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Donoho-Stark Uncertainty Principle for the Generalized Bessel Transform

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Abstract

The generalized Bessel transform satisfies some uncertainty principles similar to the Euclidean Fourier transform. A generalization of Donoho-Stark uncertainty principle is obtained for the generalized Bessel transform.

Keywords: Generalized Bessel transform; Donoho-stark's uncertainty principle.

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1 Introduction

There are many theorems known which state that a function and its classical Fourier transform on $\mathbb{R}$ cannot both be sharply localized. That it is impossible for a nonzero function and its Fourier transform to be simultaneously small. There are several manifestations of this principle. We refer the reader to the excellent survey article by Folland and Sitaram [3], and also the monograph by S. Thangavelu [5]. In this paper we are interested in a variant of Donoho-Stark's uncertainty principle. Recall that Donoho and Stark [2] paid attention to the supports of functions and gave qualitative uncertainty principles for the Fourier transforms. The purpose of this paper is to obtain uncertainty principle similar to Donoho-Stark's principle for the generalized Bessel transform. The outline of the content of this paper is as follows.
Section 2 is dedicated to some properties and results concerning the Generalized Bessel transform.
Section 3 is devoted to the Donoho-Stark's uncertainty principle for the Generalized Bessel transform.

2 Preliminaries

In this section we recapitulate some facts about harmonic analysis related to the generalized Bessel operator. We cite here, as briefly as possible, some properties. For more details we refer to [1].
Throughout this paper we assume that $\alpha > \frac{-1}{2}$.

We consider the second-order singular differential operator on the half line

$$\mathcal{L}_{\alpha,n} f(x) = \frac{d^2}{dx^2} f(x) + \frac{2\alpha+1}{x}\frac{d}{dx} f(x) - \frac{4n(\alpha+n)}{x^2} f(x).$$

The generalized Bessel transform is defined for a function $f \in L^1_{\alpha,n}(\mathbb{R}^+)$ by

$$\mathcal{F}_{\alpha,n}(f)(\lambda) = \int_0^\infty f(x)\varphi_\lambda(x)x^{2\alpha+1}dx, \quad \lambda \geq 0, \tag{2.1}$$

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where

$$\varphi_\lambda(x) = a_{\alpha+2n} x^{2n} \int_0^1 cos(\lambda t x)(1-t^2)^{\alpha+2n-\frac{1}{2}} dt$$

and

$$\varphi_\lambda(x) = a_{\alpha+2n} x^{2n} \int_0^1 cos(\lambda t x)(1-t^2)^{\alpha+2n-\frac{1}{2}} dt$$

and

$$a_{\alpha+2n} = \frac{2\Gamma(\alpha+2n+1)}{\sqrt{\pi}\Gamma(\alpha+2n+\frac{1}{2})}. \tag{2.2}$$

- The function φ_λ satisfies the differential equation

$$\mathcal{L}_{\alpha,n}\varphi_\lambda = -\lambda^2\varphi_\lambda$$

- For all $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}^+$,

$$|\varphi_\lambda(x)| \leq x^{2n} e^{|Im\lambda||x|}. \tag{2.3}$$

- For all $\lambda \in \mathbb{R}^+$ and $x \in \mathbb{R}^+$,

$$\lambda^{2n}\varphi_\lambda(x) = x^{2n}\varphi_x(\lambda). \tag{2.4}$$

We denote by

- $L^p_\alpha(\mathbb{R}^+)$ the class of measurable functions f on $[0,+\infty[$ for which

$$\|f\|_{L^p_\alpha(\mathbb{R}^+)} < \infty$$

where

$$\|f\|_{L^p_\alpha(\mathbb{R}^+)} = \left(\int_0^\infty |f(x)|^p x^{2\alpha+1} dx\right)^{\frac{1}{p}}, \quad \text{if } p < \infty,$$

and $\|f\|_{L^\infty_\alpha(\mathbb{R}^+)} = ess\ sup_{x\geq 0}|f(x)|$.

- $L^p_{\alpha,n}(\mathbb{R}^+)$ the class of measurable functions f on $\mathbb{R}^+$ for which

$$\|f\|_{L^p_{\alpha,n}(\mathbb{R}^+)} = \|x^{-2n}f\|_{L^p_{\alpha+2n}(\mathbb{R}^+)} < \infty.$$

For every $f \in L^1_{\alpha,n}(\mathbb{R}^+)\bigcap L^2_{\alpha,n}(\mathbb{R}^+)$ we have the Plancherel formula

$$\int_0^\infty |\ f(x)\ |^2\ x^{2\alpha+1}dx = \int_0^\infty |\ \mathcal{F}_{\alpha,n}(f)(\lambda)\ |^2\ d\mu_{\alpha+2n}(\lambda),$$

where

$$d\mu_{\alpha+2n}(\lambda) = \frac{1}{4^{\alpha+2n}(\Gamma(\alpha+2n+1))^2}\lambda^{2\alpha+4n+1}d\lambda. \tag{2.5}$$

The generalized Bessel transform $\mathcal{F}_{\alpha,n}$ extends uniquely to an isometric isomorphism from $L^2_{\alpha,n}(\mathbb{R}^+)$ onto $L^2_{\alpha+2n}(\mathbb{R}^+)$.
The inverse transform is given by

$$\mathcal{F}^{-1}_{\alpha,n}(f)(x) = \int_0^\infty f(\lambda)\varphi_\lambda(x)d\mu_{\alpha+2n}(\lambda), \tag{2.6}$$

where the integral converge in $L^2_{\alpha,n}(\mathbb{R}^+)$.

Let $f \in L^1_{\alpha,n}(\mathbb{R}^+)$ such that $\mathcal{F}_{\alpha,n}(f) \in L^1_{\alpha+2n}(\mathbb{R}^+)$, then the inverse generalized Fourier-Bessel transform is given by the formula

$$f(x) = \int_0^\infty \mathcal{F}_{\alpha,n}(f)(\lambda)\varphi_\lambda(x)d\mu_{\alpha+2n}(\lambda). \tag{2.7}$$

3 Donoho-Stark for the Fourier generalized transform

Throughout this section we denote by $\|.\|$ the operator norm on $L^2_{\alpha,n}(\mathbb{R}_+)$. More precisely if T is an operator then

$$\|T\| = sup_{f\in L^2_{\alpha,n}(\mathbb{R}^+)} \frac{\|Tf\|_{L^2_{\alpha,n}(\mathbb{R}^+)}}{\|f\|_{L^2_{\alpha,n}(\mathbb{R}^+)}}.$$

We say that f is ϵ-concentrated on a measurable set E if

$$||f - \mathcal{X}_E f||_{L^2_{\alpha,n}(\mathbb{R}^+)} < \epsilon,$$

where χ_E is the characteristic function of the set E.
Donoho and Stark [3] have shown that if f of unit $L^2(\mathbb{R}^+)$ norm is ϵ_T concentrated on a measurable set T and its Fourier transform $\mathcal{F}(f)$ is ϵ_W, on a measurable set W, then

$$|W|.|T| \geq (1 - \epsilon_T - \epsilon_W)^2.$$

Here, $|T|$ is the Lebesque measure of the set T. This inequality has been slightly improved in ref.[4] to

$$|W|.|T| \geq (1 - (\epsilon_T^2 + \epsilon_W^2)^{\frac{1}{2}})^2.$$

In this section, we will extend the Donoho-Stark uncertainty principle to the generalized Bessel transform.
Let P_E denote the time-limiting operator

$$(P_E f)(x) = \begin{cases} f(x), & x \in E \\ 0, & x \in \mathbb{R}^+ \backslash E \end{cases}. \tag{3.8}$$

This operator cuts off the part of f outside E. Let us now be more precise, we need to introduce some notations, so f is ϵ-concentrated on a set E if, and only if

$$||f - P_E f||_{L^2_{\alpha,n}(\mathbb{R}^+)} \leq \epsilon.$$

For simplicity, we will use P_X to $P_{[0,X]}$. Clearly $||P_E|| = 1$ because P_E is a projection. The second operator is the frequency-limiting operator

$$(Q_E f)(x) = \int_E \varphi_y(x)\mathcal{F}_{\alpha,n}(f)(y)d\mu_{\alpha+2n}(y), \tag{3.9}$$

From (2.6) we can also write Q_E as follows

$$Q_E f(x) = \mathcal{F}^{-1}_{\alpha,n}(P_E(\mathcal{F}_{\alpha,n}(f)))(x).$$

Then by (2.6) and (2.7) we deduce that $\mathcal{F}_{\alpha,n}(f)$ is ε-concentrated on F if and only if $\|f - Q_F f\|_{L^2_{\alpha,n}(\mathbb{R})} \leq \varepsilon\|f\|_{L^2_{\alpha,n}(\mathbb{R})}$.

We have from (3.8) and (3.9)

$$\begin{aligned}(P_X Q_Y f)(x) &= P_X \int_0^Y \varphi_y(x)\mathcal{F}_{\alpha,n}(f)(y)d\mu_{\alpha+2n}(y) \\ &= P_X \int_0^Y \varphi_y(x) \int_0^\infty \varphi_y(t)f(t)d\mu_\alpha(t)d\mu_{\alpha+2n}(y) \\ &= P_X \int_0^\infty f(t) \int_0^Y \varphi_y(x)\varphi_y(t)d\mu_{\alpha+2n}(y)d\mu_\alpha(t) \\ &= \int_0^\infty f(t)q(x,t)d\mu_\alpha(t),\end{aligned}$$

where

$$q(x,t) = \begin{cases} \int_0^Y \varphi_y(x)\varphi_y(t)d\mu_{\alpha+2n}(y), & x < X \\ 0, & x \geq X \end{cases}.$$

The Hilbert-Schmidt norm of $P_X Q_Y$ is

$$\|P_X Q_Y\|_{HS} = \left(\int_0^\infty \int_0^\infty |q(x,t)|^2 d\mu_\alpha(x) d\mu_\alpha(t)\right)^{\frac{1}{2}}.$$

The norm $||P_X Q_Y||$ does not exceed the Hilbert-Schmidt norm of $P_X Q_Y$, therefore

$$\begin{aligned} ||P_X Q_Y||^2 &\leq \|P_X Q_Y\|_{HS}^2 \\ &= \int_0^\infty \int_0^\infty |q(x,t)|^2 d\mu_\alpha(x) d\mu_\alpha(t) \\ &= \int_0^X \int_0^\infty |q(x,t)|^2 d\mu_\alpha(x) d\mu_\alpha(t). \end{aligned}$$

Notice that

$$\begin{aligned} q(x,t) &= \int_0^Y \varphi_y(x)\varphi_y(t) d\mu_{\alpha+2n}(y) \\ &= \int_0^Y y^{2n}\varphi_y(x) y^{2n}\varphi_y(t) d\mu_\alpha(y). \end{aligned}$$

From (2.4) we deduce that

$$\begin{aligned} &= \int_0^Y x^{2n}\varphi_x(y) t^{2n}\varphi_t(y) d\mu_\alpha(y) \\ &= \int_0^Y x^{2n} t^{2n}\varphi_x(y)\varphi_t(y) d\mu_\alpha(y) \\ &= x^{2n} t^{2n} \mathcal{F}_{\alpha,n}(\varphi_t(.)\mathcal{X}_{[0,Y]})(x), \end{aligned}$$

the Plancherel formula for the generalized Bessel transform yields

$$\begin{aligned} \int_0^\infty |q(x,t)|^2 d\mu_\alpha(x) &= \int_0^\infty |x^{2n} t^{2n} \mathcal{F}_{\alpha,n}(\varphi_t(.)\mathcal{X}_{[0,Y]})(x)|^2 d\mu_\alpha(x) \\ &= \frac{a_\alpha}{a_{\alpha+2n}} \int_0^\infty |t^{2n} \mathcal{F}_{\alpha,n}(\varphi_t(.)\mathcal{X}_{[0,Y]})(x)|^2 d\mu_{\alpha+2n}(x) \\ &= \frac{a_\alpha}{a_{\alpha+2n}} \int_0^\infty |\mathcal{F}_{\alpha,n}(t^{2n}\varphi_t(.)\mathcal{X}_{[0,Y]})(x)|^2 d\mu_{\alpha+2n}(x), \end{aligned}$$

by Plancherel formula we have

$$\begin{aligned} \frac{a_\alpha}{a_{\alpha+2n}} \int_0^\infty |\mathcal{F}_{\alpha,n}(t^{2n}\varphi_t(.)\mathcal{X}_{[0,Y]})(x)|^2 d\mu_{\alpha+2n}(x) &= \frac{a_\alpha}{a_{\alpha+2n}} \int_0^Y |t^{2n}\varphi_t(x)|^2 d\mu_\alpha(x) \\ &= \frac{a_\alpha}{a_{\alpha+2n}} \int_0^Y |x^{2n}\varphi_x(t)|^2 d\mu_\alpha(x) \\ &= \left(\frac{a_\alpha}{a_{\alpha+2n}}\right)^2 \int_0^Y |\varphi_x(t)|^2 d\mu_{\alpha+2n}(x). \end{aligned}$$

Consequently,

$$\begin{aligned} ||P_X Q_Y||^2 &\leq \left(\frac{a_\alpha}{a_{\alpha+2n}}\right)^2 \int_0^X \int_0^Y |\varphi_x(t)|^2 d\mu_{\alpha+2n}(x) d\mu_\alpha(t) \\ &\leq \left(\frac{a_\alpha}{a_{\alpha+2n}}\right)^2 \int_0^X \int_0^Y |t^{2n}|^2 d\mu_{\alpha+2n}(x) d\mu_\alpha(t) \\ &= \left(\frac{a_\alpha}{a_{\alpha+2n}}\right)^3 \int_0^X \int_0^Y d\mu_{\alpha+2n}(x) d\mu_{\alpha+2n}(t) \\ &= \left(\frac{a_\alpha}{a_{\alpha+2n}}\right)^3 \int_0^X \int_0^Y d\mu_{\alpha+2n}(x) d\mu_{\alpha+2n}(t) \\ &= \left(\frac{a_\alpha}{a_{\alpha+2n}}\right)^3 \frac{(XY)^{\alpha+2n+1}}{\alpha+2n+1}. \end{aligned}$$

We put

$$b_{\alpha,n}=\left(\frac{a_{\alpha+2n}}{a_\alpha}\right)^3(\alpha+2n+1). \tag{3.10}$$

Let $XY<(b_{\alpha,n})^{\frac{1}{\alpha+2n+1}}$. Then $||P_XQ_Y||<1$ and therefore $I-P_XQ_Y$ is invertible with

$$\begin{aligned}||(I-P_XQ_Y)^{-1}|| &\leq \sum_{k=0}^{\infty}||P_XQ_Y||^k\\ &\leq \sum_{k=0}^{\infty}\left[\frac{(XY)^{\alpha+2n+1}}{b_{\alpha,n}}\right]^k\\ &= \frac{b_{\alpha,n}}{b_{\alpha,n}-(XY)^{\alpha+2n+1}}.\end{aligned}$$

We have

$$I=P_X+P_{(X,\infty)}=P_XQ_Y+P_XQ_{(Y,\infty)}+P_{(X,\infty)}.$$

The orthogonality of P_X and $P_{(X,\infty)}$ gives

$$||P_XQ_{(Y,\infty)}f||^2_{L^2_{\alpha,n}(\mathbb{R}^+)}+||P_{(X,\infty)}f||^2_{L^2_{\alpha,n}(\mathbb{R}^+)}=||P_XQ_{(Y,\infty)}f+P_{(X,\infty)}f||^2_{L^2_{\alpha,n}(\mathbb{R}^+)}.$$

Together with $||P_X||=1$

$$\begin{aligned}||f||^2_{2,\alpha,n} &\leq ||(I-P_XQ_Y)^{-1}||^2||(I-P_XQ_Y)f||^2_{L^2_{\alpha,n}(\mathbb{R}^+)}\\ &\leq \left(\frac{b_{\alpha,n}}{b_{\alpha,n}-(XY)^{\alpha+2n+1}}\right)^2\left[||P_XQ_{(Y,\infty)}f||^2_{L^2_{\alpha,n}(\mathbb{R}^+)}+||P_{(X,\infty)}f||^2_{L^2_{\alpha,n}(\mathbb{R}^+)}\right]\\ &\leq \left(\frac{b_{\alpha,n}}{b_{\alpha,n}-(XY)^{\alpha+2n+1}}\right)^2\left[||Q_{(Y,\infty)}f||^2_{L^2_{\alpha,n}(\mathbb{R}^+)}+||P_{(X,\infty)}f||^2_{L^2_{\alpha,n}(\mathbb{R}^+)}\right].\end{aligned}$$

If f of unit norm is ϵ_X-time-limited on $[0,X]$, then $||P_{(X,\infty)}f||_{L^2_{\alpha,n}(\mathbb{R}^+)}\leq\epsilon_X$, If f of unit norm is ϵ_Y-bandlimited on $[0,Y]$, then $||Q_{(Y,\infty)}f||_{L^2_{\alpha,n}(\mathbb{R}^+)}\leq\epsilon_Y$. Then if f of unit norm is both ϵ_X-time-limited and ϵ_Y-bandlimited,

$$1\leq\left(\frac{b_{\alpha,n}}{b_{\alpha,n}-(XY)^{\alpha+2n+1}}\right)^2(\epsilon_X^2+\epsilon_Y^2)$$

or

$$XY\geq(b_{\alpha,n})^{\frac{1}{\alpha+2n+1}}\left(1-(\epsilon_X^2+\epsilon_Y^2)^{\frac{1}{2}}\right)^{\frac{1}{\alpha+2n+1}}.$$

We arrive at the Donoho-Stark uncertainty principle for the generalized Bessel transform.

Theorem 3.1. *Let a unit norm signal f be ϵ_X-time-limited on $[0,X]$ and ϵ_Y-bandlimited on $[0,Y]$. Then*

$$XY\geq(b_{\alpha,n})^{\frac{1}{\alpha+2n+1}}\left(1-(\epsilon_X^2+\epsilon_Y^2)^{\frac{1}{2}}\right)^{\frac{1}{\alpha+2n+1}}$$

where $b_{\alpha,n}$ is given by (3.10).

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