Malaya J. Mat. 5(1)(2017) 4–18

Malaya Journal of Matematik

MJM an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

General Solution and Generalized Ulam - Hyers Stability of A Additive Functional Equation Originating From N Observations of An Arithmetic Mean In Banach Spaces Using Various Substitutions In Two Different Approaches

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Abstract

In this paper, we introduce and investigate the general solution and generalized Ulam- Hyers stability of a additive functional equation

$$f\left(\frac{\sum_{k=1}^{N} x_k}{N}\right) = \frac{1}{N}\sum_{k=1}^{N} f(x_k)$$

originating from N observations of an arithmetic mean in Banach spaces using various substitutions in two different approaches with $N \geq 2$.

Keywords: Arithmetic mean, additive functional equation, Generalized Hyers-Ulam stability, fixed point.

2010 MSC: 39B52, 32B72, 32B82.

1 Introduction

In [39], Ulam proposed the general Ulam stability problem: When is it true that by slightly changing the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true? In [19], Hyers gave the first affirmative answer to the question of Ulam for additive functional equations on Banach spaces. Hyers result has since then seen many significant generalizations, both in terms of the control condition used to define the concept of approximate solution [4, 17, 28, 31].

One of the most famous functional equations is the additive functional equation

$$f(x+y) = f(x) + f(y). \tag{1.1}$$

In 1821, it was first solved by A.L. Cauchy in the class of continuous real-valued functions. It is often called an additive Cauchy functional equation in honor of Cauchy. The theory of additive functional equations is frequently applied to the development of theories of other functional equations. Moreover, the properties of additive functional equations are powerful tools in almost every field of natural and social sciences. Every solution of the additive functional equation (1.1) is called an additive function.

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The second famous Jensen functional equation is

$$f\left(\frac{x+y}{2}\right) = \frac{1}{2}\left(f(x)+f(y)\right) \tag{1.2}$$

its solution and stability was investigated by Jensen [21], Aczel [2], Aczel et.al., [3].

The Jensen functional equation (1.2) makes sense in algebraic systems that are 2-divisible (that is, division by 2 is permissible), by replacing x by $x+y$ and y by $x-y$, (1.2) goes over to

$$f(x+y)+f(x-y) = 2f(x), \tag{1.3}$$

which in a way eliminates this problem and makes sense in algebraic systems that need not be 2-divisible. The equations (1.2) and (1.3) are equivalent in 2-divisible systems. Both equations can be solved by relating them to the additive equation (1.1) (see P.K.Sahoo, Pl.Palaniappan. [37]).

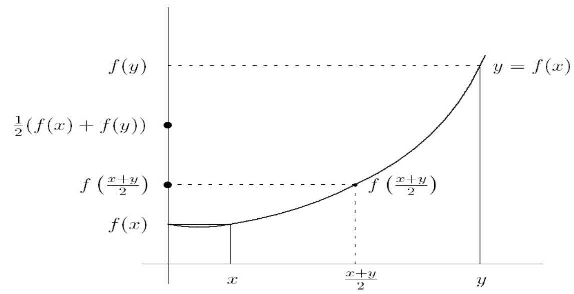

Fig: 1.1 Geometrical Interpretation of Functional Equation (1.2)

The solution and stability of various additive functional equation in various normed spaces were introduced and discussed in [5–11, 15, 26, 35] and reference cited there in.

Definition 1.1. ***Arithmetic Mean (A.M.):*** *Arithmetic mean is the total of all the items divided by their total number of items*

$$A.M. = \frac{X_1+X_2+\cdots+X_N}{N}.$$

In this paper, the authors introduce and investigate the general solution and generalized Ulam- Hyers stability of a additive functional equation

$$f\left(\frac{\sum_{k=1}^{N} x_k}{N}\right) = \frac{1}{N}\sum_{k=1}^{N} f(x_k) \tag{1.4}$$

originating from N observations of an arithmetic mean in Banach spaces using various substitutions in two different approaches with $N \geq 2$. In particular when $N=2$, we arrive the Jensen functional equation (1.2).

2 General Solution of The Functional Equation (1.4)

In this section, we present the general solution of the functional equation (1.4). For this assume U and V be real vector spaces.

Lemma 2.1. *If $f: U \to V$ be a mapping satisfying (1.1) if and only if $f: U \to V$ satisfies (1.2) for all $x, y \in U$.*

Lemma 2.2. *If $f: U \to V$ be a mapping satisfying (1.2) if and only if $f: U \to V$ satisfies (1.3) for all $x, y \in U$.*

Lemma 2.3. *If $f: U \to V$ be a mapping satisfying (1.1) if and only if $f: U \to V$ satisfies (1.3) for all $x, y \in U$.*

Remark 2.1. *If $f: U \to V$ be a mapping satisfying (1.1), (1.2) and (1.3) for all $x, y \in U$ then they are equivalent*

Theorem 2.1. *If $f: U \to V$ be a mapping satisfying (1.1) for all $x, y \in U$ if and only if $f: U \to V$ satisfies (1.4) for all $x_1, \cdots, x_N \in U$.*

Theorem 2.2. *If $f: U \to V$ be a mapping satisfying (1.2) for all $x, y \in U$ if and only if $f: U \to V$ satisfies (1.4) for all $x_1, \cdots, x_N \in U$.*

Theorem 2.3. *If $f: U \to V$ be a mapping satisfying (1.3) for all $x, y \in U$ if and only if $f: U \to V$ satisfies (1.4) for all $x_1, \cdots, x_N \in U$.*

Remark 2.2. *If $f: U \to V$ be a mapping satisfying (1.1), (1.2), (1.3) and (1.4) then they are equivalent.*

3 Generalized Ulam - Hyers Stability of (1.4) In Banach Space : Direct Method

In this section, we test the generalized Ulam - Hyers stability of the functional equation (1.4) in Banach space. To prove the stability results throughout this section, we assume Y be a Normed space and Z be a Banach space.

3.1 Substitution - 1: $N \geq 2$ N is an Integer

Theorem 3.1. *Let $\lambda : Y^N \to [0, \infty)$ and $f : Y \to Z$ are functions fulfilling the inequalities*

$$\lim_{a\to\infty} \frac{\lambda(N^{ad}x_1, N^{ad}x_2 \cdots, N^{ad}x_{N-1}, N^{ad}x_N)}{N^{ad}} = 0 \tag{3.1}$$

$$\left\| f\left(\frac{\sum_{k=1}^{N} x_k}{N}\right) - \frac{1}{N}\sum_{k=1}^{N} f(x_k)\right\| \leq \lambda(x_1, x_2 \cdots, x_{N-1}, x_N) \tag{3.2}$$

for all $x_1, \cdots, x_N \in Y$. Then there exists a unique additive function fulfilling the functional equation (1.4) and

$$\|f(x) - A_A(x)\| \leq \sum_{b=\frac{1-d}{2}}^{\infty} \frac{\Lambda_A(N^{bd}x)}{N^{bd}} \tag{3.3}$$

for all $x \in Y$, where $A_A(x)$ and $\Lambda_A(N^b x)$ is defined as

$$A_A(x) = \lim_{a\to\infty} \frac{f(N^{ad}x)}{N^{ad}} \tag{3.4}$$

and

$$\Lambda_A(N^{bd}x) = \lambda(N^{(b+1)d}x, 0 \cdots, 0, 0) \tag{3.5}$$

respectively, for all $x \in Y$ with $d = \pm 1$.

Proof. Replacing $(x_1, x_2, \cdots, x_N)$ by $(x, 0, \cdots, 0)$ in (3.2), we obtain

$$\left\| f\left(\frac{x}{N}\right) - \frac{1}{N}f(x)\right\| \leq \lambda(x, 0 \cdots, 0, 0) \tag{3.6}$$

for all $x \in Y$. Setting x by Nx in (3.6), we get

$$\left\| f(x) - \frac{1}{N}f(Nx)\right\| \leq \lambda(Nx, 0 \cdots, 0, 0) \tag{3.7}$$

for all $x \in Y$. Define $\Lambda_A(x) = \lambda(Nx, 0 \cdots, 0, 0)$ in (3.7), we have

$$\left\| f(x) - \frac{1}{N}f(Nx)\right\| \leq \Lambda_A(x) \tag{3.8}$$

for all $x \in Y$. Replacing x by Nx and divided by N in (3.8), we arrive

$$\left\| \frac{1}{N}f(Nx) - \frac{1}{N^2}f(N^2x)\right\| \leq \frac{\Lambda_A(Nx)}{N} \tag{3.9}$$

for all $x \in Y$. Combining (3.8) and (3.9), we reach

$$\left\| f(x) - \frac{1}{N^2}f(N^2x)\right\| \leq \Lambda_A(x) + \frac{\Lambda_A(Nx)}{N} \tag{3.10}$$

for all $x \in Y$. Generalizing, one can get

$$\left\| f(x) - \frac{1}{N^a}f(N^a x)\right\| \leq \sum_{b=0}^{a-1} \frac{\Lambda_A(N^b x)}{N^b} \tag{3.11}$$

for all $x \in Y$. If we put x by $N^c x$ and divided by N^c in (3.11), we obtain

$$\left\| \frac{1}{N^c} f(N^c x) - \frac{1}{N^{a+c}} f(N^{a+c} x) \right\| \to \quad 0 \quad as \quad c \quad \to \quad \infty \tag{3.12}$$

for all $x \in Y$. Thus $\left\{ \frac{1}{N^a} f(N^a x) \right\}$ is a Cauchy sequence in Z. Since Z is complete, define a mapping such that

$$A_A(x) = \lim_{a \to \infty} \frac{f(N^a x)}{N^a}$$

for all $x \in Y$. Letting a approaches to infinity in (3.11), we get (3.3) as desired. Letting $(x_1, \cdots, x_N) = (N^a x_1, \cdots, N^a x_N)$ and divided by N^a in (3.2) and using the definition the $A_A(x)$, we see that $A_A(x)$ satisfies the functional equation (1.4) for all $x_1, \cdots, x_n \in Y$. To show $A_A(x)$ is unique, let $A_B(x)$ be another mapping satisfying the functional equation (1.4) and (3.3) for all $x \in Y$. Now $\|A_A(x) - A_B(x)\| = \frac{1}{N^c} \|A_A(N^c x) - A_B(N^c x)\| \leq \frac{1}{N^c} \left\{ ||A_A(N^c x) - f(N^c x)|| + ||f(N^c x) - A_B(N^c x)|| \right\} \leq 2 \sum_{b=0}^{\infty} \frac{\Lambda_A(N^{b+c} x)}{N^{b+c}} \quad \to \quad 0 \quad as \quad c \quad \to \quad \infty$ for all $x \in Y$. Thus $A_A(x)$ is unique. Hence the theorem is true for $d = 1$.

Replacing x by $\frac{x}{N}$ and multiply by N in (3.8), we arrive

$$\left\| N f\left(\frac{x}{N}\right) - f(x) \right\| \leq N \Lambda_A \left(\frac{x}{N}\right) \tag{3.13}$$

for all $x \in Y$. The rest of the proof is similar to that of previous case. Hence the proof is complete. □

From Theorem 3.1, we prove the following corollaries concerning the Hyers- Ulam, Hyers - Ulam - TRassias and Hyers - Ulam - JRassias stabilities of the functional equation (1.4).

Corollary 3.1. *Let $f : Y \to Z$ be a mapping fulfilling the inequality*

$$\left\| f\left(\frac{\sum_{k=1}^{N} x_k}{N} \right) - \frac{1}{N} \sum_{k=1}^{N} f(x_k) \right\| \leq \beta \tag{3.14}$$

where $\beta > 0$ and for all $x_1, \cdots, x_N \in Y$. Then there exists a unique additive function satisfying the functional equation (1.4) and

$$\|f(x) - A_A(x)\| \leq \frac{\beta N}{|N-1|} \tag{3.15}$$

for all $x \in Y$.

Corollary 3.2. *Let $f : Y \to Z$ be a mapping fulfilling the inequality*

$$\left\| f\left(\frac{\sum_{k=1}^{N} x_k}{N} \right) - \frac{1}{N} \sum_{k=1}^{N} f(x_k) \right\| \leq \beta \sum_{k=1}^{N} ||x_k||^t \tag{3.16}$$

where $\beta > 0, t > 0$ and for all $x_1, \cdots, x_N \in Y$. Then there exists a unique additive function satisfying the functional equation (1.4) and

$$\|f(x) - A_A(x)\| \leq \frac{\beta ||x||^t N^{t+1}}{|N - N^t|} \tag{3.17}$$

where $t \neq 1$ for all $x \in Y$.

Corollary 3.3. *Let $f : Y \to Z$ be a mapping fulfilling the inequality*

$$\left\| f\left(\frac{\sum_{k=1}^{N} x_k}{N} \right) - \frac{1}{N} \sum_{k=1}^{N} f(x_k) \right\| \leq \beta \left\{ \prod_{k=1}^{N} ||x_k||^t + \sum_{k=1}^{N} ||x_k||^{Nt} \right\} \tag{3.18}$$

where $\beta > 0, t > 0$ and for all $x_1, \cdots, x_N \in Y$. Then there exists a unique additive function satisfying the functional equation (1.4) and

$$\|f(x) - A_A(x)\| \leq \frac{\beta ||x||^{Nt} N^{Nt+1}}{|N - N^{Nt}|} \tag{3.19}$$

where $Nt \neq 1$ for all $x \in Y$.

3.2 Substitution - 2: $N \geq 2$, N is Odd Integer

Theorem 3.2. *Let $\lambda : Y^N \to [0,\infty)$ and $f : Y \to Z$ are functions fulfilling the inequalities*

$$\lim_{a\to\infty} \frac{\lambda(N^{ad}x_1, N^{ad}x_2 \cdots, N^{ad}x_{N-1}, N^{ad}x_N)}{N^{ad}} = 0 \tag{3.20}$$

$$\left\| f\left(\frac{\sum_{k=1}^{N} x_k}{N}\right) - \frac{1}{N}\sum_{k=1}^{N} f(x_k) \right\| \leq \lambda(x_1, x_2 \cdots, x_{N-1}, x_N) \tag{3.21}$$

for all $x_1, \cdots, x_N \in Y$. Then there exists a unique additive function fulfilling the functional equation (1.4) and

$$\|f(x) - A_O(x)\| \leq \sum_{b=\frac{1-d}{2}}^{\infty} \frac{\Lambda_O(N^{bd}x)}{N^{bd}} \tag{3.22}$$

for all $x \in Y$, where $A_O(x)$ and $\Lambda_O(N^b x)$ is defined as

$$A_O(x) = \lim_{a\to\infty} \frac{f(N^{ad}x)}{N^{ad}} \tag{3.23}$$

and

$$\Lambda_O(N^{bd}x) = \lambda(N^{(b+1)d}x, \underbrace{-N^{bd}x, \cdots, -N^{bd}x}_{\frac{N-1}{2}}, \underbrace{N^{bd}x, \cdots, N^{bd}x}_{\frac{N-1}{2}}) \tag{3.24}$$

respectively, for all $x \in Y$ with $d = \pm 1$.

Proof. Replacing $(x_1, x_2, \cdots, x_N)$ by $(x, \underbrace{-x, \cdots, -x}_{\frac{N-1}{2}}, \underbrace{x, \cdots, x}_{\frac{N-1}{2}})$ in (3.21), we obtain

$$\left\| f\left(\frac{x}{N}\right) - \frac{1}{N}f(x) \right\| \leq \lambda(x, \underbrace{-x, \cdots, -x}_{\frac{N-1}{2}}, \underbrace{x, \cdots, x}_{\frac{N-1}{2}}) \tag{3.25}$$

for all $x \in Y$. Setting x by Nx in (3.25), we get

$$\left\| f(x) - \frac{1}{N}f(Nx) \right\| \leq \lambda(Nx, \underbrace{-x, \cdots, -x}_{\frac{N-1}{2}}, \underbrace{x, \cdots, x}_{\frac{N-1}{2}}) \tag{3.26}$$

for all $x \in Y$. Define $\Lambda_O(x) = \lambda(Nx, \underbrace{-x, \cdots, -x}_{\frac{N-1}{2}}, \underbrace{x, \cdots, x}_{\frac{N-1}{2}})$ in (3.26), we have

$$\left\| f(x) - \frac{1}{N}f(Nx) \right\| \leq \Lambda_O(x) \tag{3.27}$$

for all $x \in Y$. Replacing x by Nx and divided by N in (3.27), we arrive

$$\left\| \frac{1}{N}f(Nx) - \frac{1}{N^2}f(N^2x) \right\| \leq \frac{\Lambda_O(Nx)}{N} \tag{3.28}$$

for all $x \in Y$. Combining (3.27) and (3.28), we reach

$$\left\| f(x) - \frac{1}{N^2}f(N^2x) \right\| \leq \Lambda_O(x) + \frac{\Lambda_O(Nx)}{N} \tag{3.29}$$

for all $x \in Y$. Generalizing, one can get

$$\left\| f(x) - \frac{1}{N^a}f(N^a x) \right\| \leq \sum_{b=0}^{a-1} \frac{\Lambda_O(N^b x)}{N^b} \tag{3.30}$$

for all $x \in Y$. If we put x by $N^c x$ and divided by N^c in (3.30), we obtain

$$\left\| \frac{1}{N^c} f(N^c x) - \frac{1}{N^{a+c}} f(N^{a+c} x) \right\| \to \quad 0 \quad as \quad c \quad \to \quad \infty \tag{3.31}$$

for all $x \in Y$. Thus $\left\{ \frac{1}{N^a} f(N^a x) \right\}$ is a Cauchy sequence in Z. Since Z is complete, define a mapping such that

$$A_O(x) = \lim_{a \to \infty} \frac{f(N^a x)}{N^a}$$

for all $x \in Y$. The rest of proof is similar to that of Theorem 3.1. □

From Theorem 3.2, we prove the following corollaries concerning the Hyers- Ulam, Hyers - Ulam - TRassias and Hyers - Ulam - JRassias stabilities of the functional equation (1.4).

Corollary 3.4. *Let $f : Y \to Z$ be a mapping fulfilling the inequality*

$$\left\| f\left(\frac{\sum_{k=1}^{N} x_k}{N} \right) - \frac{1}{N} \sum_{k=1}^{N} f(x_k) \right\| \leq \beta \tag{3.32}$$

where $\beta > 0$ and for all $x_1, \cdots, x_N \in Y$. Then there exists a unique additive function satisfying the functional equation (1.4) and

$$\|f(x) - A_O(x)\| \leq \frac{\beta N}{|N-1|} \tag{3.33}$$

for all $x \in Y$.

Corollary 3.5. *Let $f : Y \to Z$ be a mapping fulfilling the inequality*

$$\left\| f\left(\frac{\sum_{k=1}^{N} x_k}{N} \right) - \frac{1}{N} \sum_{k=1}^{N} f(x_k) \right\| \leq \beta \sum_{k=1}^{N} ||x_k||^t \tag{3.34}$$

where $\beta > 0, t > 0$ and for all $x_1, \cdots, x_N \in Y$. Then there exists a unique additive function satisfying the functional equation (1.4) and

$$\|f(x) - A_O(x)\| \leq \frac{\beta ||x||^t (N^t + N + 1)}{|N - N^t|} \tag{3.35}$$

where $t \neq 1$ for all $x \in Y$.

Corollary 3.6. *Let $f : Y \to Z$ be a mapping fulfilling the inequality*

$$\left\| f\left(\frac{\sum_{k=1}^{N} x_k}{N} \right) - \frac{1}{N} \sum_{k=1}^{N} f(x_k) \right\| \leq \beta \prod_{k=1}^{N} ||x_k||^t \tag{3.36}$$

where $\beta > 0, t > 0$ and for all $x_1, \cdots, x_N \in Y$. Then there exists a unique additive function satisfying the functional equation (1.4) and

$$\|f(x) - A_O(x)\| \leq \frac{\beta ||x||^{Nt} N^{t+1}}{|N - N^{Nt}|} \tag{3.37}$$

where $Nt \neq 1$ for all $x \in Y$.

Corollary 3.7. *Let $f : Y \to Z$ be a mapping fulfilling the inequality*

$$\left\| f\left(\frac{\sum_{k=1}^{N} x_k}{N} \right) - \frac{1}{N} \sum_{k=1}^{N} f(x_k) \right\| \leq \beta \left\{ \prod_{k=1}^{N} ||x_k||^t + \sum_{k=1}^{N} ||x_k||^{Nt} \right\} \tag{3.38}$$

where $\beta > 0, t > 0$ and for all $x_1, \cdots, x_N \in Y$. Then there exists a unique additive function satisfying the functional equation (1.4) and

$$\|f(x) - A_O(x)\| \leq \frac{\beta ||x||^{Nt} (N^{t+1} + N^{Nt} + N + 1)}{|N - N^{Nt}|} \tag{3.39}$$

where $Nt \neq 1$ for all $x \in Y$.

3.3 Substitution - 3: $N \geq 2$, N is Even Integer

Theorem 3.3. *Let $\lambda : Y^N \to [0, \infty)$ and $f : Y \to Z$ are functions fulfilling the inequalities*

$$\lim_{a\to\infty} \frac{\lambda(N^{ad}x_1, N^{ad}x_2 \cdots, N^{ad}x_{N-1}, N^{ad}x_N)}{N^{ad}} = 0 \tag{3.40}$$

$$\left\| f\left(\frac{\sum_{k=1}^{N} x_k}{N}\right) - \frac{1}{N}\sum_{k=1}^{N} f(x_k) \right\| \leq \lambda(x_1, x_2 \cdots, x_{N-1}, x_N) \tag{3.41}$$

for all $x_1, \cdots, x_N \in Y$. Then there exists a unique additive function fulfilling the functional equation (1.4) and

$$\|f(x) - A_E(x)\| \leq \sum_{b=\frac{1-d}{2}}^{\infty} \frac{\Lambda_E(N^{bd}x)}{N^{bd}} \tag{3.42}$$

for all $x \in Y$, where $A_E(x)$ and $\Lambda_E(N^b x)$ is defined as

$$A_E(x) = \lim_{a\to\infty} \frac{f(N^{ad}x)}{N^{ad}} \tag{3.43}$$

and

$$\Lambda_E(N^{bd}x) = \lambda(N^{(b+1)d}x, \underbrace{-N^{bd}x, \cdots, -N^{bd}x}_{\frac{N-2}{2}}, \underbrace{N^{bd}x, \cdots, N^{bd}x}_{\frac{N-2}{2}}, 0) \tag{3.44}$$

respectively, for all $x \in Y$ with $d = \pm 1$.

Proof. Replacing $(x_1, x_2, \cdots, x_N)$ by $(Nx, \underbrace{-x, \cdots, -x}_{\frac{N-2}{2}}, \underbrace{x, \cdots, x}_{\frac{N-2}{2}}, 0)$ in (3.41), we obtain

$$\left\| f\left(\frac{x}{N}\right) - \frac{1}{N}f(x) \right\| \leq \lambda(Nx, \underbrace{-x, \cdots, -x}_{\frac{N-2}{2}}, \underbrace{x, \cdots, x}_{\frac{N-2}{2}}, 0) \tag{3.45}$$

for all $x \in Y$. Setting x by Nx in (3.45), we get

$$\left\| f(x) - \frac{1}{N}f(Nx) \right\| \leq \lambda\lambda(Nx, \underbrace{-x, \cdots, -x}_{\frac{N-2}{2}}, \underbrace{x, \cdots, x}_{\frac{N-2}{2}}, 0) \tag{3.46}$$

for all $x \in Y$. Define $\Lambda_E(x) = \lambda(Nx, \underbrace{-x, \cdots, -x}_{\frac{N-2}{2}}, \underbrace{x, \cdots, x}_{\frac{N-2}{2}}, 0)$ in (3.46), we have

$$\left\| f(x) - \frac{1}{N}f(Nx) \right\| \leq \Lambda_E(x) \tag{3.47}$$

for all $x \in Y$. Replacing x by Nx and divided by N in (3.47), we arrive

$$\left\| \frac{1}{N}f(Nx) - \frac{1}{N^2}f(N^2x) \right\| \leq \frac{\Lambda_E(Nx)}{N} \tag{3.48}$$

for all $x \in Y$. Combining (3.47) and (3.48), we reach

$$\left\| f(x) - \frac{1}{N^2}f(N^2x) \right\| \leq \Lambda_E(x) + \frac{\Lambda_E(Nx)}{N} \tag{3.49}$$

for all $x \in Y$. Generalizing, one can get

$$\left\| f(x) - \frac{1}{N^a}f(N^a x) \right\| \leq \sum_{b=0}^{a-1} \frac{\Lambda_E(N^b x)}{N^b} \tag{3.50}$$

for all $x \in Y$. If we put x by $N^c x$ and divided by N^c in (3.50), we obtain

$$\left\| \frac{1}{N^c} f(N^c x) - \frac{1}{N^{a+c}} f(N^{a+c} x) \right\| \rightarrow \quad 0 \quad as \quad c \quad \rightarrow \quad \infty \tag{3.51}$$

for all $x \in Y$. Thus $\left\{ \frac{1}{N^a} f(N^a x) \right\}$ is a Cauchy sequence in Z. Since Z is complete, define a mapping such that

$$A_E(x) = \lim_{a \to \infty} \frac{f(N^a x)}{N^a}$$

for all $x \in Y$. The rest of proof is similar to that of Theorem 3.1. □

From Theorem 3.3, we prove the following corollaries concerning the Hyers- Ulam, Hyers - Ulam - TRassias and Hyers - Ulam - JRassias stabilities of the functional equation (1.4).

Corollary 3.8. *Let $f : Y \rightarrow Z$ be a mapping fulfilling the inequality*

$$\left\| f\left(\frac{\sum_{k=1}^{N} x_k}{N} \right) - \frac{1}{N} \sum_{k=1}^{N} f(x_k) \right\| \leq \beta \tag{3.52}$$

where $\beta > 0$ and for all $x_1, \cdots, x_N \in Y$. Then there exists a unique additive function satisfying the functional equation (1.4) and

$$\|f(x) - A_E(x)\| \leq \frac{\beta}{|N-1|} \tag{3.53}$$

for all $x \in Y$.

Corollary 3.9. *Let $f : Y \rightarrow Z$ be a mapping fulfilling the inequality*

$$\left\| f\left(\frac{\sum_{k=1}^{N} x_k}{N} \right) - \frac{1}{N} \sum_{k=1}^{N} f(x_k) \right\| \leq \beta \sum_{k=1}^{N} ||x_k||^t \tag{3.54}$$

where $\beta > 0, t > 0$ and for all $x_1, \cdots, x_N \in Y$. Then there exists a unique additive function satisfying the functional equation (1.4) and

$$\|f(x) - A_E(x)\| \leq \frac{\beta ||x||^t (N^t + N - 2)}{|N - N^t|} \tag{3.55}$$

where $t \neq 1$ for all $x \in Y$.

Corollary 3.10. *Let $f : Y \rightarrow Z$ be a mapping fulfilling the inequality*

$$\left\| f\left(\frac{\sum_{k=1}^{N} x_k}{N} \right) - \frac{1}{N} \sum_{k=1}^{N} f(x_k) \right\| \leq \beta \left\{ \prod_{k=1}^{N} ||x_k||^t + \sum_{k=1}^{N} ||x_k||^{Nt} \right\} \tag{3.56}$$

where $\beta > 0, t > 0$ and for all $x_1, \cdots, x_N \in Y$. Then there exists a unique additive function satisfying the functional equation (1.4) and

$$\|f(x) - A_E(x)\| \leq \frac{\beta ||x||^{Nt} (N^{Nt} + N - 2)}{|N - N^{Nt}|} \tag{3.57}$$

where $Nt \neq 1$ for all $x \in Y$.

4 Generalized Ulam - Hyers Stability of (1.4) In Banach Space : Fixed Point Method

In this section, the generalized Ulam - Hyers stability of the functional equation (1.4) is proved using fixed point method.

Now, we present the following theorem due to B. Margolis and J.B. Diaz [25] for the fixed point theory.

Theorem 4.1. *[25] Suppose that for a complete generalized metric space (Ω, d) and a strictly contractive mapping $T : \Omega \to \Omega$ with Lipschitz constant L. Then, for each given $x \in \Omega$, either*

$$d(T^n x, T^{n+1} x) = \infty, \qquad \forall \qquad n \geq 0,$$

or there exists a natural number n_0 such that the properties hold:

(FP1) $d(T^n x, T^{n+1} x) < \infty$ for all $n \geq n_0$;

(FP2) The sequence $(T^n x)$ is convergent to a fixed to a fixed point y^* of T;

(FP3) y^* is the unique fixed point of T in the set $\Delta = \{y \in \Omega : d(T^{n_0} x, y) < \infty\}$;

(FP4) $d(y^*, y) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in \Delta$.

Using Theorem 4.1, we obtain the Hyers - Ulam stability of (1.4). To prove the stability results throughout this section, we assume $\mathcal{Y}$ be a Normed space and $\mathcal{Z}$ be a Banach space.

4.1 Substitution - 1: $N \geq 2$ N is an Integer

Theorem 4.2. *Let $f : \mathcal{Y} \to \mathcal{Z}$ be a mapping for which there exists a function $\lambda : \mathcal{Y}^N \to [0, \infty)$ with the condition*

$$\lim_{n \to \infty} \frac{1}{\alpha_i^n} \lambda(\alpha_i^n x_1, \cdots, \alpha_i^n x_N) = 0 \tag{4.1}$$

where

$$\alpha_i = \begin{cases} N & if \quad i = 0, \\ \frac{1}{N} & if \quad i = 1 \end{cases} \tag{4.2}$$

such that the functional inequality

$$\left\| f\left(\frac{\sum_{k=1}^N x_k}{N}\right) - \frac{1}{N}\sum_{k=1}^N f(x_k) \right\| \leq \lambda(x, y) \tag{4.3}$$

holds for all $x_1, \cdots, x_N \in \mathcal{Y}$. Assume that there exists $L = L(i)$ such that the function

$$x \to T(x) = \Lambda_A\left(\frac{x}{N}\right) \tag{4.4}$$

with the property

$$\frac{1}{\alpha_i} T(\alpha_i x) = L\, T(x) \tag{4.5}$$

for all $x \in \mathcal{Y}$. Then there exists a unique additive mapping $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ satisfying the functional equation (1.4) and

$$\| f(x) - \mathcal{A}(x) \| \leq \left(\frac{L^{1-i}}{1-L}\right) T(x) \tag{4.6}$$

for all $x \in \mathcal{Y}$.

Proof. Consider the set

$$\Psi = \{h/h : \mathcal{Y} \to \mathcal{Z},\ h(0) = 0\}$$

and introduce the generalized metric on Ψ,

$$\inf\{\rho \in (0, \infty) : \| h(x) - g(x) \| \leq \rho\, T(x), x \in \mathcal{Y}\}. \tag{4.7}$$

It is easy to see that (4.7) is complete with respect to the defined metric. Define $J : \Psi \to \Psi$ by

$$Jh(x) = \frac{1}{\alpha_i} h(\alpha_i x),$$

for all $x \in \mathcal{Y}$. Now, from (4.7) and $h, g \in \Psi$, we arrive

$$\inf\{L\rho \in (0, \infty) : \| Jh(x) - Jg(x) \| \leq L\rho T(x), x \in \mathcal{Y}\}.$$

This implies J is a strictly contractive mapping on Ψ with Lipschitz constant L (see [25]). It follows from (4.7),(4.5) and (3.8) for the case $i=1$, we reach

$$\begin{aligned}
&\inf\left\{1 \in (0,\infty) : \left\| f(x) - \frac{f(Nx)}{N} \right\| \le \Lambda_A(x), x \in \mathcal{Y}\right\} \quad or \\
&\inf\left\{1 \in (0,\infty) : \| f(x) - Jf(x) \| \le L\, T(x), x \in \mathcal{Y}\right\} \quad or \\
&\inf\left\{L^0 \in (0,\infty) : \| f(x) - Jf(x) \| \le L\, T(x), x \in \mathcal{Y}\right\} \quad or \\
&\inf\left\{L^{1-1} \in (0,\infty) : \| f(x) - Jf(x) \| \le L\, T(x), x \in \mathcal{Y}\right\}. \qquad (4.8)
\end{aligned}$$

Again replacing $x = \frac{x}{N}$ in (3.8) and it follows from (4.7), (4.5) for the case $i=0$, we get

$$\begin{aligned}
&\inf\left\{1 \in (0,\infty) : \left\| Nf\left(\frac{x}{N}\right) - f(x) \right\| \le N\Lambda_A\left(\frac{x}{N}\right), x \in \mathcal{Y}\right\} \quad or \\
&\inf\left\{1 \in (0,\infty) : \| Jf(x) - f(x) \| \le LT(x), x \in \mathcal{Y}\right\} \quad or \\
&\inf\left\{L^1 \in (0,\infty) : \| Jf(x) - f(x) \| \le LT(x), x \in \mathcal{Y}\right\} \quad or \\
&\inf\left\{L^{1-0} \in (0,\infty) : \| Jf(x) - f(x) \| \le LT(x), x \in \mathcal{Y}\right\}. \qquad (4.9)
\end{aligned}$$

Thus, from (4.8) and (4.9), we arrive

$$\inf\left\{L^{1-i} \in (0,\infty) : \| f(x) - Jf(x) \| \le L^{1-i}T(x), x \in \mathcal{Y}\right\}. \qquad (4.10)$$

Hence property (FP1) holds. It follows from property (FP2) that there exists a fixed point $\mathcal{A}$ of J in Ψ such that

$$\mathcal{A}(x) = \lim_{n\to\infty} \frac{1}{\alpha_i^n} f(\alpha_i^n x) \qquad (4.11)$$

for all $x \in \mathcal{Y}$. In order to show that $\mathcal{A}$ satisfies (1.4), replacing $(x_1, \cdots, x_N)$ by $(\alpha_i^n x_1, \cdots, \alpha_i^n x_n)$ and dividing by α_i^n in (4.3), we have

$$\| \mathcal{A}(x_1, \cdots, x_N) \| = \lim_{n\to\infty} \frac{1}{\alpha_i^n} \| f(\alpha_i^n x_1, \cdots, \alpha_i^n x_n) \| \le \lim_{n\to\infty} \frac{1}{\alpha_i^n} \lambda\left(\alpha_i^n x, \alpha_i^n y\right) = 0$$

for all $x_1, \cdots, x_N \in \mathcal{Y}$ and so the mapping $\mathcal{A}$ is additive. i.e., $\mathcal{A}$ satisfies the functional equation (1.4). By property (FP3), $\mathcal{A}$ is the unique fixed point of J in the set

$$\Delta = \{\mathcal{A} \in \Psi : d(f, \mathcal{A}) < \infty\},$$

$\mathcal{A}$ is the unique function such that

$$\inf\left\{\rho \in (0,\infty) : \| f(x) - \mathcal{A}(x) \| \le \rho T(x), x \in \mathcal{Y}\right\}.$$

Finally by property (FP4), we obtain

$$\| f(x) - \mathcal{A}(x) \| \le \| f(x) - Jf(x) \|,$$

implying

$$\| f(x) - \mathcal{A}(x) \| \le \frac{L^{1-i}}{1-L},$$

which yields

$$\inf\left\{\frac{L^{1-i}}{1-L} \in (0,\infty) : \| f(x) - \mathcal{A}(x) \| \le \left(\frac{L^{1-i}}{1-L}\right) T(x), x \in \mathcal{Y}\right\}.$$

This completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 4.2 concerning the stability of (1.4).

Corollary 4.11. *Let $f : \mathcal{Y} \to \mathcal{Z}$ be a mapping. If there exist real numbers β and t such that*

$$\left\| f\left(\frac{\sum_{k=1}^{N} x_k}{N}\right) - \frac{1}{N}\sum_{k=1}^{N} f(x_k)\right\| \leq \begin{cases} \beta, & \\ \beta \sum_{k=1}^{N} ||x_k||^t & t \neq 1; \\ \beta \left\{ \prod_{k=1}^{N} ||x_k||^t + \sum_{k=1}^{N} ||x_k||^{Nt} \right\} & Nt \neq 1; \end{cases} \tag{4.12}$$

for all $x_1, \cdots, x_N \in \mathcal{Y}$, then there exists a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ such that

$$\|f(x) - \mathcal{A}(x)\| \leq \begin{cases} \dfrac{\beta}{|N-1|}, \\ \dfrac{\beta\, N^t}{|N - N^t|}, \\ \dfrac{\beta\, N^{Nt}}{|N - N^{Nt}|} \end{cases} \tag{4.13}$$

for all $x \in \mathcal{Y}$.

Proof. Let

$$\lambda(x_1, \cdots, x_n) = \begin{cases} \beta, \\ \beta \sum_{k=1}^{N} ||x_k||^t \\ \beta \left\{ \prod_{k=1}^{N} ||x_k||^t + \sum_{k=1}^{N} ||x_k||^{Nt} \right\} \end{cases} \tag{4.14}$$

for all $x_1, \cdots, x_N \in \mathcal{Y}$. Now

$$\frac{1}{\alpha_i^n}\lambda(\alpha_i^n x_1, \alpha_i^n x_n) = \begin{cases} \dfrac{\beta}{\alpha_i^n}, \\ \dfrac{\beta}{\alpha_i^n} \displaystyle\sum_{k=1}^{N} ||\alpha_i^n x_k||^t, \\ \dfrac{\beta}{\alpha_i^n} \left\{ \displaystyle\prod_{k=1}^{N} ||\alpha_i^n x_k||^t + \sum_{k=1}^{N} ||\alpha_i^n x_k||^{Nt} \right\} \end{cases} = \begin{cases} \to 0 \text{ as } n \to \infty, \\ \to 0 \text{ as } n \to \infty, \\ \to 0 \text{ as } n \to \infty. \end{cases}$$

Thus, (4.1) holds. It follows from (4.4), (4.5) and (4.14), we have

$$T(x) = \Lambda_A\left(\frac{x}{N}\right) = \lambda(x, 0 \cdots, 0, 0) = \begin{cases} \beta \\ \beta||x||^t \\ \beta||x||^{Nt} \end{cases} \tag{4.15}$$

and

$$\frac{1}{\alpha_i}T(\alpha_i x) = \frac{1}{\alpha_i}\Lambda_A\left(\frac{\alpha_i x}{N}\right) = \lambda(\alpha_i x, 0 \cdots, 0, 0) = \begin{cases} \alpha_i^{-1}\beta, \\ \alpha_i^{t-1}\beta||x||^t, \\ \alpha_i^{Nt-1}\beta||x||^{Nt} \end{cases} = LT(x) \tag{4.16}$$

for all $x \in \mathcal{Y}$. Hence, in view of (4.16) the inequality (4.6) holds for

(i). $L = \alpha_i^{-1}$ if $i = 0$ and $L = \frac{1}{\alpha_i^{-1}}$ if $i = 1$,

(ii). $L = \alpha_i^{t-1}$ for $t < 1$ if $i = 0$ and $L = \frac{1}{\alpha_i^{t-1}}$ for $t > 1$ if $i = 1$,

(iii). $L = \alpha_i^{Nt-1}$ for $Nt > 1$ if $i = 0$ and $L = \frac{1}{\alpha_i^{Nt-1}}$ for $Nt > 1$ if $i = 1$.

Hence the proof is complete. □

The proof of the following theorems and corollaries of Sections 4.2 and 4.3 are similar tracing to that of Theorem 4.2 and Corollary 4.11. Hence the details of the proof are omitted.

4.2 Substitution - 2: $N \geq 2$, N is Odd Integer

Theorem 4.3. *Let $f : \mathcal{Y} \to \mathcal{Z}$ be a mapping for which there exists a function $\lambda : \mathcal{Y}^N \to [0, \infty)$ with the condition*

$$\lim_{n\to\infty} \frac{1}{\alpha_i^n} \lambda(\alpha_i^n x_1, \cdots, \alpha_i^n x_N) = 0 \tag{4.17}$$

where

$$\alpha_i = \begin{cases} N & if \quad i = 0, \\ \frac{1}{N} & if \quad i = 1 \end{cases} \tag{4.18}$$

such that the functional inequality

$$\left\| f\left(\frac{\sum_{k=1}^N x_k}{N}\right) - \frac{1}{N}\sum_{k=1}^{N} f(x_k) \right\| \leq \lambda(x, y) \tag{4.19}$$

holds for all $x_1, \cdots, x_N \in \mathcal{Y}$. Assume that there exists $L = L(i)$ such that the function

$$x \to T(x) = \Lambda_O\left(\frac{x}{N}\right) \tag{4.20}$$

with the property

$$\frac{1}{\alpha_i} T(\alpha_i x) = L\, T(x) \tag{4.21}$$

for all $x \in \mathcal{Y}$. Then there exists a unique additive mapping $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ satisfying the functional equation (1.4) and

$$\| f(x) - \mathcal{A}(x) \| \leq \left(\frac{L^{1-i}}{1-L}\right) T(x) \tag{4.22}$$

for all $x \in \mathcal{Y}$.

Corollary 4.12. *Let $f : \mathcal{Y} \to \mathcal{Z}$ be a mapping. If there exist real numbers β and t such that*

$$\left\|\left\| f\left(\frac{\sum_{k=1}^N x_k}{N}\right) - \frac{1}{N}\sum_{k=1}^{N} f(x_k) \right\|\right\| \leq \begin{cases} \beta, & \\ \beta \sum_{k=1}^N ||x_k||^t & t \neq 1; \\ \beta \prod_{k=1}^N ||x_k||^t & Nt \neq 1; \\ \beta \left\{ \prod_{k=1}^N ||x_k||^t + \sum_{k=1}^N ||x_k||^{Nt} \right\} & Nt \neq 1; \end{cases} \tag{4.23}$$

for all $x_1, \cdots, x_N \in \mathcal{Y}$, then there exists a unique additive function $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ such that

$$\|f(x) - \mathcal{A}(x)\| \leq \begin{cases} \dfrac{\beta}{|N-1|}, \\ \dfrac{\beta\, N^{t+1}}{|N - N^t|}, \\ \dfrac{\beta\, N^{Nt}}{|N - N^{Nt}|} \\ \dfrac{\beta\, (N^{2Nt} + 1)}{|N - N^{Nt}|} \end{cases} \tag{4.24}$$

for all $x \in \mathcal{Y}$.

4.3 Substitution - 3: $N \geq 2$, N is Even Integer

Theorem 4.4. *Let $f : \mathcal{Y} \to \mathcal{Z}$ be a mapping for which there exists a function $\lambda : \mathcal{Y}^N \to [0, \infty)$ with the condition*

$$\lim_{n\to\infty} \frac{1}{\alpha_i^n} \lambda(\alpha_i^n x_1, \cdots, \alpha_i^n x_N) = 0 \tag{4.25}$$

where

$$\alpha_i = \begin{cases} N & if \quad i = 0, \\ \frac{1}{N} & if \quad i = 1 \end{cases} \tag{4.26}$$

such that the functional inequality

$$\left\| f\left(\frac{\sum_{k=1}^{N} x_k}{N}\right) - \frac{1}{N}\sum_{k=1}^{N} f(x_k)\right\| \leq \lambda(x,y) \tag{4.27}$$

holds for all $x_1, \cdots, x_N \in \mathcal{Y}$. *Assume that there exists* $L = L(i)$ *such that the function*

$$x \to T(x) = \Lambda_E\left(\frac{x}{N}\right) \tag{4.28}$$

with the property

$$\frac{1}{\alpha_i} T(\alpha_i x) = L\, T(x) \tag{4.29}$$

for all $x \in \mathcal{Y}$. *Then there exists a unique additive mapping* $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ *satisfying the functional equation (1.4) and*

$$\| f(x) - \mathcal{A}(x) \| \leq \left(\frac{L^{1-i}}{1-L}\right) T(x) \tag{4.30}$$

for all $x \in \mathcal{Y}$.

Corollary 4.13. *Let* $f : \mathcal{Y} \to \mathcal{Z}$ *be a mapping. If there exist real numbers* β *and* t *such that*

$$\left\|\left\| f\left(\frac{\sum_{k=1}^{N} x_k}{N}\right) - \frac{1}{N}\sum_{k=1}^{N} f(x_k)\right\|\right\| \leq \begin{cases} \beta, & \\ \beta \sum_{k=1}^{N} ||x_k||^t & t \neq 1; \\ \beta \left\{\prod_{k=1}^{N} ||x_k||^t + \sum_{k=1}^{N} ||x_k||^{Nt}\right\} & Nt \neq 1; \end{cases} \tag{4.31}$$

for all $x_1, \cdots, x_N \in \mathcal{Y}$, *then there exists a unique additive function* $\mathcal{A} : \mathcal{Y} \to \mathcal{Z}$ *such that*

$$\|f(x) - \mathcal{A}(x)\| \leq \begin{cases} \dfrac{\beta}{|N-1|}, \\ \dfrac{\beta\, N^t (N-1)}{|N - N^t|}, \\ \dfrac{\beta\, N^{Nt}(N-1)}{|N - N^{Nt}|} \end{cases} \tag{4.32}$$

for all $x \in \mathcal{Y}$.

5 Application

Consider the additive functional equation

$$f\left(\frac{\sum_{k=1}^{N} x_k}{N}\right) = \frac{1}{N}\sum_{k=1}^{N} f(x_k).$$

Since $f(x) = x$ is the solution of the above functional equation, we arrive

$$f\left(\frac{\sum_{k=1}^{N} x_k}{N}\right) = f\left(\frac{x_1 + x_2 + x_3 + \cdots\cdots + x_N}{N}\right) = \frac{x_1 + x_2 + x_3 + \cdots\cdots + x_N}{N}$$

This gives the N observations of an arithmetic mean.

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Received: August 12, 2016; *Accepted*: December 22, 2016

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