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A Note on connectedness in a bispace

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Abstract

Here we have studied the idea of connectedness and totally disconnectedness in a bispace and some of its basic properties. Also it has been investigated how far several results as valid in a bitopological space are affected in a bispace.

Keywords: σ-space, bispace, connectedness, total disconnectedness.

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1 Introduction

J.C Kelly [4] introduced the notion of a bitopological space $(X, \mathcal{P}, \mathcal{Q})$ to be a set X with two topologies $\mathcal{P}$ and $\mathcal{Q}$ defined on it. It is with such spaces that some of the symmetric properties can be recovered when treating unsymmetric metrics(quasi-metrics). Other authors have expanded his results by considering analogous notions for uniform spaces. The notion of a σ- space(or simply space) was introduced by A.D Alexandroff [1] in 1940 generalizing the notion of a topological space where only countable union of open sets were taken to be open. The notion of a bispace was introduced by Lahiri and Das [7] generalizing the idea of a bitopological space in 2001. The concept of connectedness in a bitopological space was introduced by Pervin [8]. Pervin also introduced the idea of total disconnectedness in a bitopological space.The systematic study of bitopological spaces was begun by J.C Kelly [4], who introduced various separation properties into bitopological spaces and obtained generalizations of some important classical results and various other authors have contributed to the development of the theory. Here we have introduced the notion of connectedness in a bispace and studied some of its basic properties. Also we have studied some properties of totally disconnectedness in a bispace.

2 Preliminary

Definition 2.1.[1] A set X is called an Alexandroff space or σ- space(or simply space) if it is chosen a system $\mathcal{F}$ of subsets of X, satisfying the following axioms
(i)The intersection of countable number sets in $\mathcal{F}$ is a set in $\mathcal{F}$.
(ii)The union of finite number of sets from $\mathcal{F}$ is a set in $\mathcal{F}$.
(iii)The void set and X is a set in $\mathcal{F}$.
Sets of $\mathcal{F}$ are called closed sets. There complementary sets are called open.It is clear that instead of closed sets in the definition of a space, one may put open sets with subject to the conditions of countable summability, finite intersectability and the condition that X and the void set should be open.
The collection of such open sets will sometimes be denoted by $\mathcal{P}$ and the σ-space by $(X, \mathcal{P})$. It can be noted that $\mathcal{P}$ is not a topology in general as can be seen by taking $X = \mathbb{R}$, the set of real numbers and $\mathcal{P}$ as the collection of all F_σ sets in $\mathbb{R}$.

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Definition 2.2.[1] To every set M we correlate its closure $\overline{M}$= the intersection of all closed sets containing M. Generally the closure in a space is not a closed set. We denote the closure of a set M in a space $(X, \mathcal{P})$ by $\mathcal{P}$-cl(M) or cl(M) or simply $\overline{M}$ when there is no confusion about $\mathcal{P}$.

The idea of limit points, derived set, interior of a set etc. in a σ-space are similar as in the case of a topological space which have been thoroughly discussed in [6].

Definition 2.3. [2] Let $(X, \mathcal{P})$ be a space. A family of open sets B is said to form a base (open) for $\mathcal{P}$ if and only if every open set can be expressed as countable union of members of B.
Theorem 2.1. [2] A collection of subsets B of a set X forms an open base of a suitable space structure $\mathcal{P}$ of X if and only if
1) the null set $\phi \in B$
2) X is the countable union of some sets belonging to B.
3) intersection of any two sets belonging to B is expressible as countable union of some sets belonging to B.
Definition 2.4. [7] Let X be a non-empty set. If $\mathcal{P}$ and $\mathcal{Q}$ be two collection of subsets of X such that $(X, \mathcal{P})$ and $(X, \mathcal{Q})$ are two spaces, then X is called a bispace.
Definition 2.5. [7] A bispace $(X, \mathcal{P}, \mathcal{Q})$ is called pairwise T_1 if for any two distinct points x, y of X, there exist $U \in \mathcal{P}$ and $V \in \mathcal{Q}$ such that $x \in U, y \notin U$ and $y \in V, x \notin V$.
Definition 2.6. [7] A bispace $(X, \mathcal{P}, \mathcal{Q})$ is called pairwise Hausdorff if for any two distinct points x, y of X, there exist $U \in \mathcal{P}$ and $V \in \mathcal{Q}$ such that $x \in U, y \in V, U \cap V = \phi$.

3 Connectedness and Total Disconnectedness

The following two results are very useful in an arbitrary space $(X, \mathcal{P})$. For the sake of convenience we give the details proof also.
Theorem 3.1. In a σ-space $(X, \mathcal{P})$ if $A \subset X$ and $x \in \overline{A}$ then every open set U containing x intersects A.
Proof. Let $x \in \overline{A}$. If possible let there there exists an open set U containing x which does not intersect A. Then it implies that $A \subset X \setminus U$, where $X \setminus U$ is a closed set. Now $x \notin X \setminus U$ and $X \setminus U$ is a closed set containing A. This contradicts to the fact that $x \in \overline{A}$. ■
Theorem 3.2. In a σ-space $(X, \mathcal{P})$ if $A \subset X$ then $\overline{A} = A \cup A'$.
Proof. First suppose $x \in \overline{A}$. Then every open set containing x intersects A. Now if $x \notin A$, then for every open set U containing x we have, $(A \setminus \{x\}) \cap U = A \cap U \neq \emptyset$. So x becomes a limit point of A. Therefore $x \in A'$ and $x \in A \cup A'$. Hence $\overline{A} \subset A \cup A'$.
Conversely, let $y \in A \cup A'$. Now if $y \in A$ then $y \in \overline{A}$. If $y \in A'$ then if it happens that $y \notin \overline{A}$, there exists a closed set M containing A such that $y \notin M$ i.e., $y \in X \setminus M$. Now $X \setminus M$ is an open set containing y which does not intersect A and this leads to a contradiction to the fact that $y \in A'$. Therefore we must have $y \in \overline{A}$. Hence $A \cup A' \subset \overline{A}$. Thus $\overline{A} = A \cup A'$. ■

Throughout our discussion $(X, \mathcal{P}, \mathcal{Q})$ or simply X stands for a bispace, $\mathbb{R}$ stands for the set of real numbers, $\mathbb{Q}$ for the set of all rational numbers and $\mathbb{N}$ for the set of all natural numbers and sets are always subsets of X unless otherwise stated.
Definition3.1. (cf. [8]) Two non empty subsets A and B in $(X, \mathcal{P}, \mathcal{Q})$ are said to be pairwise separated if there exists a $\mathcal{P}$-open set U and a $\mathcal{Q}$-open set V such that $A \subset U$ and $B \subset V$ and $A \cap V = B \cap U = \phi$ or there exists a $\mathcal{Q}$-open set U and a $\mathcal{P}$-open set V such that $A \subset U$ and $B \subset V$ and $A \cap V = B \cap U = \phi$.
For the sake of convenience we use only the first condition.
Definition3.2. (cf. [8]) A bispace $(X, \mathcal{P}, \mathcal{Q})$ is said to be connected if and only if X can not be express as the union of two non empty separated sets.

The following theorem is a characterization of connectedness in term of open sets. Though the proof is straightforward we give the details proof for the sake of completeness.
Theorem 3.3. The following statements are equivalent for any bispace $(X, \mathcal{P}, \mathcal{Q})$:
(i) $(X, \mathcal{P}, \mathcal{Q})$ is connected.
(ii) X can not be expressed as the union of two nonempty disjoint sets A and B such that A is $\mathcal{P}$ -open and B

is $\mathcal{Q}$ -open.
(iii) X contains no nonempty proper subset which is both $\mathcal{P}$ -open and $\mathcal{Q}$ -closed (or $\mathcal{P}$ -closed and $\mathcal{Q}$ -open).
Proof. **(i) $\Rightarrow$ (ii)** First suppose that $(X, \mathcal{P}, \mathcal{Q})$ is connected. Now if possible let X can be expressed as a union of two nonempty disjoint sets A and B such that A is $\mathcal{P}$-open B is $\mathcal{Q}$-open. Therefore $X = A \cup B$, A is $\mathcal{P}$-open B is $\mathcal{Q}$-open. Then clearly X can be expressed as a union of two non-empty disjoint separated sets A and B. This is a contradiction to the fact that X is connected. Therefore X cannot be expressed as a union of two non-empty disjoint sets A and B such that A is $\mathcal{P}$-open and B is $\mathcal{Q}$-open.

(ii) $\Rightarrow$ (i) Let X can not be expressed as the union of two nonempty disjoint sets A and B such that A is $\mathcal{P}$ -open and B is $\mathcal{Q}$ -open. If possible let X be not connected then X can be expressed as the union of two non empty separated sets P and Q. So there exists $\mathcal{P}$ -open set U and $\mathcal{Q}$ -open set V such that $P \subset U$ and $Q \subset V$ and $P \cap V = Q \cap U = \phi$. If possible let $y \in U \cap V$. Then $y \in X = P \cup Q$. Now if $y \in P$ then $y \in P \cap V$, a contradiction. If $y \in Q$ we will arrive at a similar kind of contradiction. Hence $U \cap V = \phi$. Therefore X can be expressed as the union of two non empty disjoint sets U and V where U is $\mathcal{P}$ -open and V is $\mathcal{Q}$ -open, which is a contradiction. Hence X must be connected.

(ii) $\Rightarrow$ (iii) Let (ii) holds, let X contains a nonempty proper subset M which is both $\mathcal{P}$ -open and $\mathcal{Q}$ -closed. Now $X = M \cup (X \setminus M)$, where M is $\mathcal{P}$ -open and $(X \setminus M)$ is $\mathcal{Q}$ -open, which is a contradiction. Therefore X contains no nonempty proper subset proper subset which is both $\mathcal{P}$ -open and $\mathcal{Q}$ -closed.

(iii) $\Rightarrow$ (ii) Let X contains no non-empty proper subset which is both $\mathcal{P}$-open and $\mathcal{Q}$-closed. Now if possible, let X can be expressed as the union of two non-empty disjoint sets A and B, where A is $\mathcal{P}$-open and B is $\mathcal{Q}$-open. Since $X = A \cup B$ therefore $A = X \setminus B$. Since B is $\mathcal{Q}$-open, $X \setminus B$ is $\mathcal{Q}$-closed. Therefore A becomes both $\mathcal{P}$-open and $\mathcal{Q}$-closed which is a contradiction. Hence **(ii)** holds. ■

Note 3.1 We see that $(X, \mathcal{P}, \mathcal{Q})$ is connected iff X can not be expressed as the union of two non empty disjoint sets A and B such that A is $\mathcal{P}$ -open and B is $\mathcal{Q}$ -open. When X can be so expressed we write $X = A|B$ and call this separation (or disconnection) of X. ■

We see that if $(X, \mathcal{P})$ is a σ-space and if $A \subset X$ then as in the case of topological spaces the collection $\mathcal{P}_A = \{U \cap A : U \in \mathcal{P}\}$ forms a σ-space structure on A and $(A, \mathcal{P}_A)$ is called a σ-subspace of $(X, \mathcal{P})$.

Definition 3.3.(cf. [8]) A function f mapping a bispace $(X, \mathcal{P}, \mathcal{Q})$ into a bispace $(X, \mathcal{P}^*, \mathcal{Q}^*)$ is said to be continuous if and only if induced mappings $f_1 : (X, \mathcal{P}) \to (X, \mathcal{P}^*)$ and $f_2 : (X, \mathcal{Q}) \to (X, \mathcal{Q}^*)$ are continuous.

We shall denote the left-hand and right-hand topologies on $\mathbb{R}$ (having bases (∞, x) and (y, ∞) respectively) by $\mathcal{L}$ and $\mathcal{R}$. As every bitopological space is a bispace we may treat $(\mathbb{R}, \mathcal{L}, \mathcal{R})$ as a bispace also.

Theorem 3.4. A bispace $(X, \mathcal{P}, \mathcal{Q})$ is connected if and only if every continuous mapping of $(X, \mathcal{P}, \mathcal{Q})$ into $(D, \mathcal{L}_D, \mathcal{R}_D)$ is constant, where $D = \{0, 1\}$.

Proof. Let $(X, \mathcal{P}, \mathcal{Q})$ be connected. Now if such a non-constant continuous mapping f exists, then $X = f^{-1}\{0\}|f^{-1}\{1\}$ is a separation of X which is a contradiction to the fact that X is connected.
Conversely, if $(X, \mathcal{P}, \mathcal{Q})$ is not connected then let $X = A|B$ be a separation of X. So we may define a non-constant continuous function f such that $f(A) = \{0\}$ and $f(B) = \{1\}$ which is a contradiction. Hence $(X, \mathcal{P}, \mathcal{Q})$ must be connected. ■

Theorem 3.5. A bispace $(X, \mathcal{P}, \mathcal{Q})$ is connected if and only if every continuous mapping of $(X, \mathcal{P}, \mathcal{Q})$ into $(\mathbb{R}, \mathcal{L}, \mathcal{R})$ has the Durbox property (i.e,its range is an interval).

Proof. Let us suppose every such continuous function has Durbox's property. Now if X is not connected then let $X = A|B$ is a separation of X. Then the continuous function f defined by $f(A) = \{0\}$ and $f(B) = \{1\}$ does not possess the Durbox's property, which is a contradiction.
Conversely, let $(X, \mathcal{P}, \mathcal{Q})$ be connected. Now if f is continuous but does not possess the Durbox's property, then there exists a point c in (a, b) such that $\{a, b\} \subset f(X)$ but $c \notin f(X)$. Then obviously $f^{-1}((-\infty, c))|f^{-1}((c, \infty))$ is a separation of X, which is a contradiction. ■

Theorem 3.6. The continuous image of a connected set is connected.

Proof. The proof is straightforward and so is omitted. ■

Definition 3.4. (cf. [8]) A subset E in a bispace $(X, \mathcal{P}, \mathcal{Q})$ is called connected if and only if $(E, \mathcal{P}_E, \mathcal{Q}_E)$ is connected and by component we mean maximal connected set in the bispace.

Theorem 3.7. If C is connected subset of a bispace $(X, \mathcal{P}, \mathcal{Q})$ and $X = A \cup B$ where A and B are non empty pairwise separated sets, then either $C \subset A$ or $C \subset B$.

Proof. We have $X = A \cup B$, where A and B are non empty pairwise separated sets. So there exists $\mathcal{P}$-open set U and $\mathcal{Q}$-open set V such that $A \subset U$ and $B \subset V$ and $A \cap V = \emptyset = B \cap U$. Now if neither $C \subset A$ nor $C \subset B$ is true then $C \cap A \neq \emptyset$ and $C \cap B \neq \emptyset$. Now $C = (C \cap A) \cup (C \cap B)$. Again $C \cap A \subset C \cap U$, where $C \cap U$ is $\mathcal{P}_C$-open and $C \cap B \subset C \cap V$, where $C \cap V$ is $\mathcal{Q}_C$-open. We also have $(C \cap A) \cap (C \cap V) = \emptyset = (C \cap B) \cap (C \cap U)$. So we arrive at a contradiction, because C is connected. Therefore either $C \subset A$ or $C \subset B$ holds. ■

From the above result it immediately follows the corollaries.

Corollary 3.1. If every two points of a set E are contained in some connected subset of E, then E is connected.

Corollary 3.2. The union of any family of connected sets having a nonempty intersection is a connected set.

Theorem 3.8. If C is a connected set and $C \subset E \subset \mathcal{P}\text{-}cl(C) \cap \mathcal{Q}\text{-}cl(C)$, then E is is connected.

Proof. If $E = C$ then E is connected. If $E \neq C$ then if possible let E be not connected. Then there exists a $\mathcal{P}_E$-open set $U \neq \emptyset$ and $\mathcal{Q}_E$-open set $V \neq \emptyset$ such that $E = U \cup V$, $U \cap V = \emptyset$. Since C is connected, either $C \subset U$ or $C \subset V$. If $C \subset U$ then there exists a point $x \in E$ such that $x \in V$ and $x \notin U$. Hence $x \notin \mathcal{Q}_E\text{-}cl(C)$. Therefore $x \notin \mathcal{Q}\text{-}cl(C)$. Hence we have $x \notin \mathcal{P}\text{-}cl(C) \cap \mathcal{Q}\text{-}cl(C)$, which is a contradiction as $E \subset \mathcal{P}\text{-}cl(C) \cap \mathcal{Q}\text{-}cl(C)$. If $C \subset V$ then we will arrive at a similar contradiction. ■

Theorem 3.9. If C be a component in a bispace $(X, \mathcal{P}, \mathcal{Q})$, then $C = \mathcal{P}\text{-}cl(C) \cap \mathcal{Q}\text{-}cl(C)$.

Proof. Let C be a component and suppose that $p \notin C$. Then $C \cup \{p\}$ is not connected and let A, B be two non empty disjoint sets such that $C \cup \{p\} = A \cup B$, where A and B are open in $(C \cup \{p\}, \mathcal{P}_{C \cup \{p\}})$ and $(C \cup \{p\}, \mathcal{Q}_{C \cup \{p\}})$ respectively. Now since C is component, either $C \subset A$ or $C \subset B$. Hence either $\{p\} \subset A$ or $\{p\} \subset B$. So we see that $\{p\}$ is either $\mathcal{P}_{C \cup \{p\}}$-open or $\mathcal{Q}_{C \cup \{p\}}$-open. Hence either $p \notin \mathcal{P}_{C \cup \{p\}}\text{-}cl(C)$ or $p \notin \mathcal{Q}_{C \cup \{p\}}\text{-}cl(C)$. Therefore either $p \notin \mathcal{P}\text{-}cl(C)$ or $p \notin \mathcal{Q}\text{-}cl(C)$. Therefore $p \notin \mathcal{P}\text{-}cl(C) \cap \mathcal{Q}\text{-}cl(C)$. Hence we have $\mathcal{P}\text{-}cl(C) \cap \mathcal{Q}\text{-}cl(C) \subset C$. Clearly $C \subset \mathcal{P}\text{-}cl(C) \cap \mathcal{Q}\text{-}cl(C)$. ■

Definition 3.5. (cf. [11]) A bispace $(X, \mathcal{P}, \mathcal{Q})$ is said to be totally disconnected if for any two distinct points x and y there exists a disconnection $X = A|B$ with $x \in A$ and $y \in B$, where A is $\mathcal{P}$ open and B is $\mathcal{Q}$ open.

Definition 3.6. (cf. [11]) A bispace $(X, \mathcal{P}, \mathcal{Q})$ is said to be weakly totally disconnected if for any two distinct points x and y there exists a disconnection $X = A|B$, where A is $\mathcal{P}$ open and B is $\mathcal{Q}$ open such that one point belongs to A and the other point belongs to B. In this case the roles of the points need not be interchangeable.

Example3.1. Let $X = \mathbb{R}$, the set of real numbers. We now consider the collections $\mathcal{P}$ and $\mathcal{Q}$ of subsets of X as follows.

$\mathcal{P} = \{\phi, X\} \cup \{$sub sets of X whose complement is finite$\}$

$\mathcal{Q} = \{\phi, X\} \cup \{$countable sub sets of X $\}$

It is easy to examine that $(X, \mathcal{P}, \mathcal{Q})$ is a bispace. If x and y be any two distinct points in X then $\{x\}$ is a $\mathcal{Q}$-open set containing x and $(X \setminus \{x\})$ is a $\mathcal{P}$-open set containing y with $X = \{x\} \cup (X \setminus \{x\})$ and $\{x\} \cap (X \setminus \{x\}) = \phi$. Again $(X \setminus \{y\})$ is a $\mathcal{P}$-open set containing x and $\{y\}$ is a $\mathcal{Q}$-open set containing y with $X = (X \setminus \{y\}) \cup \{y\}$ and $\{y\} \cap (X \setminus \{y\}) = \phi$.

Hence $(X, \mathcal{P}, \mathcal{Q})$ is totally disconnected bispace. ■

We now give an example of a weakly totally disconnected bispace which is not totally disconnected.

Example3.2. Let $X = \mathbb{R}$, the set of real numbers. We now consider the collections $\mathcal{P}$ and $\mathcal{Q}$ of subsets of X as follows.

$\mathcal{P} = \{\phi, X\} \cup \{$sub sets of X whose complement is finite$\}$

$\mathcal{Q} = \{\phi, X\} \cup \{$countable sub sets of X not containing 2$\}$

It is easy to examine that $(X, \mathcal{P}, \mathcal{Q})$ is a weakly totally disconnected bispace. Let p be an arbitrary point in X such that $p \neq 2$. Then $\{p\} \in \mathcal{Q}$ containing p and $(X \setminus \{p\}) \in \mathcal{P}$ containing 2, such that $\{p\} \cup (X \setminus \{p\}) = X$, $\{p\} \cap (X \setminus \{p\}) = \phi$.

But as there does not exists any $\mathcal{Q}$-open set B other then X which contains 2, X can not be totally disconnected. ■

Theorem 3.10. The component of a weakly totally disconnected bispace are its points.

Proof. The proof is parallel as in the case of a bitopological space [11] and so is omitted. ■

Definition 3.7. (cf.[11]) A bispace $(X, \mathcal{P}, \mathcal{Q})$ is pairwise weakly Hausdorff if for any two distinct points x, y of X, there exist $U \in \mathcal{P}$ and $V \in \mathcal{Q}$ such that one contains x and the other contains y and $U \cap V = \phi$. The roles of the points need not be interchangeable.

Theorem 3.11. Let $(X, \mathcal{P}, \mathcal{Q})$ be a pairwise weakly Hausdorff bispace. If $\mathcal{P}$ has a base whose sets are $\mathcal{Q}$ closed or $\mathcal{Q}$ has a base whose sets are $\mathcal{P}$ closed, then $(X, \mathcal{P}, \mathcal{Q})$ is weakly totally disconnected.

Proof. The proof is parallel to the proof of theorem 2.5 [11] and so is omitted. ■

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