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Numerical Method For Variable-order Space Fractional Diffusion Equation and Applications

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Abstract

The purpose of this paper is to develop the explicit fractional order finite difference scheme for variable-order space fractional diffusion equation (VOSFDE). Furthermore, the stability and convergence of the scheme in a bounded domain are discussed. As an application of the scheme, we solve some test problems and their solutions are represented graphically by Mathematica software.

Keywords: Space fractional, diffusion equation, finite difference scheme, stability analysis, Mathematica.

2010 MSC: 46B40, 46B42, 47B60, 47B65.

1 Introduction

Now a days, the research on variable-order fractional partial differential equations is relatively new, and numerical approximation of these equations is still at an early stage of development. Lin, Liu, Anh, and Turner [9] established an equality between the variable-order RiemannLiouville fractional derivative and its GrunwaldLetnikovexpansion. Using this relationship, they defined and obtained some properties of the operator $(\frac{d^2}{dx^2})^{\alpha(x,t)}$ and devised an explicit finite difference approximation scheme for a corresponding variable-order nonlinear fractional di?usion equation. Ilic et al.[7] proposed a new matrix method for a fractional-in-space diffusion equation with homogeneous and nonhomogeneous boundary conditions on a bounded domain.

The number of scientific and engineering problems involving fractional calculus is already very large and still growing. One of the main advantages of the fractional calculus is that the fractional derivatives provide an excellent approach for the description of memory and hereditary properties of various materials and processes [6]. Many of the numerical methods using different kinds of fractional derivative operators for solving fractional partial differential equations have been proposed and are available in the literature. Anh and Leonenko presented a spectral representation of the mean-square solution of the fractional diffusion equation with random initial condition, from which the Caputo-Djrbashian regularied fractional derivative was adopted [1]. Odibat proposed two algorithms for numerical fractional integration and Caputo fractional differentiation. Using the new modification derive an algorithm to approximate fractional derivatives of arbitrary order for a given function by a weighted sum of function and its ordinary derivative values at specified points [11]. Blaszczyk focused on a numerical scheme applied for a fractional oscillator equation which includes a complex form of left- and right-sided fractional derivatives in a finite time interval [2]. Recently, more and more researchers are finding that a variety of important dynamical problems exhibit fractional order behavior that may vary with time or space. This fact indicates that variable order calculus is a natural candidate to provide an effective mathematical framework for the description of complex

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dynamical problems. The concept of a variable order operator is a much more recent development, which is a new paradigm in science. Samko and Ross directly generalized the Riemann-Liouvile and Marchaud fractional integration and differentiation of the case of variable order, and then showed some properties and an inversion formula. Lorenzo and Hartley suggested the concept of a variable order operator is allowed to vary either as a function of the independent variable of integration or differentiation (t), or as a function of some other (perhaps spatial) variable (x), they also explored more deeply the concept of variable order integration and differentiation and sought the relationship between the mathematical concepts and physical processes. Di?erent authors have proposed different definitions of variable order differential operators, each of these with a specific meaning to suit desired goals. Coimbra [4] took the Laplace-transform of Caputos definition of the fractional derivative as the starting point to suggest a novel definition for the variable order differential operator. Because of its meaningful physical interpretation, Coimbras definition is better suited for modeling physical problems. Ingman et al. [8] employed the time dependent variable order operator to model the viscoelastic deformation process. Pedro et al. studied the motion of particles suspended in a viscous fluid with drag force using variable order calculus. Sun et al. introduced a classification of variable-order fractional diffusion [2] models based on the possible physical origins that prompt the variable-order.

The variable order operator definitions recently proposed in the literature include the Riemann-Liouvile-definition, Caputo-definition, Marchaud-definition, Coimbra-definition and Grunwald-definition [4, 12]. However, to the best of the authors knowledge, detailed studies of the Grunwald-type variable order operator have not yet been performed. Samko et al. [24] compared the Riemann-Liouvile-definition and Marchaud-definition variable order operators, and noted the loss of certain properties of the Riemann-Liouvile definitions, with the Marchaud-definition being more suitable than the RiemannLiouvile-type. Ramirezetal, also compared the Riemann-Liouvile-definition, Caputo-definition, Marchaud-definition and Coimbra-definition variable order operators based on a very simple criteria: the variable order operator must return the correct fractional derivative that corresponds to the argument of the functional order. Ramirez et al. found that only the Marchaud-definition and Coimbra-definition satisfy the above elementary requirement, and the Coimbra-de?nition variable order operator is more efficient from the numerical standpoint. Soon et al. also showed that the Coimbra-definition variable order operator satisfies a mapping requirement, and it is the only definition that correctly describes position-dependent transitions between elastic and viscous regimes because it correctly returns the appropriate derivatives as a function of x(t). Ramirez, showed that the Coimbra definition is the most appropriate definition having fundamental characteristics that are desirable for physical modeling.
In our paper we consider the variable-order space fractional diffusion equation

$$\frac{\partial U(x,t)}{\partial t} = D\frac{\partial^{\alpha(x,t)}U(x,t)}{\partial x^{\alpha(x,t)}}; (x,t) \in \Omega = [0,L] \times [0,T],$$
$$1 < \alpha(x,t) \leq 2,\ t > 0$$

with initial and boundary conditions

$$\textit{initial condition}:\ U(x,0) = f(x),\ 0 \leq x \leq L$$

$$\textit{boundary conditions}:\ U(0,t) = U_L,\ U_(L,t) = U_R,\ t \geq 0$$

where $D > 0$ is the diffusivity coefficient.

We organize the paper as follows: In section 2 , we develop the explicit fractional order finite difference scheme for variable-order space fractional diffusion equation. The stability solution of VOSFDE by develop scheme is discussed in section 3 and in section 4 the convergence of the scheme is discussed upto the length. The numerical solution of variable-order space fractional diffusion equation is obtained using Mathematica software and it is represented graphically in the last section.

2 FINITE DIFFERENCE SCHEME

In this section, we develop the explicit fractional order finite difference scheme for variable-order space fractional diffusion equation. We consider the following variable-order space fractional diffusion equation

(VOSFDE) with initial and boundary conditions

$$\frac{\partial U(x,t)}{\partial t} = D\frac{\partial^{\alpha(x,t)}U(x,t)}{\partial x^{\alpha(x,t)}}; (x,t) \in \Omega = [0,L] \times [0,T], \qquad (2.1)$$

$$1 < \alpha(x,t) \leq 2,\ t > 0$$

$$\textit{initial condition}:\ U(x,0) = f(x),\ 0 \leq x \leq L \qquad (2.2)$$

$$\textit{boundary conditions}:\ U(0,t) = U_L,\ U_(L,t) = U_R,\ t \geq 0 \qquad (2.3)$$

where $D > 0$ is the diffusivity coefficient. Note that for $\alpha(x,t) = 2$, we recover the original diffusion equation.

$$\frac{\partial U(x,t)}{\partial t} = D\frac{\partial^2 U(x,t)}{\partial x^2};\ x\epsilon R;\ t \geq 0$$

Definition 2.1. *The Caputo space fractional variable-order derivative of order $\alpha(x)$, $(1 < \alpha(x) \leq 2)$ is denoted by ${}_0^C D_x^{\alpha(x)} U(x,t)$ and is defined as follows*

$$ {}_0^C D_x^{\alpha(x)} U(x,t) = \begin{cases} \frac{1}{\Gamma(n-\alpha(x))}\int_0^x \frac{1}{(x-\xi)^{\alpha(x)-n+1}}\frac{\partial^n U(\xi,t)}{\partial \xi^n}d\xi, & n < \alpha(x) < n+1 \\ \frac{\partial^2 U(x,t)}{\partial t^2}, & \alpha(x) = 2 \end{cases}$$

where $\Gamma(.)$ is a Gamma function. Adopting the discrete scheme given in [3], we discretise the variable order space fractional derivative as follows

$$\begin{aligned} \frac{\partial^{\alpha(x_j,t_n)}U(x_j,t_n)}{\partial x^{\alpha(x_j,t_n)}} &= \frac{1}{\Gamma(2-\alpha(x,t))}\int_0^x \frac{1}{(x-\xi)^{\alpha(x)-1}}\frac{\partial^2 U(\xi,t)}{\partial \xi^2}d\xi \\ &= \frac{1}{\Gamma(2-\alpha(x,t))}\sum_{k=0}^{j-1}\int_{kh}^{(k+1)h}\eta^{1-\alpha(x,t)}\frac{\partial^2 U(x-\eta,t)}{\partial \eta^2}d\eta \\ &\approx \frac{1}{\Gamma(2-\alpha(x,t))}\sum_{k=0}^{j-1}\frac{U(x-(k-1)h,t)-2U(x-kh,t)+U(x-(k+1)h)}{h^2}\times \\ &\qquad \int_{kh}^{(k+1)h}\eta^{1-\alpha(x,t)}d\eta \\ &= \frac{1}{\Gamma(2-\alpha(x,t))}\sum_{k=0}^{j-1}\frac{U(x-(k-1)h,t)-2U(x-kh,t)+U(x-(k+1)h)}{h^2} \\ &\qquad \times [(k+1)^{2-\alpha(x,t)} - k^{2-\alpha(x,t)}]h^{2-\alpha(x,t)} \\ &= \frac{h^{-\alpha(x,t)}}{\Gamma(3-\alpha(x,t))}\sum_{k=0}^{j-1}[U(x-(k-1)h,t)-2U(x-kh,t)+U(x-(k+1)h)] \\ &\qquad \times [(k+1)^{2-\alpha(x,t)} - k^{2-\alpha(x,t)}] \end{aligned} \qquad (2.4)$$

For the explicit numerical approximation scheme, we take equally spaced mesh of M points for the spatial domain $0 \leq x \leq L$, N constant time steps for the time direction domain $0 \leq t \leq T$ and denote the spatial grid points by

$$x_j = jh\ j = 1,2,...,M$$

and the time direction grid points by

$$t_n = n\tau\ n = 0,1,2,...,N,$$

where the grid spacing is $h = \frac{L}{M}$ in the spatial domain and $\tau = \frac{T}{N}$ in the temporal domain. Therefore, we have

$$U_j^n = U(x_j,t_n) = (j\Delta x, n\Delta t) = (jh, n\tau)\ j = 1,2,...,M-1,\ n = 0,1,2,...,N.$$

At the grid points $(x-j,t_n)$ equation (2.1) becomes

$$\frac{\partial U(x_j,t_n)}{\partial t} = {}_0^C D_x^{\alpha(x_j,t_n)} U(x_j,t_n) = \frac{\partial^{\alpha(x_j,t_n)}U(x_j,t_n)}{\partial x^{\alpha(x_j,t_n)}} \qquad (2.5)$$

We denote U_j^n for the numerical approximation to $U(x_j, t_n)$. From equations (2.4), (2.5) and using the forward difference to $\frac{\partial U(x,t)}{\partial t}$, we approximate the variable-order space fractional diffusion equation (2.1) as follows

$$\frac{U_j^{n+1} - U_j^n}{\tau} = \frac{Dh^{-\alpha_j^n}}{\Gamma(3-\alpha_j^n)} \sum_{k=0}^{j-1} [U_{j-k+1}^n - 2U_{j-k}^n + U_{j-k-1}^n][(k+1)^{2-\alpha_j^n} - k^{2-\alpha_j^n}] \tag{2.6}$$

For simplicity, we define

$$b_j^n = \frac{\tau Dh^{-\alpha_j^n}}{\Gamma(3-\alpha_j^n)} \; and \; g_k^n = (k+1)^{2-\alpha_j^n} - k^{2-\alpha_j^n}.$$

Then after simplification, from equation (2.6), we get

$$U_j^{n+1} = b_j^n U_{j+1}^n + (1-2b_j^n)U_j^n + b_j^n U_{j-1}^n + b_j^n \sum_{k=1}^{j-1} g_k^n [U_{j-k+1}^n - 2U_{j-k}^n + U_{j-k-1}^n] \tag{2.7}$$

where $b_j^n = \frac{\tau Dh^{-\alpha_j^n}}{\Gamma(3-\alpha_j^n)}$ *and* $g_k^n = (k+1)^{2-\alpha_j^n} - k^{2-\alpha_j^n}$, $j = 1, 2, ..., M-1$, $n = 0, 1, 2, ..., N$.
The initial condition is approximated as $U_j^0 = f(x_j)$, $j = 1, 2, ..., M-1$.
The boundary conditions are approximated as $U_0^n = U_L$ and $U_M^n = U_R$, $n = 0, 1, 2, ..., N$.
Therefore, the complete fractional approximated IBVP is

$$U_j^{n+1} = b_j^n U_{j+1}^n + (1-2b_j^n)U_j^n + b_j^n U_{j-1}^n + b_j^n \sum_{k=1}^{j-1} g_k^n [U_{j-k+1}^n - 2U_{j-k}^n + U_{j-k-1}^n] \tag{2.8}$$

$$initial\ condition: \; U_j^0 = f(x_j),\ j = 1, 2, ..., M-1 \tag{2.9}$$

$$boundary\ conditions: \; U_0^n = U_L,\ U_M^n = U_R,\ n = 0, 1, 2, ..., N. \tag{2.10}$$

where

$$b_j^n = \frac{\tau Dh^{-\alpha_j^n}}{\Gamma(3-\alpha_j^n)} \; and \; g_k^n = (k+1)^{2-\alpha_j^n} - k^{2-\alpha_j^n},\ j = 1, 2, ..., M,\ n = 0, 1, 2, ..., N.$$

Therefore, the fractional approximated IBVP (2.8) − (2.10) can be written in the following matrix equation form

$$U^{n+1} = AU^n + B \tag{2.11}$$

where $U^n = [U_1^n, U_2^n, ..., U_{M-1}^n]^T$, $B = [b_1^n U_0^n, b_2^n g_1^n U_0^n, ..., b_{M-1}^n g_{M-1}^n U_0^n]^T$ and
$A = (a_{ij})$ is an $(M-1) \times (M-1)$ matrix given by

$$a_{ij} = \begin{cases} 0, & when\ j \geq i+2 \\ b_i^n, & when\ j = i+1 \\ 1-(2-g_1^n)b_i^n, & when\ j = i = 2, 3, ..., M-1 \\ b_i^n(1-2g_1^n+g_2^n), & when\ j = i-1\ i \geq 3, 4, ..., M-1 \\ b_i^n(g_{i-j-1}^n - 2g_{i-j}^n + g_{i-j+1}^n), & when\ j \leq i-2\ i \geq 4, 5, ..., M-1 \end{cases}$$

while $a_{11} = 1 - 2b_1^n$, $a_{21} = b_2^n(1-2g_1^n)$ *and* $a_{i1} = b_i^n(g_{i-2}^n - 2g_{i-1}^n + g_{i-j+1}^n)$, $3 \leq i \leq M-1$. The above system of algebraic equations is solved by using Mathematica software in section 5.
In the next section, we discuss the stability of the solution of fractional order explicit finite difference scheme (2.8) − (2.10) developed for the time fractional diffusion equation (2.1) − (2.3).

3 STABILITY

Theorem 3.1. *The solution of the explicit type fractional order finite difference scheme* (2.8) − (2.10) *for the variable-order time fractional diffusion equation* (2.1) − (2.3) *is conditionally stable.*

Proof: We consider the equation (2.11)

$$U^{n+1} = AU^n + B. \tag{3.12}$$

Now B includes a column matrix of known values and known source term values. By Gerschgorin's first theorem [1], let λ_i be an eigenvalue of matrix A to linear system of equations (2.8) and x be the corresponding eigenvector then $Ax = \lambda x$. Choose i such that

$$|x_i| = max.\{|x_j|, j = 1, 2, ..., M-1\}$$

then

$$\sum_{j=1}^{N-1} a_{ij} x_j = \lambda x_i$$

and therefore,

$$\lambda = a_{ii} + \sum_{j=1, j\neq 1}^{N-1} a_{ij} \frac{x_j}{x_i} \tag{3.13}$$

We substitute the values of a_{ij} in equation (3.13), we get
(i) When $i = 1$

$$\begin{aligned}
\lambda &= 1 - 2b_1^n + b_1^n \frac{x_2}{x_1} \\
&\leq 1 - 2b_1^n + b_1^n = 1 - b_1^n \leq 1 \\
\lambda &\geq 1 - 2b_1^n - b_1^n = 1 - 3b_1^n \geq -1 \ when \ b_1^n \leq \frac{2}{3} \\
&\therefore 1 \leq \lambda \leq 1 \\
&\Rightarrow |\lambda| \leq 1 \ when \ b_1^n \leq \frac{2}{3}
\end{aligned}$$

(ii) When $i = 2$

$$\begin{aligned}
\lambda &= a_{22} + a_{21} \frac{x_1}{x_2} + a_{23} \frac{x_3}{x_2} \\
&= 1 - (2 - g_1^n) b_2^n + b_2^n (1 - 2g_1^n) \frac{x_1}{x_2} + b_2^n \frac{x_3}{x_2} \\
&\leq 1 - (2 - g_1^n) b_2^n + b_2^n (1 - 2g_1^n) + b_2^n \\
&\leq 1 - b_2^n g_1^n \leq 1 \\
\lambda &\geq 1 - (2 - g_1^n) b_2^n - b_2^n (1 - 2g_1^n) - b_2^n \geq 1 - b_2^n (4 - 3g_1^n) \geq -1 \\
&\qquad when \ b_2^n \leq \frac{2}{(4 - 3g_1^n)} \\
&\therefore 1 \leq \lambda \leq 1 \\
&\Rightarrow |\lambda| \leq 1 \ when \ b_2^n \leq \frac{2}{(4 - 3g_1^n)}
\end{aligned}$$

(iii) When $3 \le i \le M-1$

$$\begin{aligned}
\lambda &= 1 - b_i^n(2-g_1^n) + b_i^n(g_{i-2}^n - 2g_{i-1}^n)\frac{x_{i-2}}{x_i} \\
&\qquad + b_i^n \sum_{j=2}^{i-1}(g_{i-j-1}^n - 2g_{i-j}^n + g_{i-j+1}^n)\frac{x_{i-1}}{x_i} + b_i^n \frac{x_{i+1}}{x_i} \\
&\le 1 - b_i^n(2-g_1^n) + b_i^n(g_{i-2}^n - 2g_{i-1}^n) + b_i^n(g_{i-1}^n - g_{i-2}^n + g_0^n - g_1^n) + b_i^n \\
&\le 1 - b_i^n + b_i^n g_0^n \le 1 \\
\lambda &\ge 1 - b_i^n(2-g_1^n) - b_i^n(g_{i-2}^n - 2g_{i-1}^n) - b_i^n(g_{i-1}^n - g_{i-2}^n + g_0^n - g_1^n) - b_i^n \\
&\ge 1 - 4b_i^n + 2b_i^n g_1^n = 1 - 2b_i^n(2-g_1^n) \ge -1 \ when \ b_i^n \le \frac{1}{(2-g_1^n)} \\
&\therefore\ 1 \le \lambda \le 1 \\
&\Rightarrow\ |\lambda| \le 1 \ when \ b_i^n \le \frac{1}{(2-g_1^n)}.
\end{aligned}$$

Therefore, from $(i)-(iii)$, we prove

$$Max_{(3\le i\le M-1)}\{b_1^n, b_2^n, b_i^n\} \le \frac{1}{2}$$

then the spectral radius $\rho(A)$ of matrix A satisfies $\rho(A) \le 1$.
If

$$Max_{(3\le i\le M-1)}\{b_1^n, b_2^n, b_i^n\} \le \frac{1}{2},$$

where

$$b_1^n \le \frac{2}{3},\ b_2^n \le \frac{2}{(4-3g_1^n)}\ \ b_i^n \le \frac{1}{(2-g_1^n)}$$

then there exist a positive number $\epsilon \le C\tau$ such that

$$\|A\|_m \le \rho(A) + C\tau \le 1 + O(\tau) \le 1 + \epsilon.$$

Hence, this prove that the fractional order finite difference scheme is conditionally stable.
The next section is devoted for convergence of the finite difference scheme.

4 CONVERGENCE

Consider the another vector

$$\bar{U}^n = [U(x_0,t_n), U(x_1,t_n), \cdots, U(x_M,t_n)]^T$$

which represents the exact solution at time level t_n. The finite difference equation (2.11) becomes

$$\bar{U}^{n+1} = A\bar{U}^n + B + \tau^n \tag{4.14}$$

where τ^n is the error vector of the truncation error at time level t_n.

Theorem 4.2. *Suppose that the continuous problem* (2.1) − (2.3) *has a smooth solution* $U(x,t) \in C_{x,t}^{1+\bar{\alpha},2}$. *Let* U_j^n *be the numerical solution computed by* (2.8) − (2.10). *If* $\xi \le \frac{1}{2}$ *where* $\xi = \min_{(3\le i\le M-1)}\{\ b_1^n, b_2^n, b_i^n\ \}$, *then the fractional order explicit finite difference scheme* (2.8) − (2.10) *for IBVP* (2.1) − (2.3) *is convergent.*

Proof: Let Ω be the region $0 \le x \le L$ and $0 \le t \le T$. Take $U_j^n = U(x_j,t_n) = (j\Delta x, n\Delta t) = (jh, n\tau)$, $j = 1,2,...,M-1,\ n = 0,1,2,...,N$. with $h = \frac{L}{M}$ and $\tau = \frac{T}{N}$.
We introduced another vector

$$\bar{U}^n = [U(x_0,t_n), U(x_1,t_n), \cdots, U(x_M,t_n)]^T$$

satisfying the finite difference equation (2.11), we get

$$\bar{U}^{n+1} = A\bar{U}^n + B + \tau^n \tag{4.15}$$

where τ^n is the error vector of the truncation error at time level t_n. Now we subtract (2.11) from (4.15), we have

$$(\bar{U}^{n+1} - U^{n+1}) = A(\bar{U}^n - U^n) + \tau^n \tag{4.16}$$

where

$$U^n = [u_1, u_2, \cdots, u_M]^T \ \bar{U}^n[\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_N]^T$$

and set

$$E^n = \bar{U}^n - U^n.$$

Then from equation (4.16), we get

$$E^{n+1} = AE^n + \tau^n \tag{4.17}$$

Clearly, E^n satisfies (2.11), and if $\xi \leq \frac{1}{2}$ then from equation (4.17), we have

$$\begin{aligned}
\|E^1\|_m &= \|AE^0\|_m + \|\tau^1\|_m \\
&\leq \|A\|_m \|E^0\|_m + \|\tau^1\|_m \\
&\leq (1+\epsilon)\|E^0\|_m + \|\tau^1\|_m \\
&\leq K\|E^0\|_m + \|\tau^1\|_m \qquad (K = (1+\epsilon))
\end{aligned}$$

Suppose that

$$\|E^n\|_m \leq K\|E^n\|_m + \|\tau^n\|_m$$

Therefore,

$$\begin{aligned}
\|E^{n+1}\|_m &= \|AE^n\|_m + \|\tau^n\|_m \\
&\leq \|A\|_m \|E^n\|_m + \|\tau^n\|_m \\
&\leq (1+\epsilon)\|E^0\|_m + \|\tau^n\|_m \\
&\leq K\|E^0\|_m + \|\tau^n\|_m
\end{aligned}$$

Hence, by induction we prove that

$$\|E^n\|_m \leq K\|E^0\|_m + \|\tau^n\|_m \ for \ all \ n.$$

Since,

$$\lim_{(h,\tau)\to(0,0)} \|\tau^n\|_m = 0, \ (1 \leq M \leq n),$$

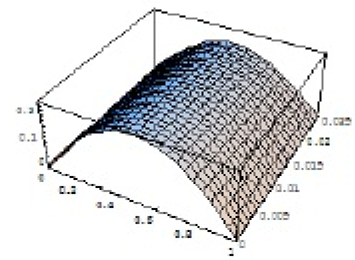
implies that $\|E^n\|_m$tends to zero uniformly in Ω as $(h,\tau) \to (0,0)$.
This show that the scheme is conditionally convergent.
Hence, the proof of the theorem is completed.

5 NUMERICAL SOLUTIONS

In this section, we obtain the approximated solution of variable-order space fractional diffusion equation (VOSFDE) with initial and boundary conditions. To obtain the numerical solution of the variable-order space fractional diffusion equation (VOSFDE) by the finite difference scheme, it is important to use some analytical model. Therefore, we present an example to demonstrate that VOSFDE can be applied to simulate behavior of a fractional diffusion equation by using Mathematica Software.
We consider the following one-dimensional variable-order space fractional diffusion equation (VOSFDE) with initial and boundary conditions

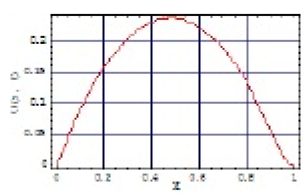
$$\begin{aligned}
\frac{\partial U(x,t)}{\partial t} &= \frac{\partial^{\alpha(x,t)} U(x,t)}{\partial x^{\alpha(x,t)}} \, 0 < x < 1, \, 1 < \alpha(x,t) \leq 2, \ t > 0 \\
\text{initial condition} : U(x,0) &= x(1-x), \ 0 \leq x \leq 1 \\
\text{boundary conditions} : U(0,t) &= 0 = U(x,t), \, t \geq 0
\end{aligned}$$

where $\alpha(x,t) = \frac{2+sin(xt)}{4}$ The numerical solutions are obtained at $t = 0.025$ by considering the parameters $\tau = 0.0025$, $h = 0.1$, is simulated in the following figure.



$Fig.1.1$: The exact solution of variable-order diffusion equation, $t = 0.025$.

$Fig.1.1$: The e



$Fig.1.2$: The approximate solution of variable-order diffusion

equation with $h = 0.1$, $\tau = 0.0025$, $t = 0.025$.

$Fig.1.2$: The approximate solution of variable-order diffusion

equation with $h = 0.1$, $\tau = 0.0025$, $t = 0.025$.

CONCLUSIONS

In the present paper, we develop a new numerical scheme for the variable order space fractional diffusion equation with the Caputo variable order space fractional operator. The proposed explicit difference approximation for space fractional variable-order diffusion equation can be reliably applied to solve any fractional order dynamical systems and controllers, minding the conditions for stability and convergence of the scheme. The numerical results are also compatible with theoretical analysis, hence showing the numerical stability of the finite difference scheme.

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