Malaya J. Mat. 5(1)(2017) 109–114

Malaya Journal of Matematik

MJM an international journal of mathematical sciences with computer applications...

www.malayajournal.org

ISSN : 2319-3786

Further nonlinear integral inequalities in two independent variables on time scales and their applications

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Abstract

Using ideas from [15], some nonlinear integral inequalities on time scales in two independent variables are established. Also, some examples are presented to show the feasibility of these results.

Keywords: Dynamic equations, time scale, integral inequality.

2010 MSC: 26D15, 26D20, 39A12.

1 Introduction

During the few years, a lot of research related to studies and the extension of some fundamental integral inequalities used in the theory of differential and integral equations on time scales. For example, we refer the reader to the papers [1-5, 8-19]. The purpose of this note is to illustrate some time scale Pachpatte-type inequalities by extending some continuous inequalities given in [15]. Inequalities of this form have in particular dominated the study of certain classes of integral equations on time scales. Throughout this work a knowledge and understanding of time scales notation is assumed; for an excellent bibliography to the time scales, see monographs of M. Bohner [6, 7] for a general review.

2 Preliminaries on time scales

In this section, we begin by giving some necessary materials for our study.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$ where $\mathbb{R}$ is the set of real numbers. The forward jump operator σ on $\mathbb{T}$ is defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \in \mathbb{T}$ for all $t \in \mathbb{T}$, C_{rd} denotes the set of rd-continuous functions and the set $\mathbb{T}^k$ which is derived from the time scale $\mathbb{T}$ as follows: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}^k = \mathbb{T}$.

Throughout this paper, we always assume that $\mathbb{T}_1$and $\mathbb{T}_2$ are time scales, and consider the time scales intervals $\overline{\mathbb{T}}_1 = [a_1, \infty) \cap \mathbb{T}_1$ and $\overline{\mathbb{T}}_2 = [a_2, \infty) \cap \mathbb{T}_2$, for $a_1 \in \mathbb{T}_1$, and $a_2 \in \mathbb{T}_2$, Ω denote the set $\overline{\mathbb{T}}_1 \times \overline{\mathbb{T}}_2$.we write $x^{\Delta_1 s}(s,t)$ the partial delta derivative of $x(s,t)$ with respect to the first variable and $x^{\Delta_2 t}(t,s)$ for the second variable.

Lemma 2.1. *[13, lemma 2] Assume that $a \geq 0$, $p \geq q \geq 0$ and $p \neq 0$, then*

$$a^{\frac{q}{p}} \leq \frac{q}{p} K^{\frac{q-p}{p}} a + \frac{p-q}{p} K^{\frac{q}{p}}, \tag{2.1}$$

for any $K > 0$.

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Lemma 2.2. *[11, Theorem 2.1] Let $u(t_1,t_2), a(t_1,t_2), f(t_1,t_2) \in C\left(\overline{\mathbb{T}}_1 \times \overline{\mathbb{T}}_2, \mathbb{R}_0^+\right)$ with $a(t_1,t_2)$ nondecreasing in each of its variables. If*

$$u(t_1,t_2) \le a(t_1,t_2) + \int_{a_1}^{t_1}\int_{a_2}^{t_2} f(s_1,s_2)\, u(s_1,s_2)\, \Delta_2 s_2 \Delta_1 s_1, \tag{2.2}$$

for $(a_1,a_2), (t_1,t_2) \in \overline{\mathbb{T}}_1 \times \overline{\mathbb{T}}_2$, then

$$u(t_1,t_2) \le a(t_1,t_2)\, e_{\int_{a_2}^{t_2} f(t_1,s_2)\, \Delta_2 s_2}(t_1,a_1), \qquad (t_1,t_2) \in \overline{\mathbb{T}}_1 \times \overline{\mathbb{T}}_2 \tag{2.3}$$

where $\mathbb{T}_1, \mathbb{T}_2$ are time scales and $\overline{\mathbb{T}}_1 = [a_1,\infty) \cap \mathbb{T}_1$, $\overline{\mathbb{T}}_2 = [a_2,\infty) \cap \mathbb{T}_2$

Lemma 2.3. *[6, Theorem 1.117] Let $a \in \mathbb{T}^k$, $b \in \mathbb{T}$ and assume $f : \mathbb{T} \times \mathbb{T}^k \to \mathbb{R}$ is continuous at (t,t), where $t \in \mathbb{T}^k$ with $t > a$. Also assume that $f^\Delta(t,.)$ is rd-continuous on $[a,\sigma(t)]$. Suppose that for each $\varepsilon > 0$ there exists a neighborhood U of t, independent of $\tau \in [a,\sigma(t)]$, such that*

$$\left| f(\sigma(t),\tau) - f(s,\tau) - f^\Delta(t,\tau)(\sigma(t)-s) \right| < \varepsilon\, |\sigma(t)-s| \ \text{ for all } s \in U,$$

where f^Δ denotes the derivative of f with respect to the first variable. Then

$$(i)\ g(t) := \int_a^t f(t,\tau)\Delta\tau \ \text{ implies } g^\Delta(t) = \int_a^t f^\Delta(t,\tau)\Delta\tau + f(\sigma(t),t);$$

Now we state the main results of this work.

3 Main result

Theorem 3.1. *Let $u(x,y), f(x,y)$ be nonnegative functions defined for $(x,y) \in \Omega$ that are right-dense continuous for $(x,y) \in \Omega$, and $L(x,y,s,t) \in C_{rd}(\Omega \times \Omega, \mathbb{R}^+)$. c, p, q, $r \in \mathbb{R}_0^+$ such that $p \ge q > 0, p \ge r > 0$.Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable increasing function on $]0,+\infty[$ with continuous decreasing first derivative on $]0,+\infty[$. If*

$$u^p(x,y) \le c + \int_{x_0}^{x}\int_{y_0}^{y} f(s,t) \left[u^q(s,t) + \int_{s_0}^{s}\int_{t_0}^{t} L(s,t,\tau,\eta) g(u^r(\tau,\eta)) \Delta_2\eta\Delta_1\tau \right] \Delta_2 t \Delta_1 s\,, \tag{3.4}$$

hold for all $(x,y) \in \Omega$, then

$$u(x,y) \le \left\{ P(x,y) e_{\int_{y_0}^{y} Q(\tau,\eta)\, \Delta_2\eta}(x,x_0) \right\}^{\frac{1}{p}}, \tag{3.5}$$

where

$$P(x,y) = c + \int_{x_0}^{x}\int_{y_0}^{y} f(s,t) \left[\frac{p-q}{p} K^{\frac{q}{p}} + g\left(\frac{p-r}{p} K^{\frac{r}{p}}\right) \int_{s_0}^{s}\int_{t_0}^{t} L(s,t,\tau,\eta)\Delta_2\eta\Delta_1\tau \right] \Delta_2 t \Delta_1 s, \tag{3.6}$$

$$Q(s,t) = f(s,t) \left[\frac{q}{p} K^{\frac{q-p}{p}} + \frac{r}{p} g'\left(\frac{p-r}{p} K^{\frac{r}{p}}\right) K^{\frac{r-p}{p}} \int_{s_0}^{s}\int_{t_0}^{t} L(s,t,\tau,\eta)\Delta_2\eta\Delta_1\tau \right], \tag{3.7}$$

and $K > 0$.

Proof. Define a function $z(x,y)$ as follows

$$z(x,y) = c + \int_{x_0}^{x}\int_{y_0}^{y} f(s,t) \left[u^q(s,t) + \int_{s_0}^{s}\int_{t_0}^{t} L(s,t,\tau,\eta) g(u^r(\tau,\eta)) \Delta_2\eta\Delta_1\tau \right] \Delta_2 t \Delta_1 s \tag{3.8}$$

then

$$z(x_0,y) = z(x,y_0) = c \tag{3.9}$$

and

$$u^p(x,y) \le z(x,y) \tag{3.10}$$

then (3.10) implies

$$u(x,y) \le z^{\frac{1}{p}}(x,y) \le \frac{1}{p}K^{\frac{1-p}{p}} z(s,t) + \frac{p-1}{p}K^{\frac{1}{p}}, \tag{3.11}$$

using (3.11) in (3.8), we get

$$z(x,y) \le c + \int_{x_0}^{x}\int_{y_0}^{y} f(s,t) \left[z^{\frac{q}{p}}(s,t) + \int_{s_0}^{s}\int_{t_0}^{t} L(s,t,\tau,\eta) g(z^{\frac{r}{p}}(\tau,\eta)) \Delta_2\eta \Delta_1\tau \right] \Delta_2 t \Delta_1 s. \tag{3.12}$$

By Lemma 2.1, the inequality (3.12) become

$$\begin{aligned} z(x,y) \quad \le \quad & c + \int_{x_0}^{x}\int_{y_0}^{y} f(s,t) \left[\frac{q}{p}K^{\frac{q-p}{p}} z(s,t) + \frac{p-q}{p}K^{\frac{q}{p}} \right. \\ & \left. + \int_{s_0}^{s}\int_{t_0}^{t} L(s,t,\tau,\eta) g\left(\frac{r}{p}K^{\frac{r-p}{p}} z(\tau,\eta) + \frac{p-r}{p}K^{\frac{r}{p}} \right) \Delta_2\eta \Delta_1\tau \right] \Delta_2 t \Delta_1 s, \end{aligned} \tag{3.13}$$

Applying the mean value theorem for the function g, then for every $x_1 \ge y_1 > 0$, there exists $c \in]y_1, x_1[$ such that

$$g(x_1) - g(y_1) = g'(c)(x_1 - y_1) \le g'(y_1)(x_1 - y_1),$$

the inequality (3.13) can be rewrite as follows

$$\begin{aligned} z(x,y) \quad \le \quad & c + \int_{x_0}^{x}\int_{y_0}^{y} f(s,t) \left[\frac{p-q}{p}K^{\frac{q}{p}} + g(\frac{p-r}{p}K^{\frac{r}{p}}) \int_{s_0}^{s}\int_{t_0}^{t} L(s,t,\tau,\eta) \Delta_2\eta \Delta_1\tau \right] \Delta_2 t \Delta_1 s \qquad (3.14) \\ & + \int_{x_0}^{x}\int_{y_0}^{y} f(s,t) z(s,t) \left[\frac{q}{p}K^{\frac{q-p}{p}} + \frac{r}{p}K^{\frac{r-p}{p}} g'(\frac{p-r}{p}K^{\frac{r}{p}}) \int_{s_0}^{s}\int_{t_0}^{t} L(s,t,\tau,\eta) \Delta_2\eta \Delta_1\tau \right] \Delta_2 t \Delta_1 s, \end{aligned}$$

replace (3.6) and (3.7) in (3.14), we obtain

$$z(x,y) \le P(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} Q(s,t) z(s,t) \Delta_2 t \Delta_1 s, \tag{3.15}$$

using Lemma 2.2 for (3.15), we get

$$z(x,y) \le P(x,y) e_{\int_{y_0}^{y} Q(s,t)\Delta_2 t}(x, x_0). \tag{3.16}$$

The required inequality (3.5) follows from (3.11) and (3.16) . □

Remark 3.1. *If we take $g(x) = x$, Theorem 3.1 will be reduced to Theorem 3.1 in [15].*

Theorem 3.2. *Assume that $u(x,y)$, $f(x,y)$ are nonnegative functions defined for $(x,y) \in \Omega$, that are right-dense continuous for $(x,y) \in \Omega$, and $L(x,y,s,t) \in C_{rd}(\Omega \times \Omega, \mathbb{R}^+)$ .Let g_1 and $g_2 : \mathbb{R}_{+\to}\mathbb{R}_+$ are a differentiable increasing functions on $]0,+\infty[$ with continuous decreasing first derivative on $]0,+\infty[$. If*

$$u^p(x,y) \le c + \int_{x_0}^{x}\int_{y_0}^{y} f(s,t) \left[g_1(u(s,t)) + \int_{s_0}^{s}\int_{t_0}^{t} L(s,t,\tau,\eta) g_2(u(\tau,\eta)) \Delta_2\eta \Delta_1\tau \right] \Delta_2 t \Delta_1 s\,, \tag{3.17}$$

hold for all $(x,y) \in \Omega$, then

$$u(x,y) \le \left\{ P_*(x,y) e_{\int_{y_0}^{y} Q_*(\tau,\eta)\ \Delta_2\eta}(x,x_0) \right\}^{\frac{1}{p}} \tag{3.18}$$

where

$$P_*(x,y) = c + \int_{x_0}^{x}\int_{y_0}^{y} f(s,t) \left[g_1(\frac{p-1}{p}K^{\frac{1}{p}}) + g_2(\frac{p-1}{p}K^{\frac{1}{p}}) \int_{s_0}^{s}\int_{t_0}^{t} L(s,t,\tau,\eta) \Delta_2\eta \Delta_1\tau \right] \Delta_2 t \Delta_1 s \tag{3.19}$$

$$Q_*(s,t) = f(s,t) \left[\frac{1}{p}K^{\frac{1-p}{p}} g_1'(\frac{p-1}{p}K^{\frac{1}{p}}) + \frac{1}{p}K^{\frac{1-p}{p}} g_2'(\frac{p-1}{p}K^{\frac{1}{p}}) \int_{s_0}^{s}\int_{t_0}^{t} L(s,t,\tau,\eta) \Delta_2\eta \Delta_1\tau \right]. \tag{3.20}$$

For $K > 0$.

Proof. Define a function $z(x,y)$ as follows

$$z(x,y) = c + \int_{x_0}^{x}\int_{y_0}^{y} f(s,t)\left[\begin{array}{c} g_1(u(s,t))+ \\ \int_{s_0}^{s}\int_{t_0}^{t} L(s,t,\tau,\eta) g_2(u(\tau,\eta))\Delta_2\eta\Delta_1\tau \end{array}\right]\Delta_2 t\Delta_1 s\,, \tag{3.21}$$

Applying the mean value theorem for the functions g_1 and g_2, from (3.11) and (3.21), we obtain

$$z\,(x,y) \le c + \int_{x_0}^{x}\int_{y_0}^{y} f\,(s,t)\left[\begin{array}{c} g_1(\frac{1}{p}K^{\frac{1-p}{p}} z(\tau,\eta) + \frac{p-1}{p}K^{\frac{1}{p}})+ \\ \int_{s_0}^{s}\int_{t_0}^{t} L(s,t,\tau,\eta) g_2(\frac{1}{p}K^{\frac{1-p}{p}} z(\tau,\eta) + \frac{p-1}{p}K^{\frac{1}{p}})\Delta_2\eta\Delta_1\tau \end{array}\right]\Delta_2 t\Delta_1 s. \tag{3.22}$$

The above inequality can be reformulated as

$$z\,(x,y) \le P_*(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} Q_*(s,t) z(s,t)\Delta_2 t\Delta_1 s, \tag{3.23}$$

where P_* and Q_* are defined by (3.19)-(3.20).
Using Lemma 2, from (3.23) we obtain

$$u(x,y) \le \left\{P_*(x,y) e_{\int_{y_0}^{y} Q_*(\tau,\eta)\ \Delta_2\eta}(x,x_0)\right\}^{\frac{1}{p}}. \tag{3.24}$$

The required inequality (3.18) follow from (3.11) and (3.24). □

Remark 3.2. *If we take $g_1(x) = x$, Theorem 3.2 will be reduced to Theorem 3.1 for $q = r = 1$.*

4 An Application

In this section we give an application of Theorem 3.1 . We consider the following partial dynamic equation on time scales

$$(u^p(x,y))^{\Delta_2 y\Delta_1 x} = F(x,y,u^q(x,y), \int_{x_0}^{x}\int_{y_0}^{y} h(s,t,\tau,\eta,u(\tau,\eta))\Delta\eta\Delta\tau), \tag{4.25}$$

with the initial boundary conditions

$$u(x,y_0) = \alpha(x), u(x_0,y) = \beta(y),\ \alpha(0) = \beta(0) = 0. \tag{4.26}$$

where $u \in C_{rd}(\Omega,\mathbb{R}), h \in C_{rd}(\Omega\times\Omega\times\mathbb{R},\mathbb{R})$ and $F \in C_{rd}(\Omega\times\mathbb{R}\times\mathbb{R},\mathbb{R})$.

Proposition 4.1. *Assume that*

$$\begin{aligned} |h(x,y,s,t,u(s,t)| &\le L(x,y,s,t)\arctan(|u(s,t)|^r) \\ |F(x,y,u,v)| &\le f(x,y)(|u|+|v|), \\ |\alpha(x)+\beta(y)| &\le c, \end{aligned} \tag{4.27}$$

where L, f, c, p, q, r are defined as in Theorem 3.1.

If $u(x,y)$ is a solution of (4.25)-(4.26), then

$$u(x,y) \le \left\{P(x,y) e_{\int_{y_0}^{y} Q(\tau,\eta)\ \Delta\eta}(x,x_0)\right\}^{\frac{1}{p}}, \tag{4.28}$$

where $P(x,y)$, $Q(x,y)$ are defined as in (3.6)-(3.7) respectively (by replacing $g(x)$ by $\arctan(x)$ and $g'(x)$ by $\frac{1}{1+x^2}$).

Proof. The solution $u(x,y)$ can be written as

$$u^p(x,y)=\alpha(x)+\beta(y)+\int_{x_0}^{x}\int_{y_0}^{y}F(s,t,u^q(s,t),\int_{s_0}^{s}\int_{t_0}^{t}h(s,t,\tau,\eta,u(\tau,\eta))\Delta_2\eta\Delta_1\tau)\Delta_2t\Delta_1s, \tag{4.29}$$

using (4.27) in (4.29), we have

$$|u^p(x,y)|\le c+\int_{x_0}^{x}\int_{y_0}^{y}f(s,t)(|u^q(s,t)|+\int_{s_0}^{s}\int_{t_0}^{t}L(s,t,\tau,\eta)\arctan|u(\tau,\eta)|^r\Delta_2\eta\Delta_1\tau)\Delta_2t\Delta_1s, \tag{4.30}$$

Now, a suitable application of Theorem 3.1 for (4.30), yields the inequality (4.28). □

Remark 4.3. *We can also replace the function* $\arctan(|u(s,t)|^r)$ *by* $\ln(|u(s,t)|^r+1)$ *in* (4.27) *to obtain another estimate of the solution of* (4.25) − (4.26).

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Received: June 07, 2016; *Accepted*: November 23, 2016

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