

# Pseudo Asymptotically Periodic Integral Solution of Partial Neutral Functional Differential Equations

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**Abstract** In this paper, we propose a new class of functions called  $\mu$ -pseudo  $\mathcal{S}$ -asymptotically periodic function on  $\mathbb{R}$  by the measure theory. Furthermore, the existence, uniqueness of  $\mu$ -pseudo  $\mathcal{S}$ -asymptotically periodic integral solution to partial neutral functional differential equations with finite delay are investigated. Here we assume that the undelayed part is not necessarily densely defined and satisfies the Hille-Yosida condition.

**Keywords:**  $\mu$ -pseudo  $\mathcal{S}$ -asymptotically periodic function, Partial neutral functional differential equations, Measure theory, Integral solution.

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## 1 Introduction

The existence of periodic solution or asymptotically periodic solution is very important in the qualitative studies of many problems. Many authors have made important contributions to the theory of periodicity, asymptotic periodicity and applications to differential equations, integral equations, integro-differential equations and partial functional differential equations. More details on this topic can be found in [10, 12, 20, 21, 25, 30].

The notion of  $\mathcal{S}$ -asymptotic periodicity is an important generalization of asymptotic periodicity, which was introduced by Henríquez et al. in [18, 19]. Since then, it attracted the attention of many researchers [7, 11, 13, 22] and this concept has undergone several interesting, natural, and powerful generalizations, such as pseudo  $\mathcal{S}$ -asymptotic periodicity [23], weighted pseudo  $\mathcal{S}$ -asymptotic periodicity [28], and so on. On the other hand, Blot et al. [9] used some results of the measure theory to establish a new concept of  $\mu$ -pseudo almost periodicity which generalizes weighted pseudo almost periodicity. Using the methods of [9], we introduce the concept of  $\mu$ -pseudo  $\mathcal{S}$ -asymptotic periodicity by measure theory in this paper.

Partial neutral functional differential equations (PNFDEs), arising from many biological, chemical, and physical systems, become an interesting and important field in dynamical systems. In the standard framework of semilinear PNFDEs, one assumes that the operator  $A$  in the linear part is densely defined. However, there are many examples in which the density condition is not satisfied [3, 15, 17, 27, 29]. Here we assume that the linear part is not necessarily densely defined and satisfies the Hille-Yosida condition. Existence, uniqueness of  $\mu$ -pseudo  $\mathcal{S}$ -asymptotically periodic integral solution to PNFDEs are investigated.

The paper is organized as follows. In Section 2, some notations are presented and we propose a new class of functions called  $\mu$ -pseudo  $\mathcal{S}$ -asymptotically periodic function by the measure theory. In Section 3, we recall some fundamental results which include the variation of constants formula and spectral decomposition. Sections 4 is devoted to the existence and uniqueness of  $\mu$ -pseudo  $\mathcal{S}$ -asymptotically periodic integral solution of PNFDEs. In Section 5, we provide an example to illustrate our main results.

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## 2 Preliminaries and basic results

Let  $(X, \| \cdot \|)$ ,  $(Y, \| \cdot \|)$  are two Banach spaces and  $\mathbb{N}, \mathbb{R}, \mathbb{R}^+$  and  $\mathbb{C}$  stand for the set of natural numbers, real numbers, nonnegative real numbers and complex numbers, respectively. For  $A$  being a linear operator on  $X$ ,  $D(A), \rho(A), R(\lambda, A), \sigma(A)$  stand for the domain, the resolvent set, the resolvent and spectrum of  $A$ . In order to facilitate the discussion below, we further introduce the following notations:

- $C(\mathbb{R}, X)$  (resp.  $C(\mathbb{R} \times Y, X)$ ): the set of continuous functions from  $\mathbb{R}$  to  $X$  (resp. from  $\mathbb{R} \times Y$  to  $X$ ).
- $\mathcal{C} = C([-r, 0], X)$ : the space of continuous functions from  $[-r, 0]$  to  $X$  endowed with the uniform norm topology.
- $BC(\mathbb{R}, X)$  (resp.  $BC(\mathbb{R} \times Y, X)$ ): the Banach space of bounded continuous functions from  $\mathbb{R}$  to  $X$  (resp. from  $\mathbb{R} \times Y$  to  $X$ ) with the supremum norm.
- $B(X, Y)$ : the Banach space of bounded linear operators from  $X$  to  $Y$  endowed with the operator topology. In particular, we write  $B(X)$  when  $X = Y$ .
- $L^p(\mathbb{R}, X)$ : the space of all classes of equivalence (with respect to the equality almost everywhere on  $\mathbb{R}$ ) of measurable functions  $f : \mathbb{R} \rightarrow X$  such that  $\|f\| \in L^p(\mathbb{R}, \mathbb{R})$ .

For  $\omega > 0$ , define

$$C_0(\mathbb{R}, X) = \{x \in BC(\mathbb{R}, X) : \lim_{|t| \rightarrow \infty} \|x(t)\| = 0\}.$$

$$C_\omega(\mathbb{R}, X) = \{x \in BC(\mathbb{R}, X) : x \text{ is } \omega\text{-pripodic}\}.$$

**Definition 2.1.** A function  $f \in BC(\mathbb{R}, X)$  is called asymptotically  $\omega$ -periodic if there exists  $g \in C_\omega(\mathbb{R}, X), \varphi \in C_0(\mathbb{R}, X)$  such that  $f = g + \varphi$ . Denote by  $AP_\omega(\mathbb{R}, X)$  the set of such functions.

**Definition 2.2.** A function  $f \in BC(\mathbb{R}, X)$  is said to be  $\mathcal{S}$ -asymptotically  $\omega$ -periodic if there exists  $\omega > 0$  such that  $\lim_{t \rightarrow \infty} (f(t + \omega) - f(t)) = 0$ . Denote by  $SAP_\omega(\mathbb{R}, X)$  the set of such functions.

**Definition 2.3.** A function  $f \in BC(\mathbb{R}, X)$  is called pseudo  $\mathcal{S}$ -asymptotically  $\omega$ -periodic if there exists  $\omega > 0$  such that

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \|f(t + \omega) - f(t)\| dt = 0.$$

Denote by  $PSAP_\omega(\mathbb{R}, X)$  the set of such functions.

Let  $U$  be the set of all functions  $\rho : \mathbb{R} \rightarrow (0, \infty)$  which are positive and locally integrable over  $\mathbb{R}$ . For a given  $T > 0$  and each  $\rho \in U$ , set

$$m(T, \rho) := \int_{-T}^T \rho(t) dt.$$

Define  $U_\infty := \{\rho \in U : \lim_{T \rightarrow \infty} m(T, \rho) = \infty\}$ .

**Definition 2.4.** Let  $\rho \in U_\infty$ . A function  $f \in BC(\mathbb{R}, X)$  is called weighted pseudo  $\mathcal{S}$ -asymptotically  $\omega$ -periodic if there exists  $\omega > 0$  such that

$$\lim_{T \rightarrow \infty} \frac{1}{m(T, \rho)} \int_{-T}^T \rho(t) \|f(t + \omega) - f(t)\| dt = 0.$$

Denote by  $WPSAP_\omega(\mathbb{R}, X)$  the set of such functions.

*Remark 2.1.* Note that in the above definitions, if the function  $f$  is limited on  $\mathbb{R}^+$ , i.e.,  $AP_\omega(\mathbb{R}^+, X), SAP_\omega(\mathbb{R}^+, X), PSAP_\omega(\mathbb{R}^+, X), WPSAP_\omega(\mathbb{R}^+, X)$  is defined in [18], [23], [24], [28], respectively.

Next, we introduce the new class of functions called  $\mu$ -pseudo  $\mathcal{S}$ -asymptotically periodic on  $\mathbb{R}$  by the measure theory.  $\mathcal{B}$  denotes the Lebesgue  $\sigma$ -field of  $\mathbb{R}$ ,  $\mathcal{M}$  stands for the set of all positive measure  $\mu$  on  $\mathcal{B}$  satisfying  $\mu(\mathbb{R}) = \infty$  and  $\mu([a, b]) < \infty$  for all  $a, b \in \mathbb{R} (a \leq b)$ . We formulate the following hypothesis:

$(H_0)$  For all  $\tau \in \mathbb{R}$ , there exist  $\beta > 0$  and a bounded interval  $I$  such that

$$\mu(\{a + \tau, a \in A\}) \leq \beta \mu(A) \quad \text{if } A \in \mathcal{B} \text{ satisfies } A \cap I = \emptyset.$$

**Definition 2.5.** Let  $\mu \in \mathcal{M}$ . A function  $f \in BC(\mathbb{R}, X)$  is called  $\mu$ -pseudo  $\mathcal{S}$ -asymptotically  $\omega$ -periodic if there exists  $\omega > 0$  such that

$$\lim_{T \rightarrow +\infty} \frac{1}{\mu([-T, T])} \int_{[-T, T]} \|f(t + \omega) - f(t)\| d\mu(t) = 0.$$

Denote by  $PSAP_\omega(\mathbb{R}, X, \mu)$  the set of such functions.

*Remark 2.2.* (i) If the measure  $\mu$  is the Lebesgue measure,  $PSAP_\omega(\mathbb{R}, X, \mu)$  is  $PSAP_\omega(\mathbb{R}, X)$ .

(ii) Let  $\rho(t) > 0$  a.e. on  $\mathbb{R}$  for the Lebesgue measure.  $\mu$  denotes the positive measure defined by

$$\mu(A) = \int_A \rho(t) dt \quad \text{for } A \in \mathcal{B},$$

where  $dt$  denotes the Lebesgue measure on  $\mathbb{R}$ , then  $PSAP_\omega(\mathbb{R}, X, \mu)$  is  $WPSAP_\omega(\mathbb{R}, X)$ . One can see [4, 8, 9] for more details.

Similarly as the proof of [9], one has the following results for  $PSAP_\omega(\mathbb{R}, X, \mu)$ .

**Lemma 2.1.** Let  $\mu \in \mathcal{M}$ , then the following properties hold:

- (i)  $f \pm g \in PSAP_\omega(\mathbb{R}, X, \mu)$  if  $f, g \in PSAP_\omega(\mathbb{R}, X, \mu)$ .
- (ii)  $\lambda f \in PSAP_\omega(\mathbb{R}, X, \mu)$  if  $\lambda \in \mathbb{R}, f \in PSAP_\omega(\mathbb{R}, X, \mu)$ .
- (iii)  $AP_\omega(\mathbb{R}, X) \subset SAP_\omega(\mathbb{R}, X) \subset PSAP_\omega(\mathbb{R}, X) \subset WPSAP_\omega(\mathbb{R}, X) \subset PSAP_\omega(\mathbb{R}, X, \mu)$ .
- (iv)  $PSAP_\omega(\mathbb{R}, X, \mu)$  is a Banach space with the supremum norm  $\|\cdot\|$ .

**Lemma 2.2.** Let  $\mu \in \mathcal{M}$  and satisfies  $(H_0)$ , then  $PSAP_\omega(\mathbb{R}, X, \mu)$  is translation invariant.

**Theorem 2.1.** Assume that  $\mu \in \mathcal{M}$ . Let  $f : \mathbb{R} \times X \rightarrow X$  be a function bounded on bounded sets of  $X$ ,  $f \in PSAP_\omega(\mathbb{R} \times X, X, \mu)$ , and there exists a constant  $L_f > 0$  such that

$$\|f(t, x) - f(t, y)\| \leq L_f \|x - y\|, \quad t \in \mathbb{R}, \quad x, y \in X,$$

then  $f(\cdot, u(\cdot)) \in PSAP_\omega(\mathbb{R}, X, \mu)$  if  $u(\cdot) \in PSAP_\omega(\mathbb{R}, X, \mu)$ .

**Lemma 2.3.** Let  $\mu \in \mathcal{M}$  and satisfies  $(H_0)$ , if  $f \in PSAP_\omega(\mathbb{R}, X, \mu)$ ,  $G \in L^1(\mathbb{R}, B(X))$ , then the convolution product  $f * G$  defined by

$$(f * G)(t) = \int_{-\infty}^{+\infty} G(s) f(t - s) ds, \quad t \in \mathbb{R}$$

lies in  $PSAP_\omega(\mathbb{R}, X, \mu)$ .

*Proof.* Let  $f \in PSAP_\omega(\mathbb{R}, X, \mu)$ , then by Lemma 2.2, one has  $f(\cdot - s) \in PSAP_\omega(\mathbb{R}, X, \mu)$  for all  $s \in \mathbb{R}$ . It is not difficult to see that  $f * G \in BC(\mathbb{R}, X)$ . Since  $\mu(\mathbb{R}) = +\infty$ , then there exists  $r_0 \geq 0$  such that  $\mu([-r, r]) > 0$  for all  $r \geq r_0$ . Hence by Fubini's theorem, one has

$$\begin{aligned} & \frac{1}{\mu([-T, T])} \int_{[-T, T]} \|(f * G)(t + \omega) - (f * G)(t)\| d\mu(t) \\ & \leq \frac{1}{\mu([-T, T])} \int_{[-T, T]} \int_{-\infty}^{+\infty} \|G(s)\| \|f(t + \omega - s) - f(t - s)\| ds d\mu(t) \\ & \leq \int_{-\infty}^{+\infty} \frac{\|G(s)\|}{\mu([-T, T])} \int_{[-T, T]} \|f(t - s + \omega) - f(t - s)\| d\mu(t) ds. \end{aligned}$$

Moreover, since  $G \in L^1(\mathbb{R}, B(X))$  and

$$0 \leq \frac{\|G(s)\|}{\mu([-T, T])} \int_{[-T, T]} \|f(t - s + \omega) - f(t - s)\| d\mu(t) \leq 2\|G\| \|f\| \quad \text{for all } s \in \mathbb{R},$$

then

$$\lim_{T \rightarrow +\infty} \int_{-\infty}^{+\infty} \frac{\|G(s)\|}{\mu([-T, T])} \int_{[-T, T]} \|f(t - s + \omega) - f(t - s)\| d\mu(t) ds = 0,$$

by Lebesgue dominated convergence theorem and  $f(\cdot - s) \in PSAP_\omega(\mathbb{R}, X, \mu)$ , one has

$$\lim_{T \rightarrow +\infty} \frac{1}{\mu([-T, T])} \int_{[-T, T]} \|(f * G)(t + \omega) - (f * G)(t)\| d\mu(t) = 0,$$

that is  $f * G \in PSAP_\omega(\mathbb{R}, X, \mu)$ . □

### 3 Variation of constants formula and spectral decomposition

In this paper, we will investigate the existence and uniqueness of  $\mu$ -pseudo  $\mathcal{S}$ -asymptotically periodic integral solution for PNFDEs:

$$\frac{d}{dt} \mathcal{D}u_t = A\mathcal{D}u_t + L(u_t) + f(t), \quad t \in \mathbb{R}, \tag{3.1}$$

where  $A$  is a linear operator on Banach space  $X$ , not necessarily densely defined and satisfies the Hille-Yosida condition. Fix  $r \geq 0$ ,  $u_t \in \mathcal{C}$  is defined by  $u_t(\theta) = u(t + \theta)$  for  $\theta \in [-r, 0]$ .  $\mathcal{D} \in B(\mathcal{C}, X)$ ,  $L \in B(\mathcal{C}, X)$ ,  $f \in PSAP_\omega(\mathbb{R}, X, \mu)$ . For the well posedness of (3.1), we assume that  $\mathcal{D}$  has the following form:

$$\mathcal{D}\psi = \psi(0) - \int_{-r}^0 [d\eta(\theta)]\psi(\theta) \quad \text{for } \psi \in \mathcal{C},$$

for a mapping  $\eta : [-r, 0] \rightarrow B(X)$  of bounded variation and nonatomic at zero, which means that there exists a continuous nondecreasing function  $\delta : [0, r] \rightarrow [0, +\infty)$  such that  $\delta(0) = 0$  and

$$\left| \int_{-s}^0 [d\eta(\theta)]\psi(\theta) \right| \leq \delta(s) \sup_{-r \leq \theta \leq 0} |\psi(\theta)| \quad \text{for } \psi \in \mathcal{C}, s \in [0, r].$$

To (3.1), we associate the following initial value problem

$$\begin{cases} \frac{d}{dt} \mathcal{D}u_t = A\mathcal{D}u_t + L(u_t) + f(t), & t \geq \sigma, \\ u_\sigma = \varphi \in \mathcal{C}. \end{cases} \tag{3.2}$$

**Definition 3.1.** [16]  $u \in C([-r + \sigma, +\infty], X)$  is said to be an integral solution of (3.2) if the following conditions hold:

- (i)  $\int_\sigma^t \mathcal{D}u_s ds \in D(A)$  for  $t \geq \sigma$ .
- (ii)  $\mathcal{D}u_t = \mathcal{D}\varphi + A \int_\sigma^t \mathcal{D}u_s ds + \int_\sigma^t (L(u_s) + f(s)) ds$  for  $t \geq \sigma$ .
- (ii)  $u_\sigma = \varphi$ .

If  $\overline{D(A)} = X$ , the integral solution coincide with the known mild solution. One can see that if  $u_t$  is an integral solution of (3.2), then  $u_t \in D(A)$  for all  $t \geq 0$ , in particular  $\mathcal{D}\varphi \in \overline{D(A)}$ . Let us introduce the part  $A_0$  of the operator  $A$  in  $\overline{D(A)}$  which defined by

$$\begin{cases} D(A_0) = \{x \in D(A) : Ax \in \overline{D(A)}\} \\ A_0x = Ax \quad \text{for } x \in D(A_0). \end{cases}$$

We make the following assumption:

( $H_1$ )  $A$  satisfies the Hille-Yosida condition: there exist  $M \geq 1, \omega \in \mathbb{R}$  such that  $(\omega, +\infty) \subset \rho(A)$  and

$$|R(\lambda, A)^n| \leq \frac{M}{(\lambda - \omega)^n}, \quad \text{for } n \in \mathbb{N}, \lambda > \omega.$$

**Lemma 3.1.** [6]  $A_0$  generates a strongly continuous semigroup  $(T_0(t))_{t \geq 0}$  on  $\overline{D(A)}$ .

The phase space  $\mathcal{C}_0$  of (3.2) is defined by

$$\mathcal{C}_0 = \{\varphi \in \mathcal{C} : \mathcal{D}\varphi \in \overline{D(A)}\}.$$

For each  $t \geq 0$ , we define the linear operator  $\mathcal{U}(t)$  on  $\mathcal{C}_0$  by

$$\mathcal{U}(t) = v_t(\cdot, \varphi),$$

where  $v(\cdot, \varphi)$  is the solution of the following homogeneous equation

$$\begin{cases} \frac{d}{dt} \mathcal{D}v_t = A\mathcal{D}v_t + L(v_t), & t \geq 0, \\ v_0 = \varphi \in \mathcal{C}. \end{cases}$$

**Proposition 3.1.** [2]  $(\mathcal{U}(t))_{t \geq 0}$  is a strongly continuous semigroup of linear operators on  $\mathcal{C}_0$ .

**Theorem 3.1.** [2] Let  $\mathcal{A}_U$  defined on  $\mathcal{C}_0$  by

$$\begin{cases} D(\mathcal{A}_U) = \{\varphi \in C^1([-r, 0], X) : \mathcal{D}\varphi \in D(A), \mathcal{D}\varphi' \in \overline{D(A)} \text{ and } \mathcal{D}\varphi' = A\mathcal{D}\varphi + L(\varphi)\} \\ \mathcal{A}_U\varphi = \varphi' \text{ for } \varphi \in D(\mathcal{A}_U). \end{cases}$$

Then  $\mathcal{A}_U$  is the infinitesimal generator of the semigroup  $(\mathcal{U}(t))_{t \geq 0}$  on  $\mathcal{C}_0$ .

Let  $X_0$  be the space defined by

$$\langle X_0 \rangle = \{X_0c : c \in X\}$$

where the function  $X_0c$  is defined by

$$(X_0c)(\theta) = \begin{cases} 0 & \text{if } \theta \in [-r, 0), \\ c & \text{if } \theta = 0. \end{cases}$$

The space  $\mathcal{C}_0 \oplus \langle X_0 \rangle$  equipped with the norm  $|\varphi + X_0c| = |\varphi|_{\mathcal{C}} + |c|$  for  $(\varphi, c) \in \mathcal{C}_0 \times X$  is a Banach space and consider the extension  $\tilde{\mathcal{A}}_U$  defined on  $\mathcal{C}_0 \oplus \langle X_0 \rangle$  by

$$\begin{cases} D(\tilde{\mathcal{A}}_U) = \{\varphi \in C^1([-r, 0], X) : \mathcal{D}\varphi \in D(A) \text{ and } \mathcal{D}\varphi' \in \overline{D(A)}\} \\ \tilde{\mathcal{A}}_U\varphi = \varphi' + X_0(A\mathcal{D}\varphi + L(\varphi) - \mathcal{D}\varphi'). \end{cases}$$

In order to compute the resolvent operator  $R(\lambda, \tilde{\mathcal{A}}_U)$ , we suppose the following assumption.

(H<sub>2</sub>)  $\mathcal{D}e^\lambda c \in D(A)$  for all  $c \in D(A)$  and all complex  $\lambda$ , where  $e^\lambda c \in \mathcal{C}$  is defined by

$$(e^\lambda c)(\theta) = e^{\lambda\theta} c, \quad \text{for } \theta \in [-r, 0].$$

**Lemma 3.2.** [2] Assume that (H<sub>1</sub>)-(H<sub>2</sub>) hold, then  $\tilde{\mathcal{A}}_U$  satisfies the Hille-Yosida condition on  $\mathcal{C}_0 \otimes \langle X_0 \rangle$ : there exists  $\tilde{M} \geq 0, \tilde{\omega} \in \mathbb{R}$  such that  $(\tilde{\omega}, +\infty) \subset \rho(\tilde{\mathcal{A}}_U)$  and

$$|(\lambda I - \tilde{\mathcal{A}}_U)^{-n}| \leq \frac{\tilde{M}}{(\lambda - \tilde{\omega})^n}, \quad \text{for } n \in \mathbb{N}, \lambda > \tilde{\omega}.$$

Moreover, the part of  $\tilde{\mathcal{A}}_U$  on  $D(\tilde{\mathcal{A}}_U) = \mathcal{C}_0$  is exactly the operator  $\mathcal{A}_U$ .

Now, we can state the variation of constants formula associated to (3.2)

**Theorem 3.2.** [2] Assume that (H<sub>1</sub>) and (H<sub>2</sub>) hold, then for  $\varphi \in \mathcal{C}_0$ , the integral solution  $x$  of (3.2) is given by the following variation of constants formula

$$u_t = \mathcal{U}(t)\varphi + \lim_{\lambda \rightarrow +\infty} \int_0^t \mathcal{U}(t-s) \tilde{B}_\lambda X_0 f(s) ds \quad \text{for } t \geq \sigma,$$

where  $\tilde{B}_\lambda = \lambda(\lambda I - \tilde{\mathcal{A}}_U)^{-1}$  for  $\lambda > \tilde{\omega}$ .

**Definition 3.2.** We say a semigroup  $(\mathcal{U}(t))_{t \geq 0}$  is hyperbolic if

$$\sigma(\mathcal{A}_U) \cap i\mathbb{R} = \emptyset.$$

**Definition 3.3.** The operator  $\mathcal{D}$  is said to be stable if there exist positive constants  $\eta, \nu$  such that the solution of the homogenous equation

$$\begin{cases} \mathcal{D}y_t = 0 & \text{for } t \geq 0 \\ y_0 = \phi, \end{cases}$$

where  $\phi \in \{\psi \in \mathcal{C} : \mathcal{D}\psi = 0\}$  satisfies

$$|y_t(\cdot, \phi)| \leq \nu e^{-\eta t} |\phi| \quad \text{for } t \geq 0.$$

Example 3.1. The operator  $\mathcal{D}$  defined by

$$\mathcal{D}\varphi = \varphi(0) - q\varphi(-r)$$

is stable if and only if  $|q| < 1$ .

For the sequel, we make the following assumptions:

(H<sub>3</sub>)  $T_0(t)$  is compact on  $\overline{D(A)}$  for every  $t > 0$ .

(H<sub>4</sub>) The operator  $\mathcal{D}$  is stable.

We get the following result on the spectral decomposition of the phase space  $\mathcal{C}_0$ .

**Theorem 3.3.** [2] Assume that (H<sub>1</sub>)-(H<sub>4</sub>) hold. If the semigroup  $(\mathcal{U}(t))_{t \geq 0}$  is hyperbolic, then the space  $\mathcal{C}_0$  is decomposed as a direct sum

$$\mathcal{C}_0 = S \oplus U$$

of two  $\mathcal{U}(t)$  invariant closed subspaces  $S$  and  $U$  such that the restricted semigroup on  $U$  is a group and there exist positive constants  $\overline{M}, \overline{\omega}$  such that

$$\begin{aligned} |\mathcal{U}(t)\varphi| &\leq \overline{M}e^{-\overline{\omega}t}|\varphi| \quad \text{for } t \geq 0, \varphi \in S, \\ |\mathcal{U}(t)\varphi| &\leq \overline{M}e^{\overline{\omega}t}|\varphi| \quad \text{for } t \leq 0, \varphi \in U, \end{aligned}$$

where  $S$  and  $U$  are called the stable and unstable space respectively.

**Theorem 3.4.** [5] Assume that (H<sub>1</sub>)-(H<sub>4</sub>) hold and the semigroup  $(\mathcal{U}(t))_{t \geq 0}$  is hyperbolic. If  $f \in BC(\mathbb{R}, X)$ , then there exists a unique bounded integral solution  $u$  of (3.1) which is given by

$$\begin{aligned} u_t &= \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s)\Pi^s(\tilde{B}_\lambda X_0 f(s))ds \\ &+ \lim_{\lambda \rightarrow +\infty} \int_{+\infty}^t \mathcal{U}^u(t-s)\Pi^u(\tilde{B}_\lambda X_0 f(s))ds \quad \text{for } t \in \mathbb{R}, \end{aligned} \tag{3.3}$$

where  $\mathcal{U}^s(t), \mathcal{U}^u(t)$  are the restrictions of  $\mathcal{U}(t)$  on  $S, U$  respectively,  $\Pi^s, \Pi^u$  are the projections of  $\mathcal{C}_0$  onto  $S, U$ , respectively.

### 4 Partial neutral functional differential equations

In what follows, we will investigate the existence, uniqueness of  $\mu$ -pseudo  $\mathcal{S}$ -asymptotically periodic integral solution of PNFDEs. First, consider following partial neutral functional differential equations

$$\frac{d}{dt}\mathcal{D}u_t = A\mathcal{D}u_t + L(u_t) + f(t), \quad t \in \mathbb{R}, \tag{4.1}$$

where  $A$  is a linear operator on Banach space  $X$ , satisfies the Hille-Yosida condition.  $u_t \in \mathcal{C}$  is defined by  $u_t(\theta) = u(t + \theta)$  for  $\theta \in [-r, 0]$ .  $\mathcal{D} \in B(\mathcal{C}, X), L \in B(\mathcal{C}, X), f \in PSAP_\omega(\mathbb{R}, X, \mu), \mu \in \mathcal{M}$ .

**Theorem 4.1.** Assume that (H<sub>1</sub>)-(H<sub>4</sub>) hold,  $f \in PSAP_\omega(\mathbb{R}, X, \mu), \mu \in \mathcal{M}$  and the semigroup  $(\mathcal{U}(t))_{t \geq 0}$  is hyperbolic, then (4.1) has a unique integral solution  $u \in PSAP_\omega(\mathbb{R}, X, \mu)$  which is given by (3.3).

*Proof.* By Theorem 3.4, (4.1) has a unique bounded integral solution  $u$  which is given by (3.3). Let

$$u_t = (\Gamma^s f)(t) + (\Gamma^u f)(t),$$

where

$$\begin{aligned} (\Gamma^s f)(t) &= \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s)\Pi^s(\tilde{B}_\lambda X_0 f(s))ds, \\ (\Gamma^u f)(t) &= \lim_{\lambda \rightarrow +\infty} \int_{+\infty}^t \mathcal{U}^u(t-s)\Pi^u(\tilde{B}_\lambda X_0 f(s))ds. \end{aligned}$$

By  $(H_1)$ , there exists a constant  $\tilde{K} > 0$  such that

$$\|(\Gamma^s f)(t)\| \leq \tilde{K} \int_{-\infty}^t e^{-\bar{\omega}(t-s)} \|f(s)\| ds. \tag{4.2}$$

Let  $G : \mathbb{R} \rightarrow \mathbb{R}$  be the function defined by

$$G(t) = e^{-\bar{\omega}t} \text{ for } t \geq 0 \text{ and } G(t) = 0 \text{ for } t < 0,$$

hence

$$\int_{-\infty}^t e^{-\bar{\omega}(t-s)} \|f(s)\| ds = \int_0^\infty e^{-\bar{\omega}s} \|f(t-s)\| ds = \int_{-\infty}^\infty G(s) \|f(t-s)\| ds.$$

Since  $\|f(t)\| \in PSAP_\omega(\mathbb{R}, \mathbb{R}, \mu)$ , by Lemma 2.3, one has

$$\int_{-\infty}^t e^{-\bar{\omega}(t-s)} \|f(s)\| ds \in PSAP_\omega(\mathbb{R}, \mathbb{R}, \mu),$$

so  $\Gamma^s f \in PSAP_\omega(\mathbb{R}, X, \mu)$  by (4.2). Proceeding in a similar manner, we have  $\Gamma^u f \in PSAP_\omega(\mathbb{R}, X, \mu)$ . The proof is complete.  $\square$

Next, consider the nonlinear equation

$$\frac{d}{dt} \mathcal{D}u_t = A\mathcal{D}u_t + L(u_t) + f(t, u(t-r)), \quad t \in \mathbb{R}, \tag{4.3}$$

where  $A$  is a linear operator on Banach space  $X$ , satisfies the Hille-Yosida condition,  $\mathcal{D} \in B(\mathcal{C}, X)$ ,  $L \in B(\mathcal{C}, X)$ ,  $f : \mathbb{R} \times X \rightarrow X$  is a function bounded on bounded sets of  $X$ .

We make the following assumption:

$(H_5)$   $f \in PSAP_\omega(\mathbb{R} \times X, X, \mu)$ ,  $\mu \in \mathcal{M}$  and and satisfies the Lipschitz condition

$$\|f(t, u) - f(t, v)\| \leq L_f \|u - v\|, \quad u, v \in X, \quad t \in \mathbb{R},$$

where  $L_f > 0$  is a constant.

**Theorem 4.2.** Assume that  $(H_0)$ - $(H_5)$  hold and the semigroup  $(\mathcal{U}(t))_{t \geq 0}$  is hyperbolic, then (4.3) has a unique integral solution  $u(t) \in PSAP_\omega(\mathbb{R}, X, \mu)$  if  $L_f$  is small enough.

*Proof.* Let  $v \in PSAP_\omega(\mathbb{R}, X, \mu)$ , by Theorem 2.1, Lemma 2.2 and  $(H_5)$ , it is easy to see that  $f(\cdot, v(\cdot - r)) \in PSAP_\omega(\mathbb{R}, X, \mu)$ . Consider the equation

$$\frac{d}{dt} \mathcal{D}u_t = A\mathcal{D}u_t + L(u_t) + f(t, v(t-r)), \quad t \in \mathbb{R}. \tag{4.4}$$

By Theorem 4.1, we deduce that (4.4) has a unique integral solution  $\mathcal{F}v$  which is given by

$$\left[ \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s) \Pi^s(\tilde{B}_\lambda X_0 f(s, v(s-r))) ds + \lim_{\lambda \rightarrow +\infty} \int_{+\infty}^t \mathcal{U}^u(t-s) \Pi^u(\tilde{B}_\lambda X_0 f(s, v(s-r))) ds \right] (0).$$

The operator  $\mathcal{F}$  is well defined on  $PSAP_\omega(\mathbb{R}, X, \mu)$ . By  $(H_5)$ , there exists a constant  $\sigma_0$  such that

$$\sup_{t \in \mathbb{R}} |(\mathcal{F}v_1)(t) - (\mathcal{F}v_2)(t)| \leq L_f \sigma_0 \sup_{t \in \mathbb{R}} |v_1(t) - v_2(t)|.$$

If we choose  $L_f \sigma_0 < 1$ , by Banach contraction mapping principle,  $\mathcal{F}$  has a unique fixed point in  $PSAP_\omega(\mathbb{R}, X, \mu)$ , which is the  $\mu$ -pseudo  $\mathcal{S}$ -asymptotically periodic integral solution to (4.3).  $\square$

### 5 Example

Consider the nonautonomous version of the model proposed in [26]

$$\begin{cases} \frac{\partial}{\partial t}[u(t, \xi) - qu(t - r, \xi)] = \frac{\partial^2}{\partial \xi^2}[u(t, \xi) - qu(t - r, \xi)] + \int_{-r}^0 \gamma(\theta)u(t + \theta, \xi)d\theta \\ \quad + \vartheta(u(t - r, \xi)) + \phi(t)g(\xi) \quad \text{for } t \in \mathbb{R}, \xi \in [0, \pi], \\ u(t, 0) - qu(t - r, 0) = u(t, \pi) - qu(t - r, \pi) = 0 \quad \text{for } t \in \mathbb{R}, \end{cases} \tag{5.1}$$

where  $q \in (0, 1)$ ,  $\gamma \in C([-r, 0], \mathbb{R})$ ,  $g \in C([-r, 0], \mathbb{R})$ ,  $\vartheta : \mathbb{R} \rightarrow \mathbb{R}$  is a Lipschitzian continuous function with Lipschitz constant  $L_\vartheta$ , and  $\phi \in PSAP_\omega(\mathbb{R}, X, \mu)$ ,  $\mu \in \mathcal{M}$  satisfying  $(H_0)$ .

Let  $X = C([0, \pi], \mathbb{R})$  and define the operator  $A$  by

$$D(A) = \{u \in C^2([0, \pi], \mathbb{R}) : u(0) = u(\pi) = 0\}, \quad \text{and } Au := u'', \quad u \in D(A).$$

**Lemma 5.1.** [14] *The operator  $A$  satisfies the Hille-Yosida condition on  $X$ :*

$$(0, +\infty) \subset \rho(A) \quad \text{and} \quad |(\lambda I - A)^{-1}| \leq \frac{1}{\lambda} \quad \text{for } \lambda > 0.$$

It is not difficult to see that  $(H_1)$  holds by Lemma 5.1. Let  $A_0$  be the part of the operator  $A$  in  $\overline{D(A)}$ ,  $A_0$  is given by

$$\begin{aligned} D(A_0) &= \{u \in C^2([0, \pi], \mathbb{R}) : u(0) = u(\pi) = u''(0) = u''(\pi) = 0\}, \\ Au &:= u'', \quad u \in D(A_0). \end{aligned}$$

$A_0$  generates a strongly continuous compact semigroup  $(T_0(t))_{t \geq 0}$  on  $\overline{D(A)}$ , which implies that  $(H_3)$  holds and  $\overline{D(A)} = \{u \in X : u(0) = u(\pi) = 0\}$ .

Define the bounded linear operator  $\mathcal{D} : \mathcal{C} \rightarrow X$  by

$$\mathcal{D}\psi = \psi(0) - q\psi(\psi - r).$$

Since  $0 < q < 1$ , then  $\mathcal{D}$  is stable and  $(H_4)$  holds. Moreover, by definitions of the operators  $A, \mathcal{D}$ , it follows that  $(H_2)$  is satisfied.

Let

$$\begin{aligned} L(\psi)(\xi) &= \int_{-r}^0 \gamma(\theta)\psi(\theta)(\xi)d\theta \quad \text{for } \xi \in [0, \pi], \psi \in \mathcal{C}. \\ f(t, y)(\xi) &= \vartheta(y(\xi)) + \phi(t)g(\xi) \quad \text{for } y \in X, t \in \mathbb{R}, \xi \in [0, \pi], \end{aligned}$$

then  $L \in B(\mathcal{C}, X)$  and  $(H_5)$  holds with the Lipschitz constant  $L_\vartheta$ . Let  $u(t) = u(t, \cdot)$ , (5.1) can be rewritten as an abstract system of the form (4.3). For the hyperbolicity, we suppose that

$$(H_6) \quad \int_{-r}^0 |\gamma(\theta)|d\theta < 1 - q.$$

**Lemma 5.2.** [16] *Assume that  $(H_6)$  holds, then the semigroup  $(\mathcal{U}(t))_{t \geq 0}$  is hyperbolic.*

By Theorem 4.2, one has

**Theorem 5.1.** *Under the above assumptions, (5.1) has a unique integral solution  $u \in PSAP_\omega(\mathbb{R}, X, \mu)$  if  $L_\vartheta$  is small enough.*

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