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Existence results for non-autonomous neutral integro-differential systems with impulsive and nonlocal conditions

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Abstract

In accordance with semigroup theory, fractional powers of operators, approximation techniques and Banach contraction principle fixed point theorem, this manuscript is primarily involved with the existence results for an impulsive non-autonomous neutral integro-differential systems with nonlocal conditions in Banach space $\mathbb{E}$.

Keywords: Integro-differential equations, Semigroup theory, Impulsive and nonlocal conditions, Evolution equations, Fixed point theorem.

2010 MSC: 34A37, 34K37, 35R11, 34G20.

1 Introduction

The dynamics of many processes in engineering, physics, population dynamics, biology, medicine and other fields are subject to sudden changes just like shocks or perturbations. These perturbations may be considered as impulses. In particular, in the periodic treatment of some diseases, impulses correspond to the administration of a drug treatment or a missing product. In environmental sciences, impulses correspond to seasonal changes of the water level of artificial reservoirs. This sort of models can be defined by impulsive differential equations. For more details on this theory and its applications, we suggest the reader to refer the books [1, 2] and the papers [3–8], and the references cited therein. These days, impulsive integro-differential equations have become an significant area of research because of their uses to numerous problems arising in communications, control technology, impact mechanics and electrical engineering, etc.

The nonlocal condition, which is a speculation of the standard initial condition, was inspired by physical issues. On many instances, problems under consideration, primarily coming up from physics phenomena, advise that the initial condition is an estimation via solving the problem in some finite sequence of times, and then we say that the initial condition is nonlocal. Evolution problems with nonlocal initial conditions in Banach spaces are now perfectly realized due to the fact it was initiated by Byszewski [9, 10], where the author demonstrated the existence and uniqueness of mild, strong and classical solution to the first-order initial value problem by utilizing the techniques of semigroups and the Banach fixed point theorem. For the importance of nonlocal conditions in diverse areas, we suggest [9, 10] and references cited therein.

Moreover, a class of equations depends on past as well as present values but which involve derivatives with delays as well as the function itself. Such equations historically have been referred to as neutral functional differential equations. For systems with neutral type, the existence of the solution has been investigated in Tsoi [11]. A great information to the literature for neutral functional differential equations is the book by Hale and Lunel [12] and the references therein.

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The existence, controllability and other qualitative and quantitative properties of non-autonomous differential and integro-differential equations with impulsive conditions) are the most advancing area of pursuit, in particular, see [13–15]. In [13], the authors studied the existence of the mild solutions to a class of abstract non-autonomous impulsive functional integro-differential equations. The existence and Ulam-Hyers-Rassias stability of mild solution of impulsive non-autonomous differential equations are studied by authors in [14]. In particular, in [15], author has demonstrated the controllability of a system of impulsive semilinear non-autonomous differential equations via Rothe's type fixed-point theorem. By applying approximation techniques and fractional operator, the existence of the mild solution for different class of impulsive functional integro-differential equations have been established by many authors[16–21]. Recently, in [19], the authors investigate the existence of a mild solution for an impulsive nonlocal non-autonomous neutral functional differential equation in Banach space by utilizing the approximation techniques, fractional powers of operators and Krasnoselskii's fixed-point theorem.

Motivated by above mentioned works [13, 19], the main purpose of this paper is to prove the existence of mild solutions for the following impulsive non-autonomous neutral partial integro-differential equations in a Banach space $\mathbb{E}$:

$$\frac{d}{dt}\left[z(t)-\mathscr{F}\left(t,z(h_1(t)),\int_0^t a_1(t,s,z(h_2(s)))ds\right)\right]=-B(t)z(t)+\mathscr{G}\left(t,z(h_3(t)),\int_0^t a_2(t,s,z(h_4(s)))ds\right)$$
$$+\mathscr{H}\left(t,z(h_5(t)),\int_0^t a_3(t,s,z(h_6(s)))ds\right),\ t\in\mathscr{I},\ t\neq t_i, \tag{1.1}$$
$$z(0)=z_0+g(z)\in\mathbb{E}, \tag{1.2}$$
$$\Delta z(t_i)=\mathcal{I}_i(z(t_i)),\quad i=1,2,\ldots,q,\ q\in\mathbb{N}, \tag{1.3}$$

where $\mathscr{I}=[0,T], 0<T<\infty, -B(t):\mathcal{D}(B(t))\subseteq\mathbb{E}\to\mathbb{E}, t\geq 0$ is a closed densely defined linear operator. Here, $h_j:\mathscr{I}\to\mathscr{I}, j=1,2,\ldots,6$ and $0=t_0<t_1<t_2<\cdots<t_q<t_{q+1}=T$ are fixed numbers, $\Delta z|_{t=t_i}=z(t_i^+)-z(t_i^-)$ and $z(t_i^-)=\lim\limits_{\epsilon\to 0-}z(t_i+\epsilon)$ and $z(t_i^+)=\lim\limits_{\epsilon\to 0+}z(t_i+\epsilon)$ denotes the left and right limits of $z(t)$ at $t=t_i$, respectively. Let $B(t)$ be the infinitesimal generator of a compact analytic semigroup of bounded linear operators on a Banach space $\mathbb{E}$. The functions $\mathscr{F},\mathscr{G},\mathscr{H},a_i, i=1,2,3$ and $I_i:\mathbb{E}\to\mathbb{E}(i=1,2,\ldots,q)$ are appropriate functions fulfilling some suitable conditions to be specified later.

The rest of this paper is organized as follows: In section 2, we recall some basic definitions and preliminary facts which will be utilized throughout this paper. Existence theorems and their proofs are given in section 3. Finally, in Section 4 an example is presented to illustrate the application of the obtained results.

2 Preliminaries

In this section, we recall some basic definitions, preliminaries, theorems and lemmas and assumptions required for establishing our results.

Throughout this manuscript, we assume that $(\mathbb{E},\|\cdot\|)$ is a Banach space and the notation $\mathcal{C}([0,T],\mathbb{E})$ stands for the space of $\mathbb{E}$-valued continuous functions on $[0,T]$ with the norm $\|y\|=\sup\{\|y(\tau)\|,\tau\in[0,T]\}$ and $\mathscr{L}^1([0,T],\mathbb{E})$ denotes the space of $\mathbb{E}$-valued Bochner integrable functions on $[0,T]$ endowed with the norm $\|\mathcal{F}\|_{\mathscr{L}^1}=\int_0^T\|\mathcal{F}(t)\|dt, \mathcal{F}\in([0,T],\mathbb{E})$. We denote by $\mathcal{C}^\beta([0,T],\mathbb{E})$ the space of all uniformly H$o$lder continuous functions from $[0,T]$ into $\mathbb{E}$ with exponent $\beta>0$. We can easily confirm that $\mathcal{C}^\beta([0,T],\mathbb{E})$ is a Banach space with the norm

$$\|z\|_{\mathcal{C}^\beta([0,T],\mathbb{E})}=\sup_{0\leq t\leq T}\|z(t)\|+\sup_{0\leq t,s\leq T,t\neq s}\frac{\|z(t)-z(s)\|}{|t-s|^\beta}.$$

To be able to define the mild solution for the impulsive problem, we define the space $PC([0,T];\mathbb{E})=\{z:[0,T]\to\mathbb{E}: y$ is continuous at $t\neq t_i$ and left continuous at $t=t_i$ and $z(t_i^+)$ exists, for all $i=1,2,\cdots,q\}$. Clearly, $PC([0,T];\mathbb{E})$ is a Banach space endowed the norm $\|z\|_{PC}=\sup\limits_{t\in[0,T]}\|z(s)\|$. For a function $z \in PC([0,T];\mathbb{E})$ and $i \in \{0,1,\cdots,q\}$, we define the function $\widetilde{z}_i \in \mathcal{C}([t_i,t_{i+1}],\mathbb{E})$ such that

$$\widetilde{z}_i(t) = \begin{cases} z(t), & \text{for} \quad t \in (t_i, t_{i+1}], \\ z(t_i^+), & \text{for} \quad t = t_i. \end{cases}$$

For $W \subset PC([0,T],\mathbb{E})$ and $i \in \{0,1,\cdots,q\}$, we have $\widetilde{W}_i = \{\widetilde{z}_i : z \in W\}$ and following Accoli-Arzela type criteria.

Lemma 2.1. *[18] A set $W \subset PC([0,T];\mathbb{E})$ is relatively compact in $PC([0,T];\mathbb{E})$ if and only if each set $\widetilde{W}_j (j = 1,2,\cdots,q)$ is relatively compact in $\mathcal{C}([t_j,t_{j+1}],\mathbb{E})(j = 0,1,2,\cdots,q)$.*

Let $\{B(t) : 0 \leq t \leq T\}, 0 < T < \infty$ be a family of closed linear operators on the Banach space $\mathbb{E}$. We impose following restrictions ([22]) as:

(P1) The domain $\mathcal{D}(B)$ of $\{B(t) : t \in [0,T]\}$ is dense in $\mathbb{E}$ and $\mathcal{D}(B)$ is independent of t.

(P2) For each $0 \leq t \leq T$ and $Re\,\lambda \leq 0$, the resolvent $R(\lambda; B(t))$ exists and there exists a positive constant K (independent of t and λ) such that

$$\|R(\lambda; B(t))\| \leq \frac{K}{(|\lambda|+1)}, \quad Re\lambda \leq 0, \quad t \in [0,T].$$

(P3) For each fixed $\xi \in [0,T]$, there exists a constant $K > 0$ and $0 < \mu \leq 1$ such that

$$\|[B(\tau) - B(s)]B^{-1}(\xi)\| \leq K|\tau - s|^{\mu}, \quad \text{for any } \tau, s \in [0,T],$$

where μ and K are independent of τ, s and ξ.

(P4) For every $t \in [0,T]$, the resolvent set of $B(t)$, the resolvent $R(\lambda, B(t))$, is a compact operator for some $\lambda \in \rho(B(t))$.

The assumptions $(P1) - (P3)$ permit that there is a unique linear evolution system (linear evolution operator) $\mathcal{S}(t,s), 0 \leq s \leq t \leq T$ which is generated by family $\{B(t) : t \in [0,T]\}$ and there exists a family of bounded linear operators $\{\Phi(t,s) : 0 \leq t \leq s \leq T\}$ such that $\|\Phi(t,s)\| \leq \dfrac{K}{|t-s|^{1-\mu}}$. We also have that $\mathcal{S}(t,s)$ can be written as

$$\mathcal{S}(t,s) = e^{-(t-s)B(t)} + \int_0^t e^{-(t-\tau)B(\tau)}\Phi(\tau,s)d\tau.$$

The assumption $(P2)$ guarantees that $-B(s), s \in [0,T]$ is the infinitesimal generator of a strongly continuous compact analytic semigroup $\{e^{-tB(s)} : t \geq 0\}$ in $\mathbb{B}(\mathbb{E})$, where the symbol $\mathbb{B}(\mathbb{E})$ stands for the Banach algebra of all bounded linear operators on $\mathbb{E}$.

By the assumptions $(P1) - (P4)$ see [[22]], it follows that there is a unique fundamental solution $\{\mathcal{S}(t,s) : 0 \leq t \leq s \leq T\}$ for the homogeneous Cauchy problem such that

(i) $\mathcal{S}(t,s) \in \mathbb{B}(\mathbb{E})$ and the mapping $(t,s) \to \mathcal{S}(t,s)y$ is continuous for $y \in \mathbb{E}$, i.e $\mathcal{S}(t,s)$ is strongly continuous in t,s for all $0 \leq s \leq t \leq T$.

(ii) For each $y \in \mathbb{E}, \mathcal{S}(t,s)y \in \mathcal{D}(B)$, for all $0 \leq s \leq t \leq T$.

(iii) $\mathcal{S}(t,\tau)\mathcal{S}(\tau,s) = \mathcal{S}(t,s)$ for all $0 \leq s \leq \tau \leq t \leq T$.

(iv) For each $0 \leq s < t \leq T$, the derivative $\dfrac{\partial \mathcal{S}(t,s)}{\partial t}$ exists in the stong operator topology and an element of $\mathbb{B}(\mathbb{E})$, and strongly continuous in t, where $s < t \leq T$.

(v) $\mathcal{S}(t,t) = I$.

(vi) $\dfrac{\partial \mathcal{S}(t,s)}{\partial t} + B(t)\mathcal{S}(t,s) = 0$ for all $0 \leq s < t \leq T$.

Further, we have also the following assumptions:

$$\|e^{-tB(\tau)}\| \le Ke^{-dt}, \quad t \ge 0;$$
$$\|B(\tau)e^{-tB(t)}\| \le \frac{Ke^{-dt}}{t}, \quad t > 0;$$
$$\|B(t)\mathcal{S}(t,\tau)\| \le K|t-\tau|^{-1}, \quad 0 \le s \le t \le T,$$

for all $\tau \in [0,T]$, where d is a positive constant. For $\alpha > 0$, we may define negative fractional powers $B(t)^{-\alpha}$ as

$$B(t)^{-\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^\infty s^{\alpha-1}e^{-sB(t)}ds.$$

Then, the operator $B(t)^{-\alpha}$ is bounded linear and one to one operator on $\mathbb{E}$ and $B^{-\alpha}(t)B^{-\beta}(t) = B^{-(\alpha+\beta)}(t)$. Therefore, it implies that there exists an inverse of the operator $B(t)^{-\alpha}$. We can define $B(t)^{\alpha} \equiv [B(t)^{-\alpha}]^{-1}$ which is the positive fractional powers of $B(t)$. The operator $B(t)^{\alpha} \equiv [B(t)^{-\alpha}]^{-1}$ is closed densely defined linear operator with domain $\mathcal{D}(B(t)^{\alpha}) \subset \mathbb{E}$ and for $\alpha < \beta$, we get $\mathcal{D}(B(t)^{\beta}) \subset \mathcal{D}(B(t)^{\alpha})$. Let $E_\alpha(t_0) = \mathcal{D}(B(t_0)^{\alpha})$ be a Banach space with a norm $\|z\|_\alpha = \|B(t_0)^{\alpha}z\|, t_0 \in [0,T]$. For $0 < \omega_1 \le \omega_2$, we have that embedding $E_{\omega_2}(t_0) \hookrightarrow E_{\omega_1}(t_0)$ is continuous and dense. For each $\alpha > 0$, we may define $E_{-\alpha}(t_0) = (E_\alpha)^*(t_0)$, which is the dual space of $E_\alpha(t_0)$. The dual space is a Banach space with natural norm $\|z\|_{-\alpha} = \|B(t_0)^{-\alpha}z\|$. In particular, by the assumption $(P3)$, we conclude a constant $K > 0$, such that

$$\|B(t)B(s)^{-1}\| \le K, \quad \text{for all} \quad 0 \le s,t \le T. \tag{2.1}$$

Now, we also have following results:

$$\|B^{\alpha}(t)B^{-\beta}(s)\| \le \mathcal{N}_{\alpha,\beta}, \tag{2.2}$$
$$\|B^{\beta}(t)e^{-sB(t)}\| \le \frac{\mathcal{N}_\beta}{s^\beta}e^{-ws}, \quad t > 0, \quad \beta \le 0, \quad w > 0, \tag{2.3}$$
$$\|B^{\beta}(t)\mathcal{S}(t,s)\| \le \mathcal{N}_\beta|t-s|^{-\beta}, \quad 0 < \beta < \mu + 1, \tag{2.4}$$
$$\|B^{\beta}(t)\mathcal{S}(t,s)B^{-\beta}(s)\| \le \mathcal{N}'_\beta, \quad 0 < \beta < \mu + 1, \tag{2.5}$$

for $s,t \in [0,T], 0 \le \alpha < \beta$ and $t > 0$, where $\mathcal{N}_{\alpha,\beta}$ is a constant related to T and μ and $\mathcal{N}_{\alpha,\beta}, \mathcal{N}_\beta, \mathcal{N}'_\beta$ show their dependence on the constants α, β. We also have following results.

Lemma 2.2. *([23, Lemma II.14.1]) Suppose that* $(P1)-(P3)$ *are satisfied. If* $0 \le \gamma \le 1, 0 \le \beta \le \alpha < 1+\mu, 0 < \alpha - \gamma \le 1$, *then for any* $0 \le \tau < t < t+\Delta t \le t_0, 0 \le \varsigma \le t_0$,

$$\|B^{\gamma}(\varsigma)[\mathcal{S}(t+\Delta t,\tau) - \mathcal{S}(t,\tau)]B^{-\beta}(\tau)\| \le \mathcal{N}_{\gamma,\beta,\alpha}(\Delta t)^{\alpha-\gamma}|t-\tau|^{\beta-\alpha}.$$

For additional details about the above mentioned concept, we refer to monographs [22–24].

Our main existence results are based on Banach contraction principle and the Kransnoselskii's fixed point theorem.

Lemma 2.3. *If* $\mathbb{E}$ *is a Banach space and* $\Gamma : \mathbb{E} \to \mathbb{E}$ *is a contraction mapping, then* Γ *has a unique fixed point.*

3 Existence Results

In this section, we present and prove the existence results for the problem (1.1)-(1.3) under different fixed point theorem. Initially, we prove the existence and uniqueness for the problem (1.1)-(1.3) on the Banach subspace $E_\alpha(t_0)$ for some $0 < \alpha < 1$ and $t_0 \in [0,T]$ under Banach fixed point theorem. To be able to use this theorem, we need to list the following conditions:

(H1) $\mathscr{F} : \mathscr{I} \times E_\alpha(t_0) \times E_\alpha(t_0) \to \mathbb{E}$ is a Lipschitz continuous function then there exists $\mathscr{L}_{\mathscr{F}}, \mathscr{L}^*_{\mathscr{F}} > 0$ and for all $t,s, \in [0,T]$ and $x,y,\overline{x},\overline{y} \in E_\alpha(t_0)$ such that

$$\|B(t)\mathscr{F}(t,x,y) - B(t)\mathscr{F}(s,\overline{x},\overline{y})\| \le \mathscr{L}_{\mathscr{F}}\left[|t-s| + \|x-\overline{x}\|_\alpha + \|y-\overline{y}\|_\alpha\right],$$

$$\|B(t)\mathscr{F}(t,x,0)\| \leq \mathscr{L}_{\mathscr{F}}\|x\| + \mathscr{L}^*_{\mathscr{F}}, \quad x \in E_\alpha(t_0),$$

and

$$\mathscr{L}^*_{\mathscr{F}} = \sup_{t\in\mathscr{I}} \|B(t)\mathscr{F}(t,0,0)\|.$$

(H2) The nonlinear function $\mathscr{G} : \mathscr{I} \times E_\alpha(t_0) \times E_\alpha(t_0) \to \mathbb{E}$ is a Lipschitz continuous function with $\mathscr{G}(\mathscr{I} \times E_\alpha(t_0) \times E_\alpha(t_0)) \subset \mathcal{D}(B)$. Then there exist constants $\mathscr{L}_{\mathscr{G}}, \mathscr{L}^*_{\mathscr{G}} > 0$ and for all $t, s, \in [0,T]$ and $x, y, \overline{x}, \overline{y} \in E_\alpha(t_0)$, such that

$$\|\mathscr{G}(t,x,y) - \mathscr{G}(s,\overline{x},\overline{y})\| \leq \mathscr{L}_{\mathscr{G}}\left[|t-s| + \|x-\overline{x}\|_\alpha + \|y-\overline{y}\|_\alpha\right],$$

and

$$\mathscr{L}^*_{\mathscr{G}} = \sup_{t\in\mathscr{I}} \|\mathscr{G}(t,0,0)\|.$$

(H3) The nonlinear function $\mathscr{H} : \mathscr{I} \times E_\alpha(t_0) \times E_\alpha(t_0) \to \mathbb{E}$ is a Lipschitz continuous function with $\mathscr{H}(\mathscr{I} \times E_\alpha(t_0) \times E_\alpha(t_0)) \subset \mathcal{D}(B)$. Then there exist constants $\mathscr{L}_{\mathscr{H}}, \mathscr{L}^*_{\mathscr{H}} > 0$ and for all $t, s, \in [0,T]$ and $x, y, \overline{x}, \overline{y} \in E_\alpha(t_0)$, such that

$$\|\mathscr{H}(t,x,y) - \mathscr{H}(s,\overline{x},\overline{y})\| \leq \mathscr{L}_{\mathscr{H}}\left[|t-s| + \|x-\overline{x}\|_\alpha + \|y-\overline{y}\|_\alpha\right],$$

and

$$\mathscr{L}^*_{\mathscr{H}} = \sup_{t\in\mathscr{I}} \|\mathscr{H}(t,0,0)\|.$$

(H4) The map $a_i : \mathscr{D} \times E_\alpha(t_0) \to E_\alpha(t_0), i = 1,2,3$; where $\mathscr{D} = \{(t,s) \in \mathscr{I} \times \mathscr{I} : t \geq s\}$ and $i = 1,2,3$ are continuous and there exist positive constants $\mathscr{L}_{a_i}, \mathscr{L}^*_{a_i} > 0$ such that

$$\left\| \int_0^t [a_i(t,s,x) - a_i(t,s,y)]ds \right\|_\alpha \leq \mathscr{L}_{a_i}\|x-y\|_\alpha, \quad x, y \in E_\alpha(t_0),$$

and

$$\mathscr{L}^*_{a_i} = \sup_{t\in[0,T]} \int_0^t \|a_1(t,s,0)\|ds.$$

(H5) The functions $\mathcal{I}_i : E_\alpha(t_0) \to E_\alpha(t_0), i = 1,2,\cdots,q$ are continuous functions and there exists a positive constant $\mathscr{L}_I > 0$ such that

$$\|B(t)\mathcal{I}_i x - B(t)\mathcal{I}_i\overline{x}\| \leq \mathscr{L}_I\|x-\overline{x}\|_\alpha.$$

(H6) The function $g : PC([0,T], E_\alpha(t_0)) \to \mathcal{D}(B)$ is a nonlinear function which satisfies that $B(t)g$ is continuous on $PC([0,T], E_\alpha(t_0))$ and there exists a constant $\mathscr{L}_g$ such that

$$\|B(t)g(z) - B(t)g(\overline{z})\| \leq \mathscr{L}_g\|z-\overline{z}\|_{PC},$$
$$\|B(t)g(z)\| \leq \mathscr{L}_g\|z\|_{PC(E_\alpha(t_0))}, \text{ for each } z \in PC(\mathscr{I}, E_\alpha(t_0)).$$

Consider the sets $\mathscr{B}_r = \{z \in E_\alpha(t_0) : \|z\|_\alpha \leq r\}$ and $\mathscr{W}_r = \{z \in PC([0,T], E_\alpha(t_0)) : z(t) \in \mathscr{B}_r, \text{for all } t \in [0,T]\}$ for each finite constant $r > 0$.

Now, we are in a position to define the mild solution for the problem (1.1)-(1.3).

Definition 3.1. *A PC fucntion $z(\cdot) : \mathscr{I} \to \mathbb{E}$ is called a mild solution for the problem if $z(0) = z_0 + g(z)$ and the following integral equation*

$$z(t) = \begin{cases} \mathcal{S}(t,0)\left[z_0 + g(z) - \mathscr{F}(0, z(h_1(0)), 0)\right] \\ +\mathscr{F}\left(t, z(h_1(t)), \int_0^t a_1(t,s,z(h_2(s))ds\right) \\ -\int_0^t \mathcal{S}(t,\tau)B(\tau)\mathscr{F}\left(\tau, z(h_1(\tau)), \int_0^\tau a_1(\tau,\xi,z(h_2(\xi))d\xi\right)d\tau \\ +\int_0^t \mathcal{S}(t,\tau)\mathscr{G}\left(\tau, z(h_3(\tau)), \int_0^\tau a_2(\tau,\xi,z(h_4(\xi)))d\xi\right)d\tau \\ +\int_0^t \mathcal{S}(t,\tau)\mathscr{H}\left(\tau, z(h_5(\tau)), \int_0^\tau a_3(\tau,\xi,z(h_6(\xi))d\xi\right)d\tau \\ +\sum_{0<t_i<t} \mathcal{S}(t,t_i)\mathcal{I}_i(z(t_i^-)), \quad t \in [0,T]. \end{cases}$$

is fulfilled.

Theorem 3.1. *Let (H1)-(H6) holds, $z_0 \in E_\beta(t_0)$ for some $\beta \in (0,1]$ and*

$$\Upsilon = \Bigg[\mathcal{N}_{\alpha,\beta}\mathcal{N}'_\beta\mathcal{N}_{\beta,1}[\mathscr{L}_{\mathscr{F}} + Kq\mathscr{L}_I] + \mathcal{N}_{\alpha,\beta}\mathcal{N}'_1\mathscr{L}_g + \mathcal{N}_{\alpha,1}\mathscr{L}_{\mathscr{F}}(1+\mathscr{L}_{a_1}) + \mathcal{N}_{\alpha,\beta}\mathcal{N}_\beta\frac{T^{1-\beta}}{1-\beta}\{\mathscr{L}_{\mathscr{F}}(1+\mathscr{L}_{a_1})$$
$$+ \mathscr{L}_{\mathscr{G}}(1+\mathscr{L}_{a_2}) + \mathscr{L}_{\mathscr{H}}(1+\mathscr{L}_{a_3})\}\Bigg] < 1 \tag{3.1}$$

then the impulsive problem (1.1)-(1.3) has a unique mild solution $x \in \mathbb{E}$.

Proof. First, we will transform the problem (1.1)-(1.3) into a fixed point problem. Recognize the operator $\Gamma : PC(J, E_\alpha(t_0)) \to PC(J, E_\alpha(t_0))$ by

$$(\Gamma z)(t) = \begin{cases} \mathcal{S}(t,0)\left[z_0 + g(z) - \mathscr{F}(0, z(h_1(0)), 0)\right] \\ +\mathscr{F}\left(t, z(h_1(t)), \int_0^t a_1(t,s,z(h_2(s))ds\right) \\ -\int_0^t \mathcal{S}(t,\tau)B(\tau)\mathscr{F}\left(\tau, z(h_1(\tau)), \int_0^\tau a_1(\tau,\xi,z(h_2(\xi))d\xi\right)d\tau \\ +\int_0^t \mathcal{S}(t,\tau)\mathscr{G}\left(\tau, z(h_3(\tau)), \int_0^\tau a_2(\tau,\xi,z(h_4(\xi)))d\xi\right)d\tau \\ +\int_0^t \mathcal{S}(t,\tau)\mathscr{H}\left(\tau, z(h_5(\tau)), \int_0^\tau a_3(\tau,\xi,z(h_6(\xi))d\xi\right)d\tau \\ +\sum_{0<t_i<t} \mathcal{S}(t,t_i)\mathcal{I}_i(z(t_i^-)), \quad t \in [0,T]. \end{cases}$$

It is evident that the fixed points of the operator Γ are mild solutions of the model (1.1)-(1.3).

Now, let us demonstrating that Γ has a unique fixed point. Initially, we show that Γ maps $\mathscr{W}_r$ into $\mathscr{W}_r$. For any $z(\cdot) \in \mathscr{W}_r$, we have

$$\begin{aligned} \|(\Gamma z)(t)\|_\alpha \leq{}& \|\mathcal{S}(t,0)z_0\|_\alpha + \|\mathcal{S}(t,0)g(z)\|_\alpha + \|\mathcal{S}(t,0)\mathscr{F}(0,z(h_1(0)),0)\|_\alpha \\ &+ \left\|\mathscr{F}\left(t, z(h_1(t)), \int_0^t a_1(t,s,z(h_2(s))ds\right)\right\|_\alpha \\ &+ \left\|\int_0^t \mathcal{S}(t,\tau)B(\tau)\mathscr{F}\left(\tau, z(h_1(\tau)), \int_0^\tau a_1(\tau,\xi,z(h_2(\xi))d\xi\right)d\tau\right\|_\alpha \\ &+ \left\|\int_0^t \mathcal{S}(t,\tau)\mathscr{G}\left(\tau, z(h_3(\tau)), \int_0^\tau a_2(\tau,\xi,z(h_4(\xi)))d\xi\right)d\tau\right\|_\alpha \\ &+ \left\|\int_0^t \mathcal{S}(t,\tau)\mathscr{H}\left(\tau, z(h_5(\tau)), \int_0^\tau a_3(\tau,\xi,z(h_6(\xi))d\xi\right)d\tau\right\|_\alpha \\ &+ \left\|\sum_{0<t_i<t} \mathcal{S}(t,t_i)\mathcal{I}_i(z(t_i^-))\right\|_\alpha \\ \leq{}& \sum_{k=1}^{8} I_k. \end{aligned} \tag{3.2}$$

Now, with the help of the above discussions along with (2.1)-(2.5), we can find the following estimations:

$$\begin{aligned}
I_1 &= \|\mathcal{S}(t,0)z_0\|_\alpha \\
&\leq \|B^\alpha(t_0)B^{-\beta}(t)\|\|B^\beta(t)\mathcal{S}(t,0)B^{-\beta}(0)\|\|B^\beta(0)z_0\| \\
&\leq \mathcal{N}_{\alpha,\beta}\mathcal{N}'_\beta\|B^\beta(0)z_0\| \\
I_2 &= \|\mathcal{S}(t,0)g(z)\|_\alpha \\
&\leq \|B^\alpha(t_0)B^{-\beta}(t)\|B^\beta(t)\mathcal{S}(t,0)B^{-1}(0)\|\|B(0)g(z)\| \\
&\leq \mathcal{N}_{\alpha,\beta}\mathcal{N}'_1\mathscr{L}_g\|z\| \\
&\leq \mathcal{N}_{\alpha,\beta}\mathcal{N}'_1\mathscr{L}_g r \\
I_3 &= \|\mathcal{S}(t,0)\mathscr{F}(0,z(h_1(0)),0)\|_\alpha \\
&\leq \|B^\alpha(t_0)B^{-\beta}(t)\|\|B^\beta(t)\mathcal{S}(t,0)B^{-\beta}(0)\|\|B^\beta(0)B^{-1}(t)\|[\|B(t)\mathscr{F}(0,z(h_1(0)),0)\|] \\
&\leq \mathcal{N}_{\alpha,\beta}\mathcal{N}'_\beta\mathcal{N}_{\beta,1}(\mathscr{L}_\mathscr{F} r+\mathscr{L}^*_\mathscr{F}) \\
I_4 &= \left\|\mathscr{F}\left(t,z(h_1(t)),\int_0^t a_1(t,s,z(h_2(s))ds\right)\right\|_\alpha \\
&\leq \|B^\alpha(t_0)B^{-1}(t)\|\left[\left\|B(t)\mathscr{F}\left(t,z(h_1(t)),\int_0^t a_1(t,s,z(h_2(s))ds\right)\right\|\right] \\
&\leq \|B^\alpha(t_0)B^{-1}(t)\|\left[\left\|B(t)\mathscr{F}\left(t,z(h_1(t)),\int_0^t a_1(t,s,z(h_2(s))ds\right)\right.\right. \\
&\quad \left.\left. - B(t)\mathscr{F}(t,0,0)+B(t)\mathscr{F}(t,0,0)\right\|\right] \\
&\leq \mathcal{N}_{\alpha,1}\left[\mathscr{L}_\mathscr{F}\left(\|z(h_1(t))\|+\left\|\int_0^t a_1(t,s,z(h_2(s)))ds\right\|\right)+\mathscr{L}^*_\mathscr{F}\right] \\
&\leq \mathcal{N}_{\alpha,1}\left[\mathscr{L}_\mathscr{F}\left(\|z(h_1(t))\|+\left\|\int_0^t a_1(t,s,z(h_2(s)))ds-\int_0^t a_1(t,s,0)ds\right\|\right.\right. \\
&\quad \left.\left.+\left\|\int_0^t a_1(t,s,0)ds\right\|\right)+\mathscr{L}^*_\mathscr{F}\right] \\
&\leq \mathcal{N}_{\alpha,1}\left[\mathscr{L}_\mathscr{F}\left(\|z(h_1(t))\|+\mathscr{L}_{a_1}\|z(h_2(s))\|+\mathscr{L}^*_{a_1}\right)+\mathscr{L}^*_\mathscr{F}\right] \\
&\leq \mathcal{N}_{\alpha,1}\left[\mathscr{L}_\mathscr{F}[(1+\mathscr{L}_{a_1})r+\mathscr{L}^*_{a_1}]+\mathscr{L}^*_\mathscr{F}\right] \\
I_5 &= \left\|\int_0^t \mathcal{S}(t,\tau)B(\tau)\mathscr{F}\left(\tau,z(h_1(\tau)),\int_0^\tau a_1(\tau,\xi,z(h_2(\xi))d\xi\right)d\tau\right\|_\alpha \\
&\leq \int_0^t \|B^\alpha(t_0)B^{-\beta}(t)\|\|B^\beta(t)\mathcal{S}(t,\tau)\|\left[\left\|\mathscr{F}\left(\tau,z(h_1(\tau)),\int_0^\tau a_1(\tau,\xi,z(h_2(\xi))d\xi\right)\right\|\right]d\tau \\
&\leq \mathcal{N}_{\alpha,\beta}\mathcal{N}_\beta\frac{T^{1-\beta}}{1-\beta}\left[\mathscr{L}_\mathscr{F}(1+\mathscr{L}_{a_1})r+\mathscr{L}_\mathscr{F}\mathscr{L}^*_{a_1}+\mathscr{L}^*_\mathscr{F}\right] \\
I_6 &= \left\|\int_0^t \mathcal{S}(t,\tau)\mathscr{G}\left(\tau,z(h_3(\tau)),\int_0^\tau a_2(\tau,\xi,z(h_4(\xi)))d\xi\right)d\tau\right\|_\alpha \\
&\leq \int_0^t \|B^\alpha(t_0)B^{-\beta}(t)\|\|B^\beta(t)\mathcal{S}(t,\tau)\|\left[\left\|\mathscr{G}\left(\tau,z(h_3(\tau)),\int_0^\tau a_2(\tau,\xi,z(h_4(\xi)))d\xi\right)\right\|\right]d\tau \\
&\leq \mathcal{N}_{\alpha,\beta}\mathcal{N}_\beta\frac{T^{1-\beta}}{1-\beta}\left[\mathscr{L}_\mathscr{G}(1+\mathscr{L}_{a_2})r+\mathscr{L}_\mathscr{G}\mathscr{L}^*_{a_2}+\mathscr{L}^*_\mathscr{G}\right] \\
I_7 &= \left\|\int_0^t \mathcal{S}(t,\tau)\mathscr{H}\left(\tau,z(h_5(\tau)),\int_0^\tau a_3(\tau,\xi,z(h_6(\xi))d\xi\right)d\tau\right\|_\alpha \\
&\leq \int_0^t \|B^\alpha(t_0)B^{-\beta}(t)\|\|B^\beta(t)\mathcal{S}(t,\tau)\|\left[\left\|\mathscr{H}\left(\tau,z(h_5(\tau)),\int_0^\tau a_3(\tau,\xi,z(h_6(\xi))d\xi\right)\right\|\right]d\tau \\
&\leq \mathcal{N}_{\alpha,\beta}\mathcal{N}_\beta\frac{T^{1-\beta}}{1-\beta}\left[\mathscr{L}_\mathscr{H}(1+\mathscr{L}_{a_3})r+\mathscr{L}_\mathscr{H}\mathscr{L}^*_{a_3}+\mathscr{L}^*_\mathscr{H}\right]
\end{aligned}$$

$$
\begin{aligned}
I_8 &= \left\| \sum_{0<t_i<t} \mathcal{S}(t,t_i)\mathcal{I}_i(z(t_i^-)) \right\|_\alpha \\
&\leq \sum_{0<t_i<t} \|B^\alpha(t_0)B^{-\beta}(t)\| \|B^\beta(t)\mathcal{S}(t,t_i)B^{-\beta}(t_i)\| \|B^\beta(t_i)B^{-1}(0)\| [\|B(0)B^{-1}(t)\| \|B(t)\mathcal{I}_i(z(t_i^-))\|] \\
&\leq \sum_{i=1}^{q} \mathcal{N}_{\alpha,\beta}\mathcal{N}'_\beta\mathcal{N}_{\beta,1}K\mathscr{L}_I(\|z(t_i)\|) \\
&\leq \mathcal{N}_{\alpha,\beta}\mathcal{N}'_\beta\mathcal{N}_{\beta,1}Kq\mathscr{L}_I r.
\end{aligned}
$$

Now, we substitute the estimations $(I_1)-(I_8)$ in (3.2), we obtain

$$
\begin{aligned}
&\|(\Gamma z)(t)\|_\alpha \\
&\leq \mathcal{N}_{\alpha,\beta}\mathcal{N}'_\beta\|B^\beta(0)z_0\| + [\mathcal{N}_{\alpha,\beta}\mathcal{N}'_\beta\mathcal{N}_{\beta,1} + \mathcal{N}_{\alpha,1}]\mathscr{L}^*_{\mathscr{F}} + \mathcal{N}_{\alpha,1}\mathscr{L}_{\mathscr{F}}\mathscr{L}^*_{a_1} + \mathcal{N}_{\alpha,\beta}\mathcal{N}_\beta\frac{T^{1-\beta}}{1-\beta}\Big[\mathscr{L}_{\mathscr{F}}\mathscr{L}^*_{a_1} + \mathscr{L}_{\mathscr{G}}\mathscr{L}^*_{a_2} \\
&\quad + \mathscr{L}_{\mathscr{H}}\mathscr{L}^*_{a_3} + \mathscr{L}^*_{\mathscr{F}} + \mathscr{L}^*_{\mathscr{G}} + \mathscr{L}^*_{\mathscr{H}}\Big] + r\Big[\mathcal{N}_{\alpha,\beta}\mathcal{N}'_\beta\mathcal{N}_{\beta,1}[\mathscr{L}_{\mathscr{F}} + Kq\mathscr{L}_I] + \mathcal{N}_{\alpha,\beta}\mathcal{N}'_1\mathscr{L}_g + \mathcal{N}_{\alpha,1}\mathscr{L}_{\mathscr{F}}(1+\mathscr{L}_{a_1}) \\
&\quad + \mathcal{N}_{\alpha,\beta}\mathcal{N}_\beta\frac{T^{1-\beta}}{1-\beta}\{\mathscr{L}_{\mathscr{F}}(1+\mathscr{L}_{a_1}) + \mathscr{L}_{\mathscr{G}}(1+\mathscr{L}_{a_2}) + \mathscr{L}_{\mathscr{H}}(1+\mathscr{L}_{a_3})\}\Big] \\
&\leq r.
\end{aligned}
$$

Therefore, the operator Γ maps $\mathscr{W}_r$ into $\mathscr{W}_r$. Finally, we show that Γ is a contraction on $PC([0,T],E_\alpha(t_0))$.

Remark 3.1. *For better readability, we find the contraction estimations are below:*

Let us consider $z,\bar{z} \in PC([0,T],E_\alpha(t_0))$ and $t \in [0,T]$, then we obtain

$$
\|\Gamma z(t) - \Gamma\bar{z}(t)\|_\alpha = \sum_{k=9}^{16} I_k,
$$

where

$$
\begin{aligned}
I_9 &= \|\mathcal{S}(t,0)z_0 - \mathcal{S}(t,0)z_0\|_\alpha \\
&\leq 0 \\
I_{10} &= \|\mathcal{S}(t,0)g(z) - \mathcal{S}(t,0)g(\bar{z})\|_\alpha \\
&\leq \|B^\alpha(t_0)B^{-\beta}(t)\| \|B^\beta(t)\mathcal{S}(t,0)B^{-\beta}(0)\| \Big[\|B^\beta(0)B^{-1}(0)\| \|B(0)[g(z)-g(\bar{z})]\|\Big] \\
&\leq \mathcal{N}_{\alpha,\beta}\mathcal{N}'_\beta\mathcal{N}_{\beta,1}\mathscr{L}_g\|z-\bar{z}\|_{PC([0,T],E_\alpha(t_0))} \\
I_{11} &= \|\mathcal{S}(t,0)\mathscr{F}(0,z(h_1(0)),0) - \mathcal{S}(t,0)\mathscr{F}(0,\bar{z}(h_1(0)),0)\|_\alpha \\
&\leq \|B^\alpha(t_0)B^{-\beta}(t)\| \|B^\beta(t)\mathcal{S}(t,0)B^{-\beta}(0)\| \|B^\beta(0)B^{-1}(t)\| \Big[\|B(t)\mathscr{F}(0,z(h_1(0)),0) - B(t)\mathscr{F}(0,\bar{z}(h_1(0)),0)\|\Big] \\
&\leq \mathcal{N}_{\alpha,\beta}\mathcal{N}^1_\beta\mathcal{N}_{\beta,1}\mathscr{L}_{\mathscr{F}}\|z-\bar{z}\|_{PC([0,T],E_\alpha(t_0))} \\
I_{12} &= \left\| \mathscr{F}\left(t, z(h_1(t)), \int_0^t a_1(t,s,z(h_2(s)))ds\right) - \mathscr{F}\left(t, \bar{z}(h_1(t)), \int_0^t a_1(t,s,\bar{z}(h_2(s)))ds\right) \right\|_\alpha \\
&\leq \|B^\alpha(t_0)B^{-1}(t)\| \left\| B(t)\mathscr{F}\left(t, z(h_1(t)), \int_0^t a_1(t,s,z(h_2(s)))ds\right) - B(t)\mathscr{F}\left(t, \bar{z}(h_1(t)), \int_0^t a_1(t,s,\bar{z}(h_2(s)))ds\right) \right\| \\
&\leq \mathcal{N}_{\alpha,1}\mathscr{L}_{\mathscr{F}}(1+\mathscr{L}_{a_1})\|z-\bar{z}\|_{PC([0,T],E_\alpha(t_0))}
\end{aligned}
$$

$$\begin{aligned}
I_{13} &= \Big\| \int_0^t \mathcal{S}(t,\tau)B(\tau)\mathscr{F}\Big(\tau, z(h_1(\tau)), \int_0^\tau a_1(\tau,\xi,z(h_2(\xi))d\xi\Big) \\
&\quad - \int_0^t \mathcal{S}(t,\tau)B(\tau)\mathscr{F}\Big(\tau, \bar{z}(h_1(\tau)), \int_0^\tau a_1(\tau,\xi,\bar{z}(h_2(\xi))d\xi\Big)\Big\|_\alpha \\
&\leq \int_0^t \|B^\alpha(t_0)B^{-\beta}(t)\|\|B^\beta(t)\mathcal{S}(t,\tau)\|\Big[\Big\|B(\tau)\mathscr{F}\Big(\tau, z(h_1(\tau)), \int_0^\tau a_1(\tau,\xi,z(h_2(\xi))d\xi\Big) \\
&\quad - B(\tau)\mathscr{F}\Big(\tau, \bar{z}(h_1(\tau)), \int_0^\tau a_1(\tau,\xi,\bar{z}(h_2(\xi))d\xi\Big)\Big\|\Big]d\tau \\
&\leq \int_0^t \mathcal{N}_{\alpha,\beta}\mathcal{N}_\beta(t-\tau)^{-\beta}\mathscr{L}_{\mathscr{F}}\Big[\|z(h_1(\tau))-\bar{z}(h_1(\tau))\| + \Big\|\int_0^\tau a_1(\tau,\xi,z(h_2(\xi)))d\xi - \int_0^\tau a_1(\tau,\xi,\bar{z}(h_2(\xi)))d\xi\Big\|\Big]d\tau \\
&\leq \mathcal{N}_{\alpha,\beta}\mathcal{N}_\beta\int_0^t (t-s)^{-\beta}\mathscr{L}_{\mathscr{F}}\left(\|z-\bar{z}\| + \mathscr{L}_{a_1}\|z-\bar{z}\|\right)d\tau \\
&\leq \mathcal{N}_{\alpha,\beta}\mathcal{N}_\beta\frac{T^{1-\beta}}{1-\beta}\mathscr{L}_{\mathscr{F}}(1+\mathscr{L}_{a_1})\|z-\bar{z}\|_{PC([0,T],E_\alpha(t_0))} \\
I_{14} &= \Big\| \int_0^t \mathcal{S}(t,\tau)\mathscr{G}\Big(\tau, z(h_3(\tau)), \int_0^\tau a_2(\tau,\xi,z(h_4(\xi)))d\xi\Big)d\tau \\
&\quad - \int_0^t \mathcal{S}(t,\tau)\mathscr{G}\Big(\tau, \bar{z}(h_3(\tau)), \int_0^\tau a_2(\tau,\xi,\bar{z}(h_4(\xi)))d\xi\Big)d\tau\Big\|_\alpha \\
&\leq \int_0^t \|B^\alpha(t_0)B^{-\beta}(t)\|\|B^\beta(t)\mathcal{S}(t,\tau)\|\Big[\Big\|\mathscr{G}\Big(\tau, z(h_3(\tau)), \int_0^\tau a_2(\tau,\xi,z(h_4(\xi)))d\xi\Big) \\
&\quad - \mathscr{G}\Big(\tau, \bar{z}(h_3(\tau)), \int_0^\tau a_2(\tau,\xi,\bar{z}(h_4(\xi)))d\xi\Big)\Big\|\Big]d\tau \\
&\leq \mathcal{N}_{\alpha,\beta}\mathcal{N}_\beta\frac{T^{1-\beta}}{1-\beta}\mathscr{L}_{\mathscr{G}}(1+\mathscr{L}_{a_2})\|z-\bar{z}\|_{PC([0,T],E_\alpha(t_0))} \\
I_{15} &= \Big\| \int_0^t \mathcal{S}(t,\tau)\mathscr{H}\Big(\tau, z(h_5(\tau)), \int_0^\tau a_3(\tau,\xi,z(h_6(\xi))d\xi\Big)d\tau \\
&\quad - \int_0^t \mathcal{S}(t,\tau)\mathscr{H}\Big(\tau, \bar{z}(h_5(\tau)), \int_0^\tau a_3(\tau,\xi,\bar{z}(h_6(\xi))d\xi\Big)\Big\|_\alpha \\
&\leq \int_0^t \|B^\alpha(t_0)B^{-\beta}(t)\|\|B^\beta(t)\mathcal{S}(t,\tau)\|\Big[\Big\|\mathscr{H}\Big(\tau, z(h_5(\tau)), \int_0^\tau a_3(\tau,\xi,z(h_6(\xi))d\xi\Big) \\
&\quad - \mathscr{H}\Big(\tau, \bar{z}(h_5(\tau)), \int_0^\tau a_3(\tau,\xi,\bar{z}(h_6(\xi))d\xi\Big)\Big\|\Big]d\tau \\
&\leq \mathcal{N}_{\alpha,\beta}\mathcal{N}_\beta\frac{T^{1-\beta}}{1-\beta}\mathscr{L}_{\mathscr{H}}(1+\mathscr{L}_{a_3})\|z-\bar{z}\|_{PC([0,T],E_\alpha(t_0))} \\
I_{16} &= \Big\|\sum_{0<t_i<t}\mathcal{S}(t,t_i)[\mathcal{I}_i(z(t_i^-)) - \mathcal{I}_i(\bar{z}(t_i^-))]\Big\|_\alpha \\
&\leq \sum_{i=1}^q \|B^\alpha(t_0)B^{-\beta}(t)\|\|B^\beta(t)\mathcal{S}(t,t_i)B^{-\beta}(t_i)\|\|B^\beta(t_i)B^{-1}(0)\|\left[\|B(0)B^{-1}(t)\|\|B(t)[\mathcal{I}_i(z(t_i^-)) - \mathcal{I}_i(\bar{z}(t_i^-))]\|\right] \\
&\leq \mathcal{N}_{\alpha,\beta}\mathcal{N}'_\beta\mathcal{N}_{\beta,1}Kq\mathscr{L}_I\|z-\bar{z}\|_{PC([0,T],E_\alpha(t_0))}.
\end{aligned}$$

Now, we enter into the main proof of this theorem. From Remark 3.1, we obtain

$$\begin{aligned}
&\|\Gamma z(t) - \Gamma\bar{z}(t)\|_\alpha \\
&\leq \Big[\mathcal{N}_{\alpha,\beta}\mathcal{N}'_\beta\mathcal{N}_{\beta,1}[\mathscr{L}_{\mathscr{F}} + Kq\mathscr{L}_I] + \mathcal{N}_{\alpha,\beta}\mathcal{N}'_1\mathscr{L}_g + \mathcal{N}_{\alpha,1}\mathscr{L}_{\mathscr{F}}(1+\mathscr{L}_{a_1}) + \mathcal{N}_{\alpha,\beta}\mathcal{N}_\beta\frac{T^{1-\beta}}{1-\beta}\{\mathscr{L}_{\mathscr{F}}(1+\mathscr{L}_{a_1}) \\
&\quad + \mathscr{L}_{\mathscr{G}}(1+\mathscr{L}_{a_2}) + \mathscr{L}_{\mathscr{H}}(1+\mathscr{L}_{a_3})\}\Big]\|z-\bar{z}\|_{PC([0,T],E_\alpha(t_0))}
\end{aligned}$$

Therefore, we take the supremum of t over $[0,T]$ and we have

$$\|\Gamma z - \Gamma\bar{z}\|_{PC([0,T],E_\alpha(t_0))} \leq \Upsilon\|z-\bar{z}\|_{PC([0,T],E_\alpha(t_0))}.$$

Since $\Upsilon < 1$ by the inequality (3.1), it indicates that the map Γ is contraction on $PC([0,T], E_\alpha(t_0))$. Hence, by Banach contraction principle, there exists a unique fixed point $z \in PC([0,T], E_\alpha(t_0))$ such that $\Gamma z(t) = z(t)$ which is a mild solution of the problem (1.1)-(1.3). The proof is now completed. □

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