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Second kind shifted Chebyshev polynomials and power series method for solving multi-order non-linear fractional differential equations

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Abstract

In this paper, we use shifted Chebyshev approximations with the second kind [25] and fractional power series method (FPSM) ([3], [8]) to solve the multi-order non-linear fractional differential equations. The fractional derivative is described in the Caputo sense. The properties of shifted Chebyshev polynomials with the second kind are utilized to reduce multi-order NFDEs. The system of non-linear of algebraic equations which solved by using Newton iteration method. We compared with FPSM. The results are compared with the traditional methods [23].

Keywords: Shifted Chebyshev polynomials with the second kind; Fractional power series method; Caputo derivative; Multi-order nonlinear fractional differential equations.

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1 Introduction

Fractional differential equations have recently been applied in various area of engineering, science, finance, applied mathematics, bio-engineering and others ([1], [4]). However, many researchers remain unaware of this field using numerical different methods ([5], [9], [10], [11]-[14], [18], [20], [24], [26]).

The collocation methods in ([6], [7], [15], [23]) based on the Chebyshev polynomials for solving multi-term linear and nonlinear fractional differential equations subject to non-homogeneous initial conditions.

The organization of this paper is as follows. In the next section, we give the definitions of fractional derivatives in fractional calculus. In the section 3, we give the fractional power series method. In the section 4, we give some properties of Chebyshev polynomials of the second kind. In section 5, we procedure of solution for the multi-order NFDEs. In section 6, numerical simulation and comparison are given to clarify the method. Also a conclusion is given in section 7. Note that we have computed the numerical results using Matlab programming.

Now, we describe some necessary definitions and mathematical preliminaries of the fractional calculus theory required for our subsequent development.

2 Definitions of fractional derivative

Definition 2.1.

The Caputo fractional derivative operator D^{α} of order α is defined in the following form [19]

$$D^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(t)}{(x-t)^{\alpha-m+1}} dt, \quad \alpha > 0,$$

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where $m-1<\alpha<m,\ m\in\mathbb{N},\ x>0$.
Similar to integer-order differentiation, Caputo fractional derivative operator is a linear operation

$$D^{\alpha}\left(\lambda f(x)+\mu g(x)\right)=\lambda D^{\alpha} f(x)+\mu D^{\alpha} g(x),$$

where λ and μ are constants.
For the Caputo's derivative we have [19]

$$D^{\alpha} C=0, \qquad \text{C is a constant,} \tag{2.1}$$

$$D^{\alpha} x^{n}=\begin{cases} 0, & \text{for} \quad n\in\mathbb{N}_0 \text{ and } n<\lceil\alpha\rceil; \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}x^{n-\alpha}, & \text{for} \quad n\in\mathbb{N}_0 \text{ and } n\geq\lceil\alpha\rceil. \end{cases} \tag{2.2}$$

We use the ceiling function $\lceil\alpha\rceil$ to denote the smallest integer greater than or equal to α. Also $\mathbb{N}_0=\{0,1,2,...\}$. Recall that for $\alpha\in\mathbb{N}$, the Caputo differential operator coincides with the usual differential operator of integer order.
For more details on fractional derivatives definitions and its properties see ([16], [19], [21]).

The main goal in this article is concerned with the application of Chebyshev pseudo-spectral method for the second kind [25] and Power series method [3] to obtain the numerical solution of multi-order fractional differential equation of the form

$$D^{\alpha}y(x)=F\left(x,y(x),D^{\beta_1}y,...,D^{\beta_n}y\right), \tag{2.3}$$

with the following initial conditions

$$y^{(k)}(0)=y_k, \quad k=0,1,...,m, \tag{2.4}$$

where $m<\alpha\leq m+1$, $0<\beta_1<\beta_2<...<\beta_n<\alpha$ and D^{α} denotes Caputo fractional derivative of order α. It should be noted that F can be nonlinear in general.

The main idea of this work is to apply the Chebyshev collocation method for the second kind to discretize (2.3) to reduce multi-order NFDEs to a system of nonlinear of algebraic equations, and use Newton iteration method to solve the resulting system.
Chebyshev polynomials of the second kind are well known family of orthogonal polynomials on the interval $[-1,1]$ that have many applications ([2], [17], [22]). They are widely used because of their good properties in the approximation of functions [17]. However, with our best knowledge, very little work was done to adapt this polynomials to the solution of fractional differential equations.

3 Fractional power series method

In this section, we use fractional power series method (FPSM) ([3], [8]) to solve multi-order fractional differential equation. Compared to the above method, the FPSM is more simple and effective.

Definition 3.2. *([3], [8])*

A power series representation of the form

$$\sum_{n=0}^{\infty} c_n(t-t_0)^{n\alpha}=c_0+c_1(t-t_0)^{\alpha}+c_2(t-t_0)^{2\alpha}+\cdots, \tag{3.5}$$

where $0\leq m-1<\alpha\leq m,\ m\in N^{+}$ and $t\geq t_0$ is called a fractional power series (FPS) about t_0 where t is a variable and c_n are the coefficients of the series.

Theorem 3.1. *([3], [8])*

Suppose that the FPS $\sum_{n=0}^{\infty} c_n t^{n\alpha}$ has radius of convergence $R>0$. If $f(t)$ is a function defined by $f(t)=\sum_{n=0}^{\infty} c_n t^{n\alpha}$, $0\leq t<R$, then for $m-1<\alpha\leq m$ and $0<t\leq R$, we have

$$D^{\alpha} f(t)=\sum_{n=1}^{\infty} c_n \frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)} t^{(n-1)\alpha}. \tag{3.6}$$

4 Some properties of Chebyshev polynomials of the second kind

4.1 Chebyshev polynomials of the second kind

The Chebyshev polynomials $U_n(x)$of the second kind are orthogonal polynomials of degree n in x defined on the $[-1,1]$ ([17], [25])

$$U_n(x) = \frac{\sin(n+1)^{\theta}}{\sin\theta}$$

where $x = \cos\theta$ and $\theta \in [0,\pi]$.

The polynomials $U_n(x)$ are orthogonal on $[-1,1]$ with respect to the inner products

$$(U_n(x), U_m(x)) = \int_{-1}^{1} \sqrt{1-x^2}U_n(x)U_m(x)dx = \{^{0,\ n\neq m,}_{\frac{\pi}{2},\ n=m,} \tag{4.7}$$

where $\sqrt{1-x^2}$ is weight function.

$U_n(x)$ may be generated by using the recurrence relations

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x), \quad n = 2,3,...$$

with

$U_0(x) = 1. \quad U_1(x) = 2x.$

The analytical form of the Chebyshev polynomials of the second kind $U_n(x)$ of degree n is given by:

$$U_n(x) = \sum_{i=0}^{\lceil\frac{\pi}{2}\rceil} (-1)^i \binom{n-i}{i} (2x)^{n-2i}, \ \ n > 0$$

Using the properties of Gamma function the previous equation can be rewritten as:

$$U_n(x) = \sum_{i=0}^{\lceil\frac{\pi}{2}\rceil} (-1)^i 2^{n-2i} \frac{\Gamma(n-i+1)x^{^{n-2i}}}{\Gamma(i+1)\Gamma(n-2i+1)}, \ \ n > 0, \tag{4.8}$$

where $\lceil\frac{\pi}{2}\rceil$ denotes the integral part of $n/2$.

4.2 Shifted Chebyshev polynomials of the second kind

In order to use these polynomials on the interval $x \in [0,1]$ ([17], [25]). We define the so called shifted Chebyshev polynomials of the second kind $U_n^*(x)$ by introducing the change variable $z = 2x-1$. This means that the shifted Chebyshev polynomials of the second kind defined as:

$$U_n^*(x) = U_n(2x-1),$$

also there are important relation between the shifted and second kind Chebyshev polynomials as follows:

$$2xU_{n-1}^*(x^2) = U_{2n-1}(x),$$

these polynomials are orthogonal on the support interval $[0,1]$ as the following inner product:

$$(U_n^*(x), U_m^*(x)) = \int_{0}^{1} \sqrt{x-x^2}U_n^*(x)U_m^*(x)dx = \{^{0,\ n\neq m,}_{\frac{\pi}{8},\ n=m,} \tag{4.9}$$

where $\sqrt{x-x^2}$ is weight function.

$U_n^*(x)$ may be generated by using the recurrence relations

$$U_n^*(x) = 2(2x-1)U_{n-1}^*(x) - U_{n-2}^*(x), \ \ n = 2,3,...$$

with start values

$U_0^*(x) = 1, \ \ U_1^*(x) = 4x-2.$

The analytical form of the shifted Chebyshev polynomials of the second kind $U_n^*(x)$ of degree n is given by

$$U_n^*(x) = \sum_{i=0}^{n} (-1)^i 2^{2n-2i} \frac{\Gamma(2n-i+2)x^{^{n-i}}}{\Gamma(i+1)\Gamma(2n-2i+2)}, \ \ n > 0, \tag{4.10}$$

The function which may be appear in solution of the model problem (nonlinear multi-order fractional differential equations) can be written as series of $U^*(x)$.

Let $g(x)$ be a square integrable in $[0,1]$ it can be expressed in terms of the shifted Chebyshev polynomials of the second kind as follows:

$$g(x) = \sum_{i=0}^{\infty} a_i U_i^*(x), \tag{4.11}$$

where the coefficients $a_i, i = 0, 1, ...,$ are given by:

$$a_i = \frac{2}{\pi} \int_{-1}^{1} g(\frac{x+1}{2}) \sqrt{1-x^2} U_i(x) dx, \tag{4.12}$$

or

$$a_i = \frac{8}{\pi} \int_{0}^{1} g(x) \sqrt{x-x^2} U_i^*(x) dx, \tag{4.13}$$

In practice, only the first $(m+1)$ terms of shifted Chebyshev polynomials of the second kind are considered in the approximate case. Then we have:

$$g_m(x) = \sum_{i=0}^{m} a_i U_i^*(x), \tag{4.14}$$

The main approximate formula of the fractional derivative of $g_m(x)$ is given in the following theorem.

Theorem 4.2.

Let $g(x)$ be approximated by shifted Chebyshev polynomials of the second kind as (4.14) and also suppose $\alpha > 0$, then

$$D^{\alpha}(g_m(x)) = \sum_{i=\lceil \alpha \rceil}^{m} \sum_{k=0}^{i-\lceil \alpha \rceil} b_i \, N_{i,k}^{(\alpha)} \, x^{i-k-\alpha}, \tag{4.15}$$

where $N_{i,k}^{(\alpha)}$ is given by

$$N_{i,k}^{(\alpha)} = (-1)^k \frac{2^{2i-2k} \, (2n+1)\Gamma(2i-k+1)\Gamma(i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+2)\Gamma(i-k+1-\alpha)}. \tag{4.16}$$

Proof. see([25]). □

5 Procedure of solution for the multi-order NFDEs

Consider the multi-order nonlinear fractional differential equation of type given in Eq.(2.3). In order to use Chebyshev collocation method for the second kind, we first approximate $y(x)$ as

$$y_m(x) = \sum_{i=0}^{m} c_i \, U_i^*(x). \tag{5.17}$$

From Eqs.(2.3), (5.17) and Theorem 2 we have

$$\begin{aligned} &\sum_{i=\lceil \alpha \rceil}^{m} \sum_{k=0}^{i-\lceil \alpha \rceil} c_i \, N_{i,k}^{(\alpha)} \, x^{i-k-\alpha} = \\ &F\left(x, \sum_{i=0}^{m} c_i \, U_i^*(x), \sum_{i=\lceil \beta_1 \rceil}^{m} \sum_{k=0}^{i-\lceil \beta_1 \rceil} c_i \, N_{i,k}^{(\beta_1)} \, x^{i-k-\beta_1}, ..., \sum_{i=\lceil \beta_n \rceil}^{m} \sum_{k=0}^{i-\lceil \beta_n \rceil} c_i \, N_{i,k}^{(\beta_n)} \, x^{i-k-\beta_n} \right), \end{aligned} \tag{5.18}$$

we now collocate Eq.(5.18) at $(m+1-\lceil \alpha \rceil)$ points x_p as

$$\begin{aligned} &\sum_{i=\lceil \alpha \rceil}^{m} \sum_{k=0}^{i-\lceil \alpha \rceil} c_i \, N_{i,k}^{(\alpha)} \, x_p^{i-k-\alpha} = \\ &F\left(x_p, \sum_{i=0}^{m} c_i \, U_i^*(x_p), \sum_{i=\lceil \beta_1 \rceil}^{m} \sum_{k=0}^{i-\lceil \beta_1 \rceil} c_i \, N_{i,k}^{(\beta_1)} \, x_p^{i-k-\beta_1}, ..., \sum_{i=\lceil \beta_n \rceil}^{m} \sum_{k=0}^{i-\lceil \beta_n \rceil} c_i \, N_{i,k}^{(\beta_n)} \, x_p^{i-k-\beta_n} \right). \end{aligned} \tag{5.19}$$

For suitable collocation points we use roots of shifted Chebyshev polynomial $U^*_{m+1-\lceil\alpha\rceil}(x)$.
Also, by substituting Eqs.(5.17) in the initial conditions, we can find $\lceil\alpha\rceil$ equations. By substituting Eqs.(5.17) in the initial conditions (2.4) we obtain

$$\sum_{i=0}^{m}(-1)^i c_i = 0, \qquad \sum_{i=0}^{m} c_i U^{*(k)}(0) = 0, \qquad k = 1, 2, ..., m \tag{5.20}$$

Equation (5.19), together with $\lceil\alpha\rceil$ equations of the initial conditions (5.20), give $(m+1)$ of nonlinear algebraic equations which can be solved, for the unknown $c_i, i = 0, 1, ..., m$, using Newton iteration method, as described in the following section.

6 large Numerical simulation and comparison

In this section, we implement the proposed method to solve the muti-order NFDEs (2.3)-(2.4) with different two examples.

Example 1

Consider the following nonlinear initial value problem [23]

$$D^3 y(x) + D^{2.5} y(x) + y^2(x) = x^4, \tag{6.21}$$

with the following initial conditions

$$y(0) = y'(0) = 0, \quad y''(0) = 2. \tag{6.22}$$

6.1 SCP2K

We apply the suggested method with $m = 3$, and approximate the solution $y(x)$ as follows

$$y_3(x) = \sum_{i=0}^{3} c_i\, U_i^*(x). \tag{6.23}$$

Using Eq.(5.19), $\alpha = 2$, $\beta_1 = 1.5$, and for $p = 0$, we have

$$\sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=0}^{i-\lceil\alpha\rceil} c_i\, N_{i,k}^{(\alpha)}\, x_p^{i-k-\alpha} + \sum_{i=\lceil\beta_1\rceil}^{m}\sum_{k=0}^{i-\lceil\beta_1\rceil} c_i\, N_{i,k}^{(\beta_1)}\, x_p^{i-k-\beta_1} + \left(\sum_{i=0}^{m} c_i\, U_i^*(x_p)\right)^2 = x_p^4, \tag{6.24}$$

where x_p are roots of the shifted Chebyshev polynomial for the second kind $U_1^*(x)$, i.e., $x_0 = 0.5$.
By using Eqs.(6.24) and (5.20) we obtain the following nonlinear system of algebraic equations

$$c_3(N_{3,0}^{(\alpha)} + N_{3,0}^{(\beta_1)}\, x_0^{0.5}) + (s_0c_0 + s_1c_1 + s_2c_2 + s_3c_3)^2 = x_0^4, \tag{6.25}$$

$$c_1 - c_1 + c_2 - c_3 = 0, \tag{6.26}$$

$$k_0c_0 + k_1c_1 + k_2c_2 + k_3c_3 = 0, \tag{6.27}$$

$$r_0c_0 + r_1c_1 + r_2c_2 + r_3c_3 = 2, \tag{6.28}$$

where

$$s_i = U_i^*(x_0), \qquad k_i = U_i^{*(1)}(0), \qquad r_i = U_i^{*(2)}(0).$$

By solving Eqs.(6.25)-(6.28) we obtain

$$c_0 = \frac{3}{8}, \qquad c_1 = \frac{4}{8}, \qquad c_2 = \frac{1}{8}, \qquad c_3 = 0.$$

Therefore

$$y(x) = \begin{pmatrix} \frac{3}{8}, & \frac{4}{8}, & \frac{1}{8}, & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2x-1 \\ 8x^2-8x+1 \\ 32x^3-48x^2+18x-1 \end{pmatrix} = x^2,$$

which is the exact solution of this problem [23].

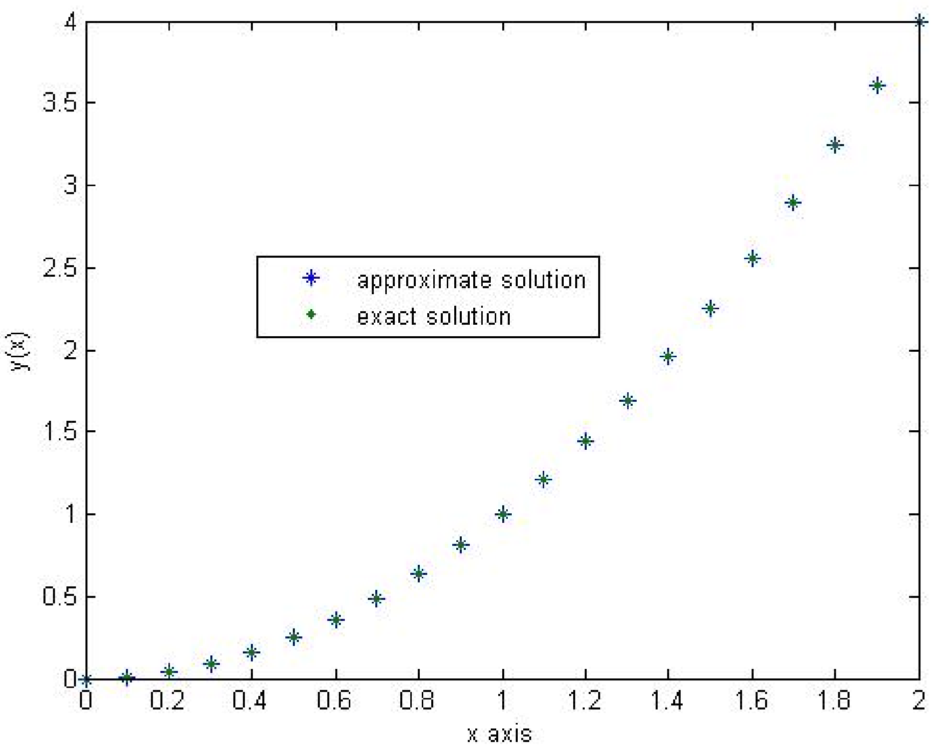

Figure 4.1. Comparison between the exact solution and the numerical solution.

From this Figure 4.1, we can conclude that the numerical results are excellent agreement with the exact solution. Also, it is evident that the overall errors can be made smaller by adding new terms from the series (5.17).

6.2 FPSM

To apple FPSM, we suppose that the solution the form

$$y(x) = \sum_{k=0}^{\infty} a_k x^{\alpha k} = a_0 + a_1 x^{\alpha} + a_2 x^{2\alpha} + ... \tag{6.29}$$

$$\begin{aligned} y^2(x) &= (\sum_{k=0}^{\infty} a_k x^{\alpha k})^2 = (a_0 + a_1 x^{\alpha} + a_2 x^{2\alpha} + ...)^2 \\ &= a_0^2 + 2a_0 a_1 x^{\alpha} + (2a_0 a_2 + a_1^2) x^{2\alpha} + 2a_1 a_2 x^{3\alpha} + a_2^2 x^{4\alpha} + ... \end{aligned} \tag{6.30}$$

$$D^3 y(x) = \alpha(\alpha-1)(\alpha-2) a_1 x^{\alpha-3} + 2\alpha(2\alpha-1)(2\alpha-2) a_2 x^{2\alpha-3} + ... \tag{6.31}$$

by theorem 1

$$D^{\alpha} y(x) = \sum_{k=1}^{\infty} a_k \frac{\Gamma(k\alpha+1)}{\Gamma((k-1)\alpha+1)} x^{\alpha(k-1)} \tag{6.32}$$

the equation (6.21) can be written as

$$D^{2.5} y(x) = x^4 - D^3 y(x) - y^2(x) \tag{6.33}$$

substituting (6.30), (6.31), (6.32) into (6.33)and comparing the cofficients of x^{α},we get

$$\sum_{k=1}^{\infty} a_k \frac{\Gamma(k\alpha+1)}{\Gamma((k-1)\alpha+1)} x^{\alpha(k-1)} =$$

$$\begin{aligned} & x^4 - (\alpha(\alpha-1)(\alpha-2) a_1 x^{\alpha-3} + 2\alpha(2\alpha-1)(2\alpha-2) a_2 x^{2\alpha-3} + ...) \\ & -(a_0^2 + 2a_0 a_1 x^{\alpha} + (2a_0 a_2 + a_1^2) x^{2\alpha} + 2a_1 a_2 x^{3\alpha} + a_2^2 x^{4\alpha} + ...) \end{aligned}$$

using initial condition $y(o) = 0$

we have $a_0 = y(0) = 0$

Next, we determine the $a_k (k = 1, 2, ...)$.
For example, if $a_0 = 0$ then
$a_1 \Gamma(\alpha + 1) = x^4 - a_0$

$$a_1 = \frac{x^4}{\Gamma(\alpha+1)}$$

$a_2 = 0$
therefore we obtain the approximate solution of equation

$$y(x) = \sum_{k=0}^{\infty} a_k x^{\alpha k} = 0 + \frac{x^4}{\Gamma(\alpha+1)} x^{\alpha} + 0 + ...$$

Example 2

In this example, we consider the following nonlinear differential equation [23]

$$D^4 y(x) + D^{3.5} y(x) + y^3(x) = x^9, \tag{6.34}$$

subject to the initial conditions

$$y^{(k)}(0) = 0, \quad k = 0, 1, 2, 3. \tag{6.35}$$

6.3 SCP2K

To solve the above problem, by applying the proposed technique described in Section 5 with $m = 4$, we approximate the solution as

$$y(x) = c_0 U_0^*(x) + c_1 U_1^*(x) + c_2 U_2^*(x) + c_3 U_3^*(x) + c_4 U_4^*(x).$$

Using Eq.(5.19), $\alpha = 4$, $\beta_1 = 3.5$, and for $p = 0$, we have

$$\sum_{i=\lceil\alpha\rceil}^{m} \sum_{k=0}^{i-\lceil\alpha\rceil} c_i\, N_{i,k}^{(\alpha)}\, x_p^{i-k-\alpha} + \sum_{i=\lceil\beta_1\rceil}^{m} \sum_{k=0}^{i-\lceil\beta_1\rceil} c_i\, N_{i,k}^{(\beta_1)}\, x_p^{i-k-\beta_1} + \left(\sum_{i=0}^{m} c_i\, U_i^*(x_p)\right)^3 = x_p^9, \tag{6.36}$$

where x_p are roots of the shifted Chebyshev polynomial for the second kind $U_1^*(x)$, i.e., $x_0 = 0.5$.
By using Eqs.(6.36) and (5.20) we obtain the following nonlinear system of algebraic equations

$$c_4 \left(N_{4,0}^{(\alpha)} + N_{4,0}^{(\beta_1)}\, x_0^{0.5}\right) + (s_0 c_0 + s_1 c_1 + s_2 c_2 + s_3 c_3 + s_4 c_4)^3 = x_0^9, \tag{6.37}$$

$$c_1 - c_1 + c_2 - c_3 + c_4 = 0, \tag{6.38}$$

$$k_0 c_0 + k_1 c_1 + k_2 c_2 + k_3 c_3 + k_4 c_4 = 0, \tag{6.39}$$

$$r_0 c_0 + r_1 c_1 + r_2 c_2 + r_3 c_3 + r_4 c_4 = 0, \tag{6.40}$$

$$z_0 c_0 + z_1 c_1 + z_2 c_2 + z_3 c_3 + z_4 c_4 = 0, \tag{6.41}$$

where

$$s_i = U_i^*(x_0), \qquad k_i = U_i^{*(1)}(0), \qquad r_i = U_i^{*(2)}(0), \qquad z_i = U_i^{*(3)}(0).$$

By solving Eqs.(6.37)-(6.41) we obtain

$$c_0 = \frac{10}{32}, \quad c_1 = \frac{15}{32}, \quad c_2 = \frac{6}{32}, \quad c_3 = \frac{1}{32}, \quad c_4 = 0.$$

Therefore

$$y(x) = \left(\tfrac{10}{32}, \ \tfrac{15}{32}, \ \tfrac{6}{32}, \ \tfrac{1}{32}, \ 0 \right) \begin{pmatrix} 1 \\ 2x - 1 \\ 8x^2 - 8x + 1 \\ 32x^3 - 48x^2 + 18x - 1 \\ 128x^4 - 256x^3 + 160x^2 - 32x + 1 \end{pmatrix} = x^3,$$

which is the exact solution of this problem [23].

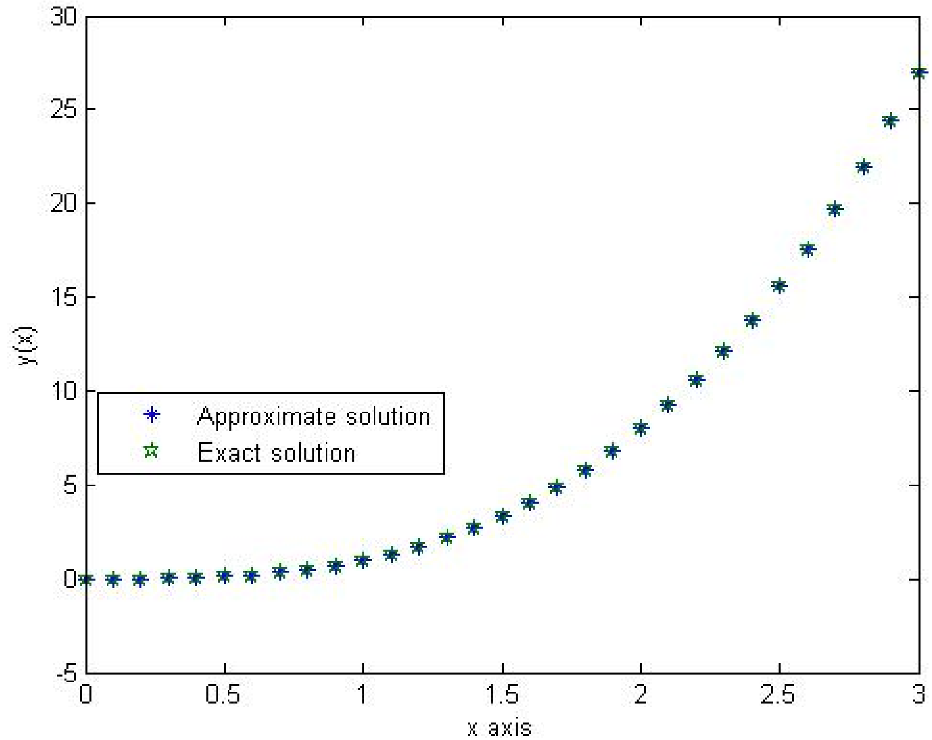

Figure 4.2. Comparison between the exact solution and the numerical solution.

From this Figure 4.2, we can conclude that the numerical results are excellent agreement with the exact solution. Also, it is evident that the overall errors can be made smaller by adding new terms from the series (5.17).

6.4 FPSM

To apple FPSM,we suppose that the solution the form

$$y(x) = \sum_{k=0}^{\infty} a_k x^{\alpha k} = a_0 + a_1 x^{\alpha} + a_2 x^{2\alpha} + ..., \tag{6.42}$$

$$\begin{aligned} y^3(x) &= (\sum_{k=0}^{\infty} a_k x^{\alpha k})^3 = (a_0 + a_1 x^{\alpha} + a_2 x^{2\alpha} + ...)^3, \\ &= a_0^3 + 3a_0^2 a_1 x^{\alpha} + (3a_0^2 a_2 + 3a_0 a_1^2) x^{2\alpha} + (6a_0 a_1 a_2 + a_1^3) x^{3\alpha} \\ &\quad + (3a_1^2 a_2 + 3a_0 a_2^2) x^{4\alpha} + 3a_1 a_2^2 x^{5\alpha} + a_2^3 x^{6\alpha} + ... \end{aligned} \tag{6.43}$$

$$D^4 y(x) = \alpha(\alpha-1)(\alpha-2)(\alpha-3) a_1 x^{\alpha-4} + 2\alpha(2\alpha-1)(2\alpha-2)(2\alpha-3) a_2 x^{2\alpha-4} + ..., \tag{6.44}$$

by theorem

$$D^{\alpha} y(x) = \sum_{k=0}^{\infty} a_k \frac{\Gamma(k\alpha+1)}{\Gamma((k-1)\alpha+1)} x^{\alpha(k-1)}, \tag{6.45}$$

the equation(6.34)can be written as

$$D^{3.5} y(x) = x^9 - D^4 y(x) - y^3(x), \tag{6.46}$$

substituting (6.43),(6.44),(6.45) into (6.46)and comparing the cofficients of x^{α},we get

$\sum_{k=1}^{\infty} a_k \frac{\Gamma(k\alpha+1)}{\Gamma((k-1)\alpha+1)} x^{\alpha(k-1)} =$

$$x^9-[\alpha(\alpha-1)(\alpha-2)(\alpha-3)a_1x^{\alpha-4}+2\alpha(2\alpha-1)(2\alpha-2)(2\alpha-3)a_2x^{2\alpha-4}+...]$$
$$-[a_0^3+3a_0^2a_1x^{\alpha}+(3a_0^2a_2+3a_0a_1^2)x^{2\alpha}+(6a_0a_1a_2+a_1^3)x^{3\alpha}+$$
$$+(3a_1^2a_2+3a_0a_2^2)x^{4\alpha}+3a_1a_2^2x^{5\alpha}+a_2^3x^{6\alpha}+...]$$

using initial condition $y^k(o)=0$

we have $a_0=y(0)=0$

Next, we determine the $a_k(k=1,2,...)$.
For example, if $a_0=0$ then
$a_1\Gamma(\alpha+1)=x^9-a_0^3$

$$a_1=\frac{x^9}{\Gamma(\alpha+1)}$$

therefore we obtain the approximate solution of equation

$$y(x)=\sum_{k=0}^{\infty}a_kx^{\alpha k}=0+\frac{x^9}{\Gamma(\alpha+1)}x^{\alpha}+....$$

7 Conclusion

In this paper, we use shifted Chebyshev approximations with the second kind [25] and fractional power series method (FPSM) ([3], [8]) to solve the multi-order nonlinear fractional differential equations. The properties of the shifted Chebyshev polynomials for the second kind are used to reduce the nonlinear multi-order fractional differential equations to the solution of non-linear system of algebraic equations. The resulting system is solved by using Newton iteration method. The fractional derivative is considered in the Caputo sense. From the solutions obtained using the suggested method, we can conclude that these solutions are in excellent agreement with the already existing ones and show that this approach can be solve the problem effectively. Comparisons are made between approximate solutions and exact solutions and other methods to illustrate the validity and the great potential of the technique. All numerical results are obtained using Matlab 12b.

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