - B_1) \left(\frac{1 + \lambda\alpha + l}{1 + \lambda(\alpha - 1) + l} \right)^{2n} - (\alpha - 1)(1 - \beta)B_1^2 \left(\frac{1 + \lambda(\alpha + 1) + l}{1 + \lambda(\alpha - 1) + l} \right)^n \right]}{32\alpha^2 \left(\frac{1 + \lambda\alpha + l}{1 + \lambda(\alpha - 1) + l} \right)^{2n} \left(\frac{1 + \lambda(\alpha + 1) + l}{1 + \lambda(\alpha - 1) + l} \right)^n} - \mu \left[\frac{(1 - \beta)B_1}{4\alpha \left(\frac{1 + \lambda\alpha + l}{1 + \lambda(\alpha - 1) + l} \right)^n} \right]^2. \quad (3.48)$$

Simplifying (3.48), we have

$$a_3 - \mu a_2^2 = \frac{1 - \beta}{32} \left[\frac{B_1^2(\beta - 1)(\alpha + 2\mu - 1) \left(\frac{1 + \lambda(\alpha + 1) + l}{1 + \lambda(\alpha - 1) + l} \right)^n - 2\alpha(B_1 - B_2) \left(\frac{1 + \lambda\alpha + l}{1 + \lambda(\alpha - 1) + l} \right)^{2n}}{\alpha^2 \left(\frac{1 + \lambda(\alpha + 1) + l}{1 + \lambda(\alpha - 1) + l} \right)^n \left(\frac{1 + \lambda\alpha + l}{1 + \lambda(\alpha - 1) + l} \right)^{2n}} \right] \quad (3.49)$$

which completes the proof. \square

Taking $\mu = 1$, we obtain

Corollary 3.6. If $f(z)^\alpha \in T_n^\alpha(\lambda, \beta, l, \Phi)$ is given by (1.12), then

$$|a_3 - a_2^2| \leq \frac{1 - \beta}{32} \left| \frac{B_1^2(\beta - 1)(\alpha + 1) \left(\frac{1 + \lambda(\alpha + 1) + l}{1 + \lambda(\alpha - 1) + l} \right)^n - 2\alpha(B_1 - B_2) \left(\frac{1 + \lambda\alpha + l}{1 + \lambda(\alpha - 1) + l} \right)^{2n}}{\alpha^2 \left(\frac{1 + \lambda(\alpha + 1) + l}{1 + \lambda(\alpha - 1) + l} \right)^n \left(\frac{1 + \lambda\alpha + l}{1 + \lambda(\alpha - 1) + l} \right)^{2n}} \right|. \quad (3.50)$$

4 Conclusion

By varying other parameters that are involved, many corollaries can be generated.

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